Momentum Gauge Fields and Non-Commutative Space-Time

(and Momentum field theory and Braneworlds)

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Quantum Phenomenology 3rd COST conference Napoli, July 13th 2022

In the standard formulation of a gauge theory one starts with a space-time dependent matter field $\Psi(x)$ which satisfies some matter field equation (*e.g.* Schrödinger equation, Klein-Gordon equation, Dirac equation) and requires that this matter field satisfy a local phase symmetry of the form $\Psi(x) \to e^{-i\lambda(x)}\Psi(x)$. The gauge function, $\lambda(x)$, can depend on space and time. Along with this local phase symmetry of the matter field, one needs to introduce the kinetic momentum/gauge covariant derivative $p_i \to p_i - eA_i(x)$ or $\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} - ieA_i(x)$, where the vector potential obeys $A_i(x) \to A_i(x) - \frac{1}{e}\frac{\partial \lambda(x)}{\partial x_i}$. This standard construction is done in position space: the matter field, Ψ is a function of position, the momentum operator is given as a derivative of position ($p_i = -i\frac{\partial}{\partial x_i}$ and we take $\hbar = 1$), and the vector potential and gauge function are functions of space and time coordinates.

However, quantum mechanics can be carried out in momentum space as well with the matter field being a function of momentum, $\Psi(p)$, and the position operator being given by $x_i = i\frac{\partial}{\partial p_i}$. In this construction the momentum operator is just multiplication by p_i just as the position operator in position space is multiplication by x_i. The momentum space gauge transformation of the matter field should be

$$\Psi(p) \rightarrow e^{-i\eta(p)}\Psi(p) \ . \tag{1}$$

The equivalent of the generalized position/gauge covariant derivative is

$$x_i \rightarrow x_i - gC_i(p) \quad \text{or} \quad \frac{\partial}{\partial p_i} \rightarrow \frac{\partial}{\partial p_i} + igC_i(p). \tag{2}$$

We have used $x_i = i\frac{\partial}{\partial p_i}$, g is some momentum-space coupling, and $C_i(p)$ is a momentum-space gauge function which must satisfy

$$C_i(p) \rightarrow C_i(p) + \frac{1}{g}\frac{\partial \eta(p)}{\partial p_i}. \tag{3}$$

Finally one can construct a momentum-space field strength tensor which is invariant under just (3), namely

$$G_{ij} = \frac{\partial C_i}{\partial p_j} - \frac{\partial C_j}{\partial p_i}. \tag{4}$$

This is the $p_i p_j$ component of the momentum gauge field, field strength tensor. It is the analog of $x_i x_j$ component of the standard gauge field, field strength tensor $F_{ij} = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}$. The 4-vector version of the standard gauge potential and field strength tensor are $A_i \to A_\mu$ and $F_{ij} \to F_{\mu\nu}$. One needs to make a similar 4-vector/4-tensor extension for the momentum gauge field and associated field strength tensor via $C_i(p) \to C_\mu(p)$ and $G_{ij} \to G_{\mu\nu}$.

II. CONNECTION TO NON-COMMUTATIVE SPACE-TIME

A. Constant non-commutativity parameter

In this subsection we point out the connection of the above momentum gauge theory with non-commutative geometry, by which we mean coordinates obeying

$$[x_i, x_j] = i\Theta_{ij} \ , \tag{7}$$

where Θ_{ij} is an anti-symmetric, *constant* rank-2 tensor

The construction from the previous section leads exactly to this kind of non-commutativity between the coordinates. We begin with equation (2) and define a generalized, gauge invariant coordinate $X_i = x_i - gC_i(p) = i\partial_{p_i} - gC_i(p)$. In its first form this looks like coordinate translation by $gC_i(p)$. Calculating the commutator of X_i and X_j gives

$$[X_i, X_j] = igG_{ij} \ , \tag{8}$$

with the momentum-space field strength G_{ij} defined in (4). Equation (8) is of the form (7) with $\Theta_{ij} = gG_{ij}$.

