Total momentum from dynamics in quantum space-time

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Quantum space-times

Quantum gravity proposals suggests that space-time should be “fuzzy” at short distance scales ($\ell \simeq l_p$), a feature encoded in models equipped with space-time non-commutativity.

Much effort has been put in understanding the kinematics inspired from the non-commutative framework.

We will investigate particle dynamics on these non-commutative spaces, gaining some insight on their properties.

Hopf-Algebra Symmetries for empty 1+1D κ-Minkowski

Coproducts

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 \qquad \Delta P_1 = P_1 \otimes 1 + e^{-\ell P_0} \otimes P_1 \qquad \Delta N = N \otimes 1 + e^{-\ell P_0} \otimes N$$

Commutators

$$[P_0, P_1] = 0 \qquad [N, P_0] = iP_1 \qquad [N, P_1] = \frac{i}{2\ell}\left(1 - e^{-2\ell P_0}\right) - \frac{i\ell}{2} P_1^2$$

Casimir Operator

$$C = \frac{4}{\ell^2} \sinh^2\left(\frac{\ell}{2} P_0\right) - (P_1)^2 e^{\ell P_0}$$

Non-commutative space-time

$$[x_1, x_0] = i\ell x_1$$

S.Majid and H.Ruegg, "Bicrossproduct structure of kappa Poincare group and noncommutative geometry,'' Phys. Lett. B {334} (1994), 348-354

Total momentum…

The standard interpretation is to regard the generators P_0, P_1, N as conserved charges of free particle motion

The coproducts inspire the composition law of said charges

$$(p \oplus k)_0 = p_0 + k_0$$

$$(p \oplus k)_1 = p_1 + k_1 + e^{-\ell p_0} k_1$$

$$\left(N_p \oplus N_k\right) = N_p + N_k + e^{-\ell p_0} N_k$$

The composition law is the fundamental building block needed to construct total momentum…

…but which one?

…but what is total momentum?

In the free particle case, even just the single momentum of each particle is a conserved charge!

Even if one commits to the $\oplus$ interpretation, there are still intrinsic ambiguities

$$(p \oplus k)_1 = p_1 + k_1 + e^{-\ell p_0} k_1 \neq (k \oplus p)_1 = p_1 + k_1 + e^{-\ell k_0} p_1$$

Can dynamics teach us something about total momentum?

Spatial "(2+0)D" κ-Minkowski (1)

Let us make the redefinitions

$$\ell \to i\ell \qquad x_0 \to ix_2 \qquad P_0 \to -iP_2 \qquad N \to -iR$$

The Hopf Algebra becomes

$$[P_1, P_2] = 0 \qquad [R, P_1] = \frac{i}{2\ell}\left(1 - e^{-2\ell P_2}\right) + \frac{i\ell}{2} P_1^2 \qquad [R, P_2] = -iP_1$$

$$\Delta P_1 = P_1 \otimes 1 + e^{-\ell P_2} \otimes P_1 \qquad \Delta P_2 = P_2 \otimes 1 + 1 \otimes P_2$$

$$\Delta R = R \otimes 1 + e^{-\ell P_2} \otimes R$$

G.Amelino-Camelia, ``Planck-scale soccer-ball problem: a case of mistaken identity,'' Entropy {19} (2017) no.8, 400

Spatial "(2+0)D" $\boldsymbol{\kappa}$-Minkowski (2)

The Casimir operator now reads

$$C = \frac{4}{\ell^2} \sinh^2\left(\frac{\ell}{2} P_2\right) + (P_1)^2 e^{\ell P_2}$$

and the coordinate non-commutativity becomes

$$[x_1, x_2] = i\ell x_2$$

To ensure Jacobi identities are satisfied, the symplectic structure is also deformed

$$[P_1, x_1] = i \qquad [P_2, x_2] = i \qquad [P_1, x_2] = 0 \qquad [P_1, x_2] = -i\ell P_1$$

Now that x_0 is a commutative variable, we can set up standard Hamiltonian analysis in the deformed Galilean Relativistic regime.

A case study: the elastic potential

In standard QM, a system of two particles interacting via an elastic potential is characterized by conservation of total momentum and total angular momentum

$$H = \frac{(\vec{p}_A)^2}{2m} + \frac{(\vec{p}_B)^2}{2m} + \frac{1}{2}\rho(\vec{q}_A - \vec{q}_B)^2$$

The conserved charges read

$$\vec{P} = \vec{p}_A + \vec{p}_B \qquad \vec{L} = \vec{L}_A + \vec{L}_B \qquad \vec{L}_I = \vec{x}_I \wedge \vec{p}_I$$

Indeed

$$[H, \vec{L}] = \left[H, \vec{P}\right] = 0$$

Our objective will be to construct a deformed Hamiltonian that commutes with total momentum and total angular momentum, as inspired from non-commutativity

