Geometrize and conquer: Klein–Gordon and Dirac equations in Doubly Special Relativity

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- In most of them a minimal length appears $\Longrightarrow$ Planck length (l_P)??
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- Problem: there are no experimental evidences of a fundamental QGT

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- New effects $\rightarrow$ Micro black holes creation?
- Spacetime can be regarded as a "foam"

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- There is a privileged observer $\rightarrow$ physical laws depending on the observer
- Formulated in the quantum field theory framework $\rightarrow$ standard model extension (SME)

Doubly Special Relativity (DSR)

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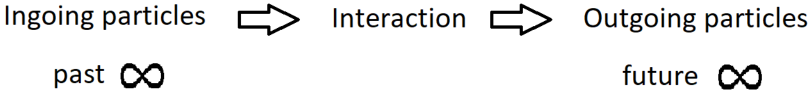
- There is a relativity principle
- Two invariants in every inertial frame: speed of light c and Planck length l_P


Ingoing particles
Interaction
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past ∞
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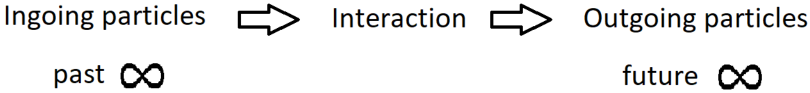

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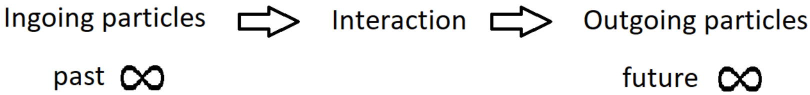

- Ingoing and outgoing particles movement is described by the dispersion relation
- In the interaction, the conservation of total momentum holds

Kinematics in DSR

- Dispersion relation

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- Dispersion relation and conservation law compatible with relativity principle $\rightarrow$ deformed Lorentz transformations

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- Deformed conservation law (DCL)

$$(p \oplus q)_0 = p_0 + q_0 \,, \qquad (p \oplus q)_i = p_i e^{q_0/2\Lambda} + q_i e^{-p_0/2\Lambda}$$

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- Dispersion relation → Squared distance from the origin to k [Amelino-Camelia et al., 2011]
- Translations, deformed "Lorentz" generators → 10 isometries of the metric!
- Only a maximally symmetric momentum space (MSS) satisfies this! → Minkowski, de Sitter or anti de Sitter

Construction of kinematics: de Sitter momentum space

- Start by a momentum metric

$$g_{00}(p) = 1\,, \quad g_{0i}(p) = g_{i0}(p) = \frac{p_i}{2\Lambda}\,, \quad g_{ij}(p) = -\delta^i_j e^{-p_0/\Lambda} + \frac{p_i p_j}{4\Lambda^2}$$

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- Compute the Casimir using [Relancio and Liberati, 2020]

$$C_{\mathrm{D}}(p) = f^\mu g_{\mu\nu}(p) f^\nu\,, \qquad f^\mu(p) := \frac{1}{2}\frac{\partial C_{\mathrm{D}}(p)}{\partial p_\mu}$$

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- Using this metric one obtains the same kinematics of κ-Poincaré in the symmetric basis!

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- Snyder kinematics [Battisti and Meljanac, 2010]

$$(p \oplus q)_\mu^{\text{Snyder}} = p_\mu \left(\sqrt{1 + \frac{q^2}{\Lambda^2}} + \frac{p_\mu \eta^{\mu\nu} q_\nu}{\Lambda^2 \left(1 + \sqrt{1 + p^2/\Lambda^2}\right)} \right) + q_\mu$$

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- κ-Poincaré in classical basis [Borowiec and Pachol, 2010]

$$(p \oplus q)_\mu^{\kappa-\textbf{Poincaré}} = p_\mu \left(\sqrt{1 + \frac{q^2}{\Lambda^2}} + \frac{q_0}{\Lambda} \right) + q_\mu + n_\mu \left[\frac{\sqrt{1 + p^2/\Lambda^2} - p_0/\Lambda}{1 - \vec{p}^2/\Lambda^2} \left(q_0 + \frac{q_\alpha \eta^{\alpha\beta} p_\beta}{\Lambda} \right) - q_0 \right]$$

where $n_\mu := (1, 0, 0, 0)$.