The result in (8) is reminiscent of the non-commutativity of the covariant derivative for regular, minimally coupled fields, $\pi_i = p_i - eA_i(x) = -i\partial_{x_i} - eA_i(x)$. Calculating the commutator of π_i with π_j gives

$$[\pi_i, \pi_j] = ieF_{ij} = ie\epsilon_{ijk}B^k \ , \tag{9}$$

where $B^k = \frac{1}{2}\epsilon^{kij}(\partial_{x_i}A_j - \partial_{x_j}A_i) = \frac{1}{2}\epsilon^{kij}F_{ij}$ is the regular magnetic field. Comparing (8) with

(9) one can define a momentum gauge field "magnetic field" as $\mathcal{B}^k = \frac{1}{2}\epsilon^{kij}(\partial_{p_i}C_j - \partial_{p_j}C_i) = \frac{1}{2}\epsilon^{kij}G_{ij}$. This in turn defines the non-commutation parameter of the spatial coordinates on the right hand side of (8) to be constant only if the momentum "magnetic" field is constant.

One can easily arrange for such a constant "magnetic" field solution via

$$C^0 = 0 \quad , \quad C^i = \frac{1}{2}\epsilon^{ijk}\mathcal{B}^j p^k \tag{10}$$

with $\mathcal{B}^j$ being a constant. Taking the curl of (10), using momentum derivatives, and doing index gymnastics yields $\epsilon^{lmi}\partial^{p^m}C^i = \mathcal{B}^l$ – one gets a constant "magnetic" field. This gives a constant non-commutative tensor $\Theta_{ij} = gG_{ij} = g\epsilon_{ijk}\mathcal{B}^k$ *i.e.* in this way one recovers a constant non-commutative parameter which is the most common assumption in the literature

A fully 4-vector version of the spatial coordinate non-commutativity in (7) is accomplished by promoting the 3-Latin indices to 4-Greek indices giving

$$[x_\mu, x_\nu] = i\Theta_{\mu\nu} \ , \tag{11}$$

where $\Theta_{\mu\nu}$ is an anti-symmetric 4-tensor. In conjunction with (11) the 4-tensor version of (8) becomes

$$[X_\mu, X_\nu] = igG_{\mu\nu} \ , \tag{12}$$

In order to get a constant $\Theta_{\mu\nu}$ for a component with one space index (*e.g.* $\mu = i$) and one time index (*i.e.* $\nu = 0$) we need to have a constant momentum gauge field, "electric" field. This is accomplished by selecting the momentum gauge field as

$$C^0 = -\mathcal{E}^j p^j \quad ; \quad C^j = 0 \tag{13}$$

The momentum gauge "electric" field is given by $G_{0i} = \partial_{p^0} C_i - \partial_{p^i} C_0 = \mathcal{E}^i$ which is the sought after constant momentum gauge field "electric" field. Using equations (11) and (12) this gives the connection between the non-commutativity parameter and momentum gauge field electric field of $\Theta_{0i} = gG_{0i} = g\mathcal{E}_i$.

B. Variable non-commutativity parameter

In the previous subsection we looked at momentum gauge field configuration with constant "magnetic" and constant "electric fields" in equations (10) and (13) respectively. In this subsection we examine momentum gauge field configurations which are variable. These variable momentum gauge fields then imply a varying of the non-commutativity parameter via the connection $\Theta_{uv} \propto G_{\mu\nu}$.

We first write down two common, ordinary gauge field solutions which have gauge fields that vary with space and time and then construct the varying momentum gauge field analogs. The two ordinary gauge field solutions we consider are a plane wave and a static points charge. The Lagrange density for standard gauge fields is $\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ with $F^{\mu\nu} = \partial^{x^\mu} A^\nu - \partial^{x^\nu} A^\mu$. The equations of motion from $\mathcal{L}_F$ are

$$\partial_{x_\mu}(\partial^{x_\mu} A^\nu - \partial^{x_\nu} A^\mu) = 4\pi J^\nu(x) \rightarrow \partial_{x_\mu} \partial^{x_\mu} A^\nu = 4\pi J^\nu(x) \rightarrow \nabla_x^2 A^\nu = 4\pi J^\nu(x) \; , \qquad (14)$$

with $J^\nu(x)$ being a conserved 4-current coming from some matter source, and ∇_x^2 is the Laplacian with respect to the position coordinates. In the last line we have taken the Lorenz gauge $\partial_{x_\mu} A^{x_\mu} = 0$. Let us look at two common solutions to (14): the plane wave vacuum solution and the point charge solution.

