Quantum Euler angles and agency-dependent properties of spacetime

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Introduction

- In classical Minkowski spacetime the connection between two inertial observers reference frame is determined by an element of the Poincaré group.
- Quantum space: we study the relative state of the reference frames of two observers.
- We focus on the case in which the two reference frames are just rotated.
- We will assume that the classical rotation group $SO(3)$ at quantum level is replaced by its quantum group deformation $SO_q(3)$.

$SU(2)$ coordinatization and homomorphism with $SO(3)$

- In classical and quantum mechanics, rotations are described by the group $SU(2)$.

$$SU(2) \ni U = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix} \quad a, c \in \mathbb{C} \ : \ |a|^2 + |c|^2 = 1$$

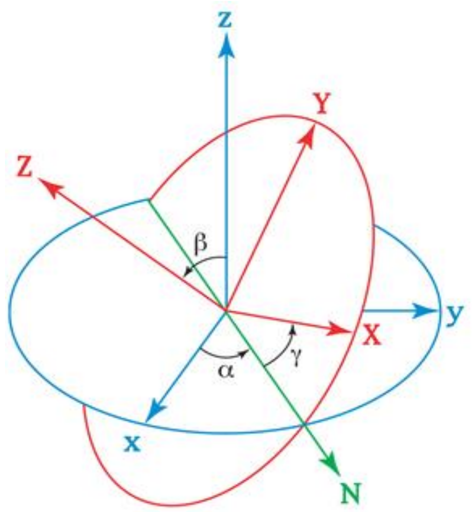
$$a = e^{i\chi} \cos\left(\frac{\theta}{2}\right) \quad c = e^{i\phi} \sin\left(\frac{\theta}{2}\right)$$

$$R = \begin{pmatrix} \frac{1}{2}(a^2 - c^2 + (a^*)^2 - (c^*)^2 & \frac{i}{2}(-a^2 + c^2 + (a^*)^2 - (c^*)^2 & a^*c + c^*a \\ \frac{i}{2}(a^2 + c^2 - (a^*)^2 - (c^*)^2 & \frac{1}{2}(a^2 + c^2 + (a^*)^2 + (c^*)^2 & -i\,(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1 - 2cc^* \end{pmatrix}$$

Classical Euler Angles

The $SU(2)$ parameters are linked to Euler angles as follows:

$$\begin{cases} \theta = \beta \\ \chi = \frac{\alpha+\gamma}{2} \\ \phi = \frac{\pi}{2} - \frac{\alpha-\gamma}{2} \end{cases}$$