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Deformed Klein–Gordon equation: construction

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Deformed Klein–Gordon equation: construction

- Klein–Gordon and Dirac equations already obtained in Hopf algebras [Lukierski et al., 1992, Nowicki et al., 1993]
- Our aim $\rightarrow$ geometrical derivation of these equations [Franchino-Viñas and Relancio, 2022]
- We are able to reproduce them from a curved momentum space!

Klein–Gordon equation: construction

- Klein-Gordon equation derived from the Casimir (squared distance)

$$\left(\Lambda^2 \operatorname{arccosh}^2\left(\cosh\left(\frac{p_0}{\Lambda}\right) - \frac{\vec{p}^2}{2\Lambda^2}\right) - m^2\right)\phi(p) = 0$$

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- Different Casimirs: possible different behavior at ultraviolet regime.

Klein–Gordon equation: invariance

- Action in momentum space

$$S_{\mathrm{KG}} := \int \mathrm{d}^4p \, \sqrt{-g} \, \phi^*(p) \left(C_{\mathrm{D}}(p) - m^2 \right) \phi(p)$$

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- $\sqrt{-g}$ guarantees invariance under a change of momentum basis.
- Invariance under deformed Lorentz transformations of the metric assuming the field transforms as a scalar

$$\phi'(p') = \phi(p)$$

since

$$C_{\mathrm{D}}(p) = C_{\mathrm{D}}(p')$$

Dirac equation: formulation

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- The new gamma matrices satisfy

$$\{\underline{\gamma}^{\mu}, \underline{\gamma}^{\nu}\} = 2 g^{\mu\nu}(p) \mathbb{1}$$

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- This equation can be obtained from the action

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- Klein–Gordon equation is obtained straightforwardly from

$$\left(\underline{\gamma}^{\nu} f_{\nu}(p) - m \right) \left(\underline{\gamma}^{\nu} f_{\nu}(p) + m \right) \, = \, C_{\mathrm{D}}(p) - m^2$$

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\begin{aligned}
\mathcal{P}_0 &:= \mathrm{i}\gamma^0\,, & \tilde{\psi}_{\mathcal{P}} &:= \mathrm{i}\gamma^0\tilde{\psi}(p_0,-\vec{p})\,,\\
\mathcal{T}_0 &:= \mathrm{i}\gamma^1\gamma^3\mathcal{K}\,, & \tilde{\psi}_{\mathcal{T}} &:= \mathrm{i}\gamma^1\gamma^3\tilde{\psi}^*(p_0,-\vec{p})\,,\\
\mathcal{C}_0 &:= \mathrm{i}\gamma^2\mathcal{K}\,, & \tilde{\psi}_{\mathcal{C}} &:= \mathrm{i}\gamma^2\tilde{\psi}^*(-p)\,.
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- Invariant under $\mathcal{P}$ and $\mathcal{T}$
- Invariant under $\mathcal{C}$ when $\Lambda \to -\Lambda$

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so for $q\rightarrow 0$

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- We can construct the Dirac equation for different relativistic kinematics!

Dirac equation in κ-Poincaré

- For the symmetric basis we find

$$\mathcal{D}_{\mathrm{D}}^{(S)} := \frac{\sqrt{\frac{C_{\mathbf{D}}^{(S)}(p)}{\Lambda^2}}}{2\Lambda \sinh\left(\sqrt{\frac{C_{\mathbf{D}}^{(S)}(p)}{\Lambda^2}}\right)} \left[2\Lambda e^{-\frac{p_0}{2\Lambda}} \gamma^i p_i + \gamma^0 \left(2\Lambda^2 \sinh\left(\frac{p_0}{\Lambda}\right) - \vec{p}^2\right)\right]$$

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- If we use instead $C^{(S)}_{\mathrm{A}}(p)$

$$\mathcal{D}^{(S)}_{\mathrm{A}} := \gamma^0 \left(\Lambda \sinh\left(\frac{p_0}{\Lambda}\right) - \frac{\vec{p}^2}{2\Lambda}\right) + e^{-p_0/2\Lambda} p_i \gamma^i$$

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- Our construction leads to

$$\left(\mathcal{D}_{\mathrm{D}}^{(S)}\right)^{2} = C_{\mathrm{D}}^{(S)}$$

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- We have made a first attempt into the identification of the relevant Hilbert space in a quantization process

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- We obtain the Klein–Gordon and Dirac equations in κ-Poincaré from a de Sitter momentum space, which are the same results obtained in the Hopf algebra scheme
- Analogous equations can be obtained for other kinematics, such as Snyder model
- We have made a first attempt into the identification of the relevant Hilbert space in a quantization process
- Future work: include interactions

Thanks for your attention!

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