- In vacuum ($J^\nu = 0$) (14) has the solution $A^\nu \propto e^{i(px-Et)} \varepsilon^\nu \delta(p^2 - E^2/c^2)$ where the δ-function enforces the mass shell condition $\frac{E^2}{p^2} = c^2$ and ε^ν is the polarization vector.

- For a point charge at rest one has the current $J^\nu = (q\delta^3(r), 0, 0, 0)$, which has the solution $A^0 = \frac{q}{r}$ and $\vec{A} = 0$, since $\nabla_x^2 \left(\frac{1}{r}\right) = 4\pi\delta(r)$.

We now examine how the above plays out for the momentum gauge fields. The momentum gauge field Lagrange density is $\mathcal{L}_G = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu}$ with $G^{\mu\nu} = \partial^{p_\mu} C^\nu - \partial^{p_\nu} C^\mu$. The equations of motions that follow from this Lagrange density are

$$\partial_{p_\mu}(\partial^{p_\mu} C^\nu - \partial^{p_\nu} C^\mu) = 4\pi \mathcal{J}^\nu(p) \rightarrow \partial_{p_\mu} \partial^{p_\mu} C^\nu = 4\pi \mathcal{J}^\nu(p) \rightarrow \nabla_p^2 C^\nu = 4\pi \mathcal{J}^\nu(p) \ , \qquad (15)$$

with $\mathcal{J}^\nu(p)$ being a 4-current matter source that is a function of p, and ∇_p^2 is the Laplacian with respect to the momentum. In the last expression we use the momentum space equivalent of the Lorenz gauge $\partial_{p_\mu} C^{p_\mu} = 0$. The current conservation in momentum space reads $\partial_{p_\mu} \mathcal{J}^{p_\mu} = 0$,

We now repeat the two types of solutions listed above for the standard gauge theory, but for the momentum gauge theory.

- In vacuum ($\mathcal{J}^\nu = 0$) (15) has solution $C^\nu \propto e^{i(px-Et)} \varepsilon^\nu \delta(x^2 - c^2 t^2)$ where the δ-function enforces the light-cone condition $\frac{x^2}{t^2} = c^2$, and ε^ν is the polarization vector.

- The momentum gauge equivalent of the charge at rest is given by $\mathcal{J}^\nu = (g\delta^3(p), 0, 0, 0)$, with $C^0 = \frac{g}{p}$ and $\vec{C} = 0$, since $\nabla_p^2 \left(\frac{1}{p}\right) = 4\pi\delta(p)$.

Notice that the point source in momentum space, that is $\mathcal{J}^\nu = (g\delta^3(p), 0, 0, 0)$ is a totally homogeneous solution in coordinate space, since it is concentrated at zero momentum, which means indeed the assumption of a totally homogeneous state. More generally, it is interesting to observe that any current of the form $\mathcal{J}^\nu = (f(p), 0, 0, 0)$, with $\mathcal{J}^0 = f(\vec{p})$ being p^0 independent, will satisfy the current conservation law of $\partial_{p_\mu} \mathcal{J}^\mu = 0$. Doing a Fourier transformation on this to coordinate space yields $x_\mu \tilde{\mathcal{J}}^\mu = 0$, where $\tilde{\mathcal{J}}^\mu$ is the Fourier transformation of $\mathcal{J}^\mu$. The equivalent statements for a a regular 4-source would be $J^\nu = (f(\vec{x}), 0, 0, 0)$, which satisfies the conservation law $\partial_{x_\mu} J^\mu = 0$ or Fourier transforming to momentum space $p_\mu \tilde{J}^\mu = 0$.

One can construct other conserved current sources for momentum gauge fields that satisfy $x_\mu \tilde{\mathcal{J}}^\mu = 0$. Starting with any 4-vector V^μ, we construct $\tilde{\mathcal{J}}^\mu = V^\mu - x^\mu V^\nu x_\nu / x^2$ which is easily seen to satisfy $x_\mu \tilde{\mathcal{J}}^\mu = 0$. That is, we can start with any vector and subtract its component in the x^μ direction and we get a conserved current.