Classical protocol for the alignement of reference frames

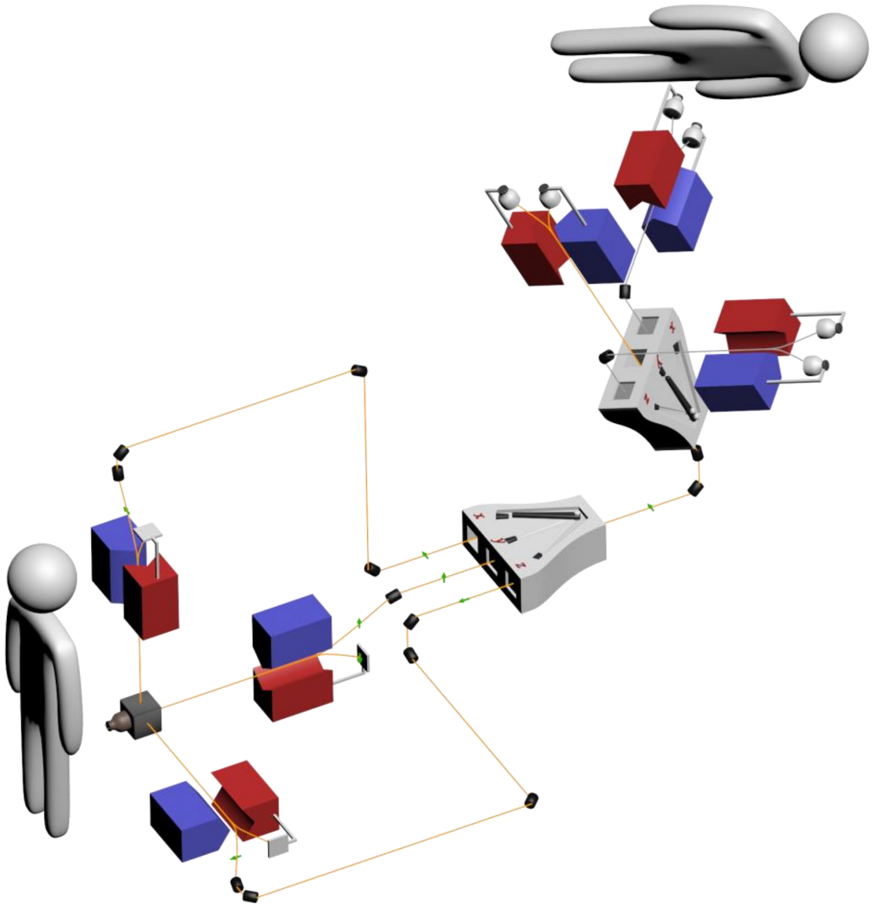
Qubit state: $\rho = \frac{1}{2}(1 + \vec{r} \cdot \vec{\sigma})$ Bloch vector: $r^i = Tr(\rho\, \sigma^i)$

Consider now an *SU*(2) transformation of the quantum state:

$$\rho \rightarrow U\rho U^\dagger = \frac{1}{2}(1 + \vec{r}\,' \cdot \vec{\sigma}) = \frac{1}{2}(1 + R\vec{r} \cdot \vec{\sigma})$$

$$R^i_j = \frac{1}{2} Tr\left(U\sigma_j U^\dagger \sigma^i\right)$$

Classical protocol for the alignement of reference frames (2)



$SU_q(2)$ algebra

- The quantum group $SU_q(2)$ is defined by considering the algebra of complex functions on $SU(2)$, denoted by $C(SU(2))$ and deforming it in a non commutative way.

- $U_q = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$ $\quad a, c \in \ C(SU_q(2))$and q$\in (0,1)$

where :

- $ac = qca; \quad ac^* = qc^*a; \quad cc^* = c^*c; \quad c^*c + a^*a = 1; \quad aa^* - a^*a = (1 - q^2)c^*c$

. Podles, “Symmetries of quantum spaces. subgroups and quotient spaces of quantumsu (2) andso (3) groups,” Communications in Mathematical Physics, vol. 170, no. 1, pp. 1–20, 1995

Homomorphism with $SO_q(3)$

- As in the classical case we can construct a homomorphism between $SU_q(2)$ and $SO_q(3)$ and we obtain the following q-deformed rotation matrix:

- $$R_q = \begin{pmatrix} \frac{1}{2}(a^2 - qc^2 + (a^*)^2 - q(c^*)^2 & \frac{i}{2}(-a^2 + qc^2 + (a^*)^2 - q(c^*)^2 & \frac{1}{2}(1+q^2)(a^*c + c^*a) \\ \frac{i}{2}(a^2 + qc^2 - (a^*)^2 - q(c^*)^2 & \frac{1}{2}(a^2 + qc^2 + (a^*)^2 + q(c^*)^2 & -\frac{i}{2}(1+q^2)\,(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1 - (1+q^2)cc^* \end{pmatrix}$$