The deformed Hamiltonian

Keeping our analysis up to second order in the deformation parameter ℓ^2, the deformed kinetic term is given by

$$H_K^I = \frac{(p_1^I)^2}{2m} + \frac{(p_2^I)^2}{2m} + \ell \frac{(p_1^I)^2 p_2^I}{2m} + \frac{\ell^2 (p_1^I)^2 (p_2^I)^2}{4m} + \frac{\ell^2 (p_2^I)^4}{24m}$$

while the most general ansatz for the quadratic potential reads

$$V(\vec{x}^A, \vec{x}^B) = \frac{1}{2}\rho(\vec{x}^A - \vec{x}^B)^2 + \ell\rho \sum \alpha_{ijk}^{IJK} p_i^I x_j^J x_k^K + \ell^2 \rho \sum \beta_{ijkh}^{IJKH}\, p_i^I p_j^J x_k^K x_h^H$$

The full Hamiltonian for two particles is simply $H = H_K^A + H_K^B + V(\vec{x}^A, \vec{x}^B)$

Vanilla κ-Minkowski : two particle dynamics

Adopting the usual physical interpretation, for the total charges to be symmetries of the dynamics we require

$$[(p^A \oplus p^B)_1, H] = 0 \qquad [(p^A \oplus p^B)_2, H] = 0 \qquad [(R^A \oplus R^B), H] = 0$$

This is equivalent to a system of equations involving coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. At first order, we find the solution

$$H = H_K^A + H_K^B + \frac{1}{2}\rho(\vec{x}_A - \vec{x}_B)^2 + \frac{1}{2}\ell\rho\left[p_2^A x_1^A\left(x_1^A - x_1^B\right) - \left(p_1^A x_1^A - 2p_1^B x_1^B\right)\left(x_2^A - x_2^B\right)\right] + O(\ell^2)$$

Vanilla κ-Minkowski : three particle dynamics

A possible separable three particle Hamiltonian for this model simply reads

$$H^{ABC} = H_K^A + H_K^B + H_K^C + V(\vec{x}^A, \vec{x}^B) + V(\vec{x}^A, \vec{x}^C) + V(\vec{x}^B, \vec{x}^C) + O(\ell^2)$$

And it can be explicitly checked that

$$[(p^I \oplus p^J \oplus p^K)_i, H^{ABC}] \neq 0 \quad [(R^I \oplus R^J \oplus R^K), H^{ABC}] \neq 0 \quad I, J, K = A, B, C \quad i = 1,2$$

Even the most general ansatz for a non-separable Hamiltonian fails ⇒ No three-particle dynamics for the coproduct inspired total momentum!

Proper-dS framework

Momentum space model with spatial κ-Minkowski algebra structure but composition laws given by

$$(p^A \oplus p^B)_1 = p_1^A + p_1^B - \ell\left(p_2^A p_1^B + p_1^A p_2^B\right) + \frac{\ell^2}{2}\left[\left(p_2^A p_1^B + p_1^A p_2^B\right)\left(p_2^A + p_2^B\right) - p_1^A (p_1^B)^2 - \left(p_1^A\right)^2 p_1^B\right]$$

$$(p^A \oplus p^B)_2 = p_2^A + p_2^B + \ell p_1^A p_1^B - \frac{\ell^2}{2}\left[\left(p_1^A\right)^2 p_2^B + (p_1^B)^2 p_2^A - p_1^A p_1^B\left(p_2^A + p_2^B\right)\right]$$

$$(R^A \oplus R^B) = R^A + R^B$$

The composition law for momentum is commutative but non-associative.

G.Amelino-Camelia, G.Gubitosi and G.Palmisano, ``Pathways to relativistic curved momentum spaces: de Sitter case study,'' Int. J. Mod. Phys. D {25} (2016) no.02, 1650027

Proper-dS: dynamics

Requiring that the total charges are symmetries of the dynamics, we obtain infinitely many solutions up to second order in $\ell^2 \Rightarrow$ A potential $V(\vec{x}^A, \vec{x}^B)$ exists.

The three-particle separable Hamiltonian

$$H^{ABC} = H_K^A + H_K^B + H_K^C + V(\vec{x}^A, \vec{x}^B) + V(\vec{x}^A, \vec{x}^C) + V(\vec{x}^B, \vec{x}^C)$$

doesn't work with total charges $\left((p^A \oplus p^B) \oplus p^C\right)_i$ nor $(p^A \oplus (p^B \oplus p^C))_i$. We need to add a non-separable term of the form

$$H^{NS} = \frac{1}{2} l^2 \rho [\left(p_2^C p_1^A - p_1^C p_2^A\right)\left(x_2^C x_1^B - x_1^C x_2^B\right) + p_2^B (p_1^C x_1^C x_2^A - p_1^C x_2^C x_1^A + p_1^A\left(p_2^A p_1^C\left(2x_1^C - x_1^A - x_1^B\right) - 2x_1^C x_2^C + x_1^C x_2^B + x_2^C x_1^A\right) +$$
$$+ p_1^B\left(p_2^C\left(x_2^C x_1^A - x_1^C x_2^A\right) + p_1^A x_2^C (2x_2^C - x_2^A - x_2^B\right) + p_2^A x_1^C\left(x_2^A - 2x_2^C\right) + p_2^A x_2^C x_1^B]$$

for the first combination to be conserved.

Conclusions

With the coproduct-inspired composition law, we have no compatible dynamics for three particle-systems.

By a method of reverse engineering, we have found a Hamiltonian compatible with the proper-dS scenario for multiparticle-systems.

Take-home message: we should regard dynamics as a fundamental element in determining total momentum for non-commutative inspired models.

Thank you for your attention!