III. GENERALIZED LANDAU LEVELS

In this section we work on the case of generalized Landau levels with a particle of mass m in a constant ordinary magnetic field and constant momentum "magnetic" field. We take both the ordinary and momentum magnetic field to point in the $3/z$-direction. We want to take these magnetic fields and minimally couple them to the free particle in equation (6). Applying minimal coupling for both coordinate gauge fields and momentum gauge fields, leads to $p_i \to p_i - eA_i$, and $x_i \to x_i - gC_i$. Having a constant, ordinary magnetic field and a constant, momentum magnetic fields in the $3/z$-direction can be obtained in the symmetric gauge with A_1 and A_2 given by,

$$A_0 = 0 \quad , \quad A_1 = -\frac{1}{2} B y \quad , \quad A_2 = \frac{1}{2} B x \ , \tag{16}$$

and with C_1 and C_2 also in the symmetric gauge given by,

$$C_0 = 0 \quad , \quad C_1 = -\frac{1}{2} \mathcal{B} p_y \quad , \quad C_2 = \frac{1}{2} \mathcal{B} p_x \ , \tag{17}$$

The constant values of the ordinary magnetic field and momentum magnetic field from (16) and (17) are B and $\mathcal{B}$ respectively.

So the equation of motion for the double gauged harmonic oscillator reads,

$$\begin{aligned} H &= \frac{1}{2m}\left(p_x + \frac{eBy}{2}\right)^2 + \frac{1}{2m}\left(p_y - \frac{eBx}{2}\right)^2 \\ &+ \frac{m\omega^2}{2}\left(x + \frac{g\mathcal{B}p_y}{2}\right)^2 + \frac{m\omega^2}{2}\left(y - \frac{g\mathcal{B}p_x}{2}\right)^2 + \frac{p_z^2}{2m} + \frac{m\omega^2}{2}z^2 \end{aligned} \tag{18}$$

or (we drop the part of the Hamiltonian associated with the kinetic energy and harmonic oscillator in the z-direction)

$$H = \left(1 + \frac{(gm\omega\mathcal{B})^2}{4}\right)\left(\frac{p_x^2}{2m} + \frac{p_y^2}{2m}\right) + \left(1 + \frac{(eB)^2}{4m^2\omega^2}\right)\frac{m\omega^2}{2}(x^2 + y^2) + L_z(-g_1 B + g_2\mathcal{B})\ . \tag{19}$$

Here $L_z = xp_y - yp_x$, this the angular momentum in the z-direction. and $g_1 = \frac{e}{2m}$ and $g_2 = \frac{gm\omega^2}{2}$ are the coupling strengths of the angular momentum to the coordinate magnetic field B and the momentum magnetic field $\mathcal{B}$ respectively.

The coupling between B and L_z is exactly what one has from the standard analysis of Landau levels. The coupling between L_z and $\mathcal{B}$ is a new feature arising from the momentum gauge fields, but the two coupling terms to L_z have a dual symmetry between the regular magnetic field, B, and momentum gauge "magnetic" field, $\mathcal{B}$.

The first term in (19) shows that the system has now developed a new, effective mass given by

$$m_{eff} = \frac{m}{1 + \frac{(gm\omega\mathcal{B})^2}{4}} \ . \tag{20}$$

The effective mass depends on the momentum "magnetic" field and is always less than m *i.e.* $m_{eff} < m$. In addition the second terms in (19) implies a new effective frequency. Taking into account the effective mass in (20) to write this second term in the form $\frac{m_{eff}\omega_{eff}^2}{2}$ gives a new effective frequency of

$$\omega_{eff} = \omega\sqrt{\left(1 + \frac{(gm\omega\mathcal{B})^2}{4}\right)\left(1 + \frac{e^2B^2}{4m^2\omega^2}\right)} \ . \tag{21}$$

Using this new effective frequency and the new effective mass one can define an effective magnetic field as a mixture between the B and the $\mathcal{B}$ fields

$$B_{eff} = \frac{-g_1 B + g_2 \mathcal{B}}{\sqrt{g_1^2 + g_2^2}} \ .$$

This resembles the definition of the physical photon or Z^0 fields as rotations of two fields in the standard electroweak model [2–4]. The coupling between this effective magnetic field and the z-component of angular momentum is now $\sqrt{g_1^2+g_2^2}B_{eff}L_z$. The total Hamiltonian is then,

$$H = \frac{1}{2m_{eff}}(p_x^2+p_y^2) + \frac{1}{2}\omega_{eff}^2 m_{eff}(x^2+y^2) + \sqrt{g_1^2+g_2^2}B_{eff}L_z \tag{23}$$