$SU_q(2)$ representations

- The Hilbert space containing the two unique irreducible representations of the $SU_q(2)$ algebra is $H = H_\pi \oplus H_\rho$ where $H_\pi = L^2(S^1) \otimes L^2(S^1) \otimes \ell$ and $H_\rho = L^2(S^1)$
- $\rho(a)|\eta\rangle = e^{i\eta}|\eta\rangle;$ $\rho(a^*)|\eta\rangle = e^{-i\eta}|\eta\rangle;$ $\rho(c)|\eta\rangle = 0;$ $\rho(c^*)|\eta\rangle = 0;$
- $\pi(a)|n,\delta,\epsilon\rangle = e^{i\epsilon}\sqrt{(1-q^{2n})}|n-1,\delta,\epsilon\rangle;$ $\pi(a^*)|n,\delta,\epsilon\rangle = e^{-i\epsilon}\sqrt{(1-q^{2n+2})}|n+1,\delta,\epsilon\rangle;$
- $\pi(c)|n,\delta,\epsilon\rangle = e^{i\delta}q^n|n,\delta,\epsilon\rangle;$ $\pi(c^*)|n,\delta,\epsilon\rangle = e^{-i\delta}q^n|n,\delta,\epsilon\rangle;$
- $a = e^{i\chi}\cos\left(\frac{\theta}{2}\right)$ $c = e^{i\phi}\sin\left(\frac{\theta}{2}\right)$ (Classical case)

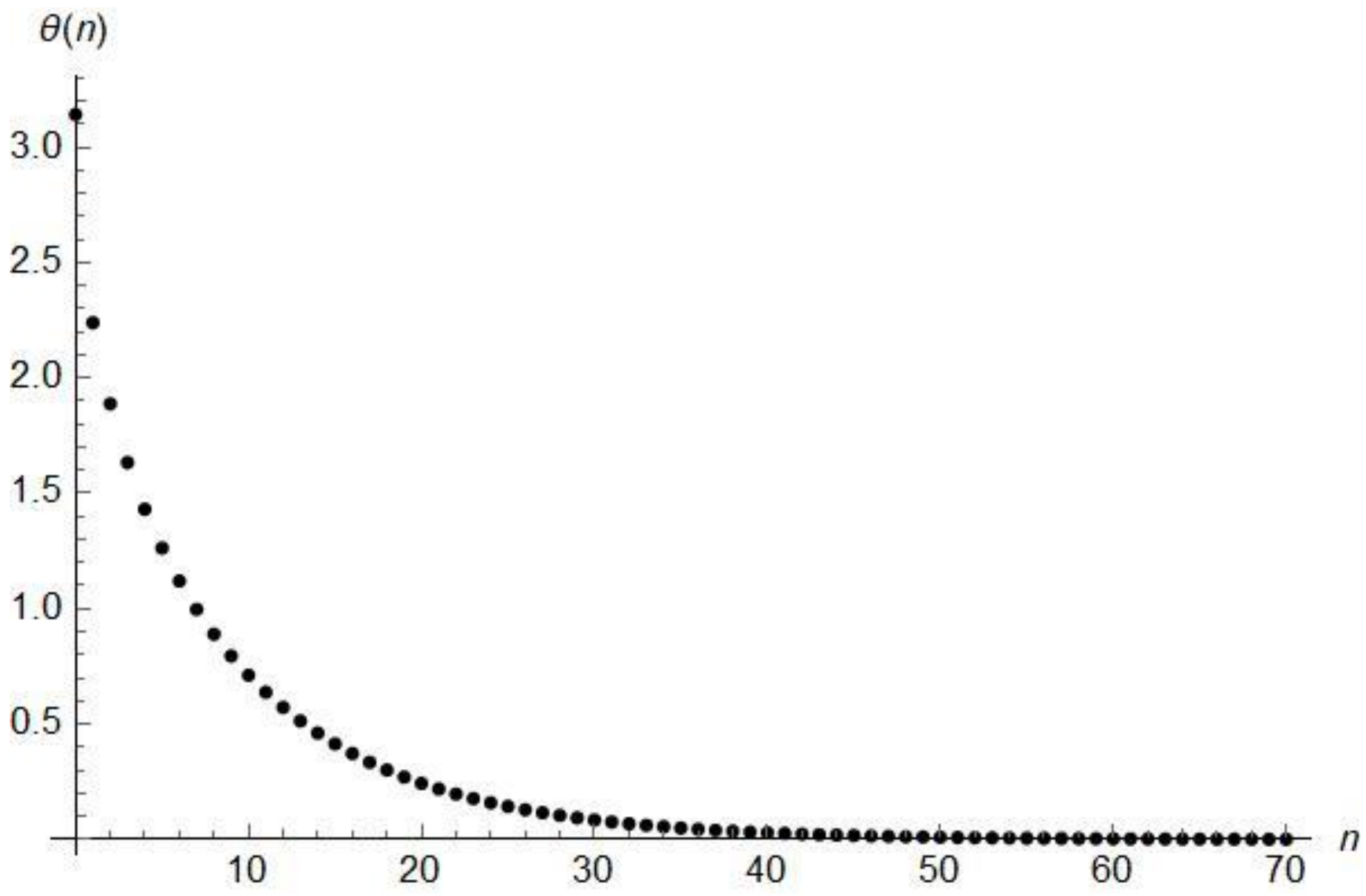
Quantum Euler angles (1)