Following [11] one can define creation/annihilation operators in terms of p_x, p_y and x, y as

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2\omega_{eff}m_{eff}}}\left(a_1+a_1^\dagger\right) \quad ; \quad y = \sqrt{\frac{\hbar}{2\omega_{eff}m_{eff}}}\left(a_2+a_2^\dagger\right) \\ &\text{and} \\ p_x &= i\sqrt{\frac{\hbar\omega_{eff}m_{eff}}{2}}\left(a_1^\dagger-a_1^\dagger\right) \quad ; \quad p_y = i\sqrt{\frac{\hbar\omega_{eff}m_{eff}}{2}}\left(a_2^\dagger-a_2^\dagger\right) \ . \end{aligned} \tag{24}$$

The creation and annihilation operators obey the usual relationship $[a_i, a_j^\dagger] = \delta_{ij}$. With these definitions we find $L_z = xp_y - yp_x = i\hbar(a_1a_2^\dagger - a_2a_1^\dagger)$ and the Hamiltonian in (23) becomes $H = \hbar\omega_{eff}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) + i\hbar\sqrt{g_1^2+g_2^2}B_{eff}(a_1a_2^\dagger - a_2a_1^\dagger)$. The first two terms can be seen

IV. MOMENTUM DEPENDENT NON-COMMUTATIVITY PARAMETER

In this section we examine two simple examples where the non-commutativity parameter, $\Theta_{\mu\nu}$, is not a constant but depends on the momentum. Recently, other authors [12] have considered momentum dependent non-commutative parameters. However, in this work the inspiration is quite different as it exploits some geometry in momentum space. Also the non-commutativity parameter in [12] depends on both momentum and position, while our in our construction below the non-commutativity parameter depends only on momentum, which is closer to the energy-momentum dependence of masses and couplings in QFT that one finds from the renormalization group.

The examples we choose are the momentum gauge field version of a capacitor and solenoid, with the momentum gauge fields being piece-wise constant in different momentum ranges, leading to different, $\Theta_{\mu\nu}$'s in these different ranges.

A. Capacitor-type momentum electric field configuration

The standard, infinite parallel plate capacitor has a 4-current source of

$$J^\nu = (f(z), 0, 0, 0) \quad \text{with} \quad f(z) = \sigma[\delta(z+a) - \delta(z-a)] \tag{25}$$

This source represents two infinite planes of surface charge $\pm\sigma$ placed perpendicular to the z-axis at $z = \mp a$. This source gives an electric field of

$$E_z = 4\pi\sigma \quad \text{for} \quad -a \leq z \leq a \quad \text{and} \quad E_z = 0 \quad \text{for} \quad |a| \leq |z| \; , \tag{26}$$

i.e. non-zero between the planes and zero outside the planes.

The momentum gauge field analog of this standard capacitor system has a constant momentum "electric" field similar to that in equation (13), but it should be restricted in momentum rather than position as is the case in equation (26). Actually for the momentum gauge field system we want the inverse of the above standard capacitor – we want the momentum "electric" field to be zero between the planes (*i.e* at small momentum) and non-zero outside the planes (*i.e* at large momentum). The capacitor-like configuration for the momentum gauge fields that we want has a 4-current source of

$$\mathcal{J}^\nu = (f(p), 0, 0, 0) \text{ with } f(p) = \Sigma[\delta(p_z + p_a) + \delta(p_z - p_a)]. \tag{27}$$

The planes are symmetrically placed at $p_z = \pm p_a$ and, in contrast to the sources for the standard capacitor in (25), the momentum planes now have the **same** "surface charge", Σ. This same "surface charge" set up leads to a momentum "electric" field in the p_z direction given by

$$\begin{aligned} \mathcal{E}_z &= 4\pi\Sigma \text{ for } p_z \geq p_a \ , \ \mathcal{E}_z = -4\pi\Sigma \text{ for } p_z \leq -p_a, \\ \text{and } \mathcal{E}_z &= 0 \text{ for } -p_a \leq p_z \leq p_a. \end{aligned} \tag{28}$$

The momentum "electric" field of (28) is zero between the plates and non-zero outside the plates, which is the inverse of the standard capacitor (26).

The reason for building our momentum gauge field capacitor system as the **inverse** of the normal capacitor is due to the connection between the non-commutativity parameter, $\Theta_{\mu\nu}$ and the momentum gauge field tensor, $G_{\mu\nu}$, as given equations (11) and (12) *i.e.* $\Theta_{\mu\nu} = gG_{\mu\nu}$. We want to have a normal position-position commutator (*i.e.* $[X_\mu, X_\nu] = 0$) for momenta near zero (*i.e.* for $-p_a \leq p_z \leq p_a$) but we want non-commutative space-time effects for large momenta *i.e.* we want $\Theta_{\mu\nu} \propto G_{\mu\nu} \neq 0$ for large momenta, $|p_a| \leq |p_z|$. This is different from the usual non-commutative space-time approach where the non-commutative parameter is "turned on" for all momentum. Here the non-commutativity, at least for the Θ_{0i} components, is turned on only for z-momentum magnitude satisfying $|p_a| < |p_z|$.

B. Current sheet-type momentum magnetic field

In this subsection we carry out a similar construction as in the preceding subsection, but for the space/space components of $\Theta_{\mu\nu}$ and $G_{\mu\nu}$. In this case the standard gauge field system we want to build a momentum gauge field analog of is two infinite plane sheet currents located at $z = \pm a$. These current sheets are symmetrically placed on the z-axis around $z = 0$. The explicit surface currents are

$$\vec{K} = \pm J \hat{\mathbf{y}} \quad \text{at} \quad z = \mp a \tag{29}$$

This leads a regular magnetic field of

$$\vec{B} = 4\pi J \hat{\mathbf{x}} \quad \text{for} \quad -a \leq z \leq a \quad \text{and} \quad \vec{B} = 0 \ \text{ for } \ |a| \leq |z| \tag{30}$$

i.e. the magnetic field is a non-zero constant between the sheets and zero outside the sheets.

The momentum gauge field analog of this is two momentum gauge field current sheets at the momentum planes, $p_z = \pm p_a$. These planes are symmetric around the origin through the p_z-axis. Explicitly the "momentum" current sheets are

$$\vec{\mathcal{K}} = \mathcal{J}\hat{\mathbf{y}} \quad \text{at} \quad p_z = \pm p_a \tag{31}$$

Note that here we have the currents in the same direction, rather than opposite direction as for the regular gauge field current sheets of (29). The reason for this is the same as for the momentum gauge field, capacitor-like system of the preceding subsection: we want the non-commutativity parameter to be zero for momentum in the range $-p_a \leq p_z \leq p_a$ and we want a non-zero non-commutativity parameter for momentum in the range $|p_a| \leq |p_z|$. Putting this all together the momentum gauge field "magnetic" field is

$$\begin{aligned} \vec{\mathcal{B}} &= 4\pi\mathcal{J}\hat{\mathbf{x}} \;\; \text{for} \;\; p_a \leq p_z \quad \text{and} \quad \vec{\mathcal{B}} = -4\pi\mathcal{J}\hat{\mathbf{x}} \;\; \text{for} \;\; p_z \leq -p_a \\ \text{and} \;\; \vec{\mathcal{B}} &= 0 \quad \text{for} \quad -p_a \leq p_z \leq p_a \; . \end{aligned} \tag{32}$$

The momentum gauge "magnetic" field is a non-zero, constant outside the current sheets and zero between the current sheets. This implies that the space/space non-commutativity parameter, Θ_{ij}, is zero for momenta in the range $-p_a \leq p_z \leq p_a$, while for large magnitude momenta (*i.e.* $|p_a \leq |p_z|$) the space/space component $\Theta_{yz} = gG_{yz} = g\epsilon_{yzx}\mathcal{B}_x = \pm g\mathcal{B}$ is a non-zero constant. Both this simple example and the example from the preceding subsection show that one can construct non-commutative space-times where the non-commutativity only "turns" on at some large enough momentum, rather than being on all the time.