- Comparing the phases of operators a and c and their classical analogues, we identify ε with χ and δ with φ.

- Then, exploiting the fact that c is diagonal, we are led to a significant result:

- $$q^n = \sin\left(\frac{\theta(n)}{2}\right) \Longleftrightarrow \theta(n) = 2\arcsin(q^n)$$

Quantum Euler angles (2)



Quantum Rotations

- A state $|\psi\rangle \in H$ is representative of the relative orientation between two reference frames, A and B.

- Our proposal is that the mean value of R_q on $|\psi\rangle$ will give an estimate of the entries of the rotation matrix that connects A and B

$$\langle\psi|R_q|\psi\rangle_{\mathrm{ij}}$$

- However, due to the non commutatitvity we will have in general a non vanishing variance for the matrix elements:

$$\Delta_{ij} = \sqrt{\langle\psi|R_q^2|\psi\rangle_{\mathrm{ij}} - \langle\psi|R_q|\psi\rangle_{\mathrm{ij}}^2}$$

Quantum Rotations (around z-axis)

- Consider a state $|\chi\rangle$ in representation ρ.

- The mean value of the rotation matrix is:

$$\langle\chi|R_q|\chi\rangle_{\text{ij}} = \begin{pmatrix} \cos(2\chi) & -\sin(2\chi) & 0 \\ \sin(2\chi) & \cos(2\chi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which coincides with a classical rotation matrix that describes a rotation of 2χ around the *z*-axis.

- $\Delta_{ij} = 0 \rightarrow$ Classical rotations

Quantum Rotations (general case)

- Consider a state $|\psi\rangle$ in representation π.
- Since (ϕ, χ) are mapped in a trivial way to the classical angles, it is reasonable to assume that the state that better describes a rotation with θ, ϕ, χ as Euler angles is of the form:

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n, \phi, \chi\rangle$$

- In general if we calculate Δ_{ij} on such a state, they do not vanish simultaneously, and in this sense we might say that in this framework a fuzzyness is introduced.

Rotation of π about the x-axis (Example)

- $|\psi\rangle = \left|0, \frac{\pi}{2}, 0\right\rangle$

- $\langle\psi|R_q|\psi\rangle = \begin{pmatrix} 1-(1-q) & 0 & 0 \\ 0 & -1+(1-q) & 0 \\ 0 & 0 & -1+2(1-q) \end{pmatrix} + \mathrm{o}(1-\mathrm{q})$

- $\langle\psi|\Delta R_q|\psi\rangle = \begin{pmatrix} \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2(1-q)} & \sqrt{2(1-q)} & 0 \end{pmatrix} + o(1-q)$

Agency dependent properties of spacetime

- Because the z-axis is «special», in the sense that rotation around it are sharp the space that an observer sees depends on the choice of this axis $\Longleftarrow\!\!\Longrightarrow$ agency dependent spacetime.

Thank you