Momentum Gauge Theory and braneworlds we have considered only

the momentum gauge field version of it. For simplicity let us just consider however the momentum Klein Gordon equation.

$$\partial_{p_\mu}\partial^{p_\mu}\phi + m^2\phi = 0 \tag{33}$$

We now Fourier transform (33) to energy-momentum space, in order to make the connection with brane-world models. the Fourier transform of (33) is

$$x_\mu x^\mu \tilde{\phi} - m^2\tilde{\phi} = 0\ . \tag{34}$$

where $\tilde{\phi}$ is the Fourier transformed scalar field. Equation (40) is solved by

$$\tilde{\phi} = \delta(x_\mu x^\mu - m^2)F(x) \tag{35}$$

which suggests use of these solutions for brane world scenarios, taking μ to run over more than four dimensions, eq. (35) tells us that nevertheless the wavefunction is restricted to a lower dimensional surface, like in the brane-world scenarios [18–28] . There will be restoration of translation type symmetries, since surfaces like $x_\mu x^\mu - m^2 = 0$ embeded in a flat Minkowski space have maximal symmetry and some of these transformations, the quasi translations, allow us to get us from one point in the manifold to any other point in the manifold, for example can take the origin $x^\mu = 0$ to any other point a^μ [29].

B. The non-commutative Massive Klein Gordon Momentum Field Theories Braneworlds

In order to make the braneworld coordinates non commutative, we minimally couple the Klein Gordon momentum equation of motion (33) to a momentum gauge field, so that

$$x_\mu \to x_\mu - gC_\mu(p) = X_\mu \quad \text{or} \quad \frac{\partial}{\partial p_\mu} \to \frac{\partial}{\partial p_\mu} + igC_\mu(p). \tag{36}$$

So now the braneworld manifold becomes

$$X_\mu X^\mu - m^2 = 0 \tag{37}$$

With X_μ being the noncommutative space time coordinate as defined in (36) , Equation (37) is meant to be understood in a weak sense, that is that $(X_\mu X^\mu - m^2)\phi = 0$.

C. The Massive Dirac Momentum Field Theories Braneworlds

We could use the Dirac momentum equation instead of the Klein Gordon momentum equation, that is

$$(i\gamma^{\mu}\partial_{p_{\mu}} - m)\psi = 0 \tag{38}$$

multiplying both sides of (38) by $(i\gamma^{\mu}\partial_{p_{\mu}} + m)$ and taking into account the γ^{μ} algebra we obtain

$$\partial_{p_{\mu}}\partial^{p_{\mu}}\psi + m^2\psi = 0 \tag{39}$$

which in coordinate space translates into

$$x_{\mu}x^{\mu}\psi - m^2\psi = 0 \tag{40}$$

which is solved by

$$\psi = \delta(x_{\mu}x^{\mu} - m^2)F(x) \tag{41}$$

which again leads to a Brane World scenario.

D. The Non Commutative Massive Dirac Momentum Field Theories and associated split Brane worlds

In order to consider the non commutative situation for Dirac Branes, we consider now the gauged version of (38)

$$(i\gamma^\mu(\partial_{p_\mu} + igC_\mu(p)) - m)\psi = 0 \tag{42}$$

multiplying both sides of (38) by $(i\gamma^\mu(\partial_{p_\mu} + igC_\mu(p)) + m)$ and taking into account the γ^μ algebra following similar steps to those usually done with the squaring of the ordinary Dirac equation, we obtain now,

$$(\partial_{p_\mu} + igC_\mu(p))(\partial^{p_\mu} + igC^\mu(p))\psi + \frac{ig}{2}\sigma^{\mu\nu}G_{\mu\nu}\psi + m^2\psi = 0 \tag{43}$$

From this, from the presence of the $\frac{ig}{2}\sigma^{\mu\nu}G_{\mu\nu}\psi$ term, we obtain that the brane world now splits into many braneworlds which depend on the spin state of the Dirac field, so if we were to look at a given brane world some polarizations or spin states of the Dirac field may be lost. We notice that the field strength $G_{\mu\nu}$ is the noncommutativity tensor of the spacetime coordinates, so that we found a coupling that splits the braneworlds which is the coupling between the noncommutativity tensor and spin.

Reference:

Momentum Gauge Fields and Non-Commutative Space-Time

E. Guendelman and D. Singleton, e-Print: 2206.02638 [quant-ph]

(Braneworlds part not included there however)

Thank you for your attention!!