κ-Braided non-commutative field theory

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[Lizzi,Mercati, PRD 103 (2021)]

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κ- Minkowski

[Celeghini, Giachetti, Sorace, Tarlini, JMP 31 (1990)]
[Lukierski, Nowicki, Ruegg, PLB 264 (1991), 293 (1992)]
[Majid, Ruegg, PLB 334 (1994)] [Zakrzewski, JPA 27 (1993)]

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(v^\mu x^\nu - v^\nu x^\mu), \quad \mu = 0, \dots, d \qquad v^\mu \in \mathbb{R}^{d+1},$$

$(x^\mu)^\dagger = x^\mu, \quad [\kappa] = \ell^{-1} = (\text{length})^{-1}$ scale,

$x^\mu \in A =$coordinate algebra= 'non-commutative functions'=scalar fields

κ- Poincaré

Poincaré group described through $C[ISO(3,1)]$

$$\Delta[\Lambda^{\mu}{}_{\nu}] = \Lambda^{\mu}{}_{\alpha} \otimes \Lambda^{\alpha}{}_{\nu}, \quad \Delta[a^{\mu}] = \Lambda^{\mu}{}_{\nu} \otimes a^{\nu} + a^{\mu} \otimes 1,$$

$$S[\Lambda^{\mu}{}_{\nu}] = (\Lambda^{-1})^{\mu}{}_{\nu}, \qquad S[a^{\mu}] = -(\Lambda^{-1})^{\mu}{}_{\nu} a^{\nu}, \qquad \epsilon[\Lambda^{\mu}{}_{\nu}] = \delta^{\mu}_{\nu}, \qquad \epsilon[a^{\mu}] = 0$$

Non-commutative deformation of $C[ISO(3,1)]$ --> $C_{\kappa}[ISO(3,1)]$

$$[\Lambda^{\mu}{}_{\nu}, a^{\gamma}] = \frac{i}{\kappa}[(\Lambda^{\mu}{}_{\alpha} v^{\alpha} - v^{\mu})\Lambda^{\gamma}{}_{\nu} + (\Lambda^{\alpha}{}_{\nu} g_{\alpha\beta} - g_{\nu\beta}) v^{\beta} g^{\mu\gamma}]$$

$$[\Lambda^{\mu}{}_{\nu}, \Lambda^{\alpha}{}_{\beta}] = 0, \quad [a^{\mu}, a^{\nu}] = \frac{i}{\kappa}(v^{\mu} a^{\nu} - v^{\nu} a^{\mu}),$$

$$\Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} g^{\alpha\beta} = g^{\mu\nu}, \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} g_{\mu\nu} = g_{\alpha\beta}$$

Hopf algebra $\forall g_{\mu\nu}$: $g_{\mu\nu} g^{\nu\gamma} = \delta^{\gamma}_{\mu}$

Different models $\leftrightarrow$ $v^{\mu} v^{\nu} g_{\mu\nu}$

Relativity principle

Non-commutative Poincaré transformation

Left co-action $\overline{(\cdot)} : \mathrm{A} \to \mathrm{C}_\kappa[\mathrm{ISO}(3,1)] \otimes \mathrm{A}$

$$\bar{x}^{\,\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}, \quad \left[x^{\mu}, \Lambda^{\rho}{}_{\sigma}\right] = [x^{\mu}, a^{\nu}] = 0$$

Given the $\mathrm{C}_\kappa[\mathrm{ISO}(3,1)]$ algebra

$$[\bar{x}^{\mu}, \bar{x}^{\nu}] = \frac{i}{\kappa}(v^{\mu}\bar{x}^{\nu} - v^{\nu}\bar{x}^{\mu}) = \overline{[x^{\mu}, x^{\nu}]}$$

the commutation relations are the same to all observers.

Field theory on κ-Minkowski

Multilocal functions

QFT requires the concept of N-point functions. In the commutative case, given the Abelian algebra of coordinates F, two-point functions are elements of $F \otimes F$, generated by

$$x_1^\mu = x^\mu \otimes 1, \quad x_2^\mu = 1 \otimes x^\mu, \quad 1 = 1 \otimes 1$$

Starting from the non-abelian algebra A, the canonical algebra structure on the tensor product $\mathrm{A} \otimes \mathrm{A}$ is

$$[x_1^\mu, x_1^\nu] = \frac{i}{\kappa}(v^\mu x_1^\nu - v^\nu x_1^\mu), \; [x_2^\mu, x_2^\nu] = \frac{i}{\kappa}(v^\mu x_2^\nu - v^\nu x_2^\mu),$$

$$[x_1^\mu, 1] = [x_2^\mu, 1] = 0, \quad \boxed{[x_1^\mu, x_2^\nu] = 0}$$ → Not covariant!

Braided tensor product algebra

We look for a deformation $A \otimes_\kappa A$ of the tensor product algebra, such that:

$$[x_1^\mu, x_1^\nu] = \frac{i}{\kappa}(v^\mu x_1^\nu - v^\nu x_1^\mu),\ [x_2^\mu, x_2^\nu] = \frac{i}{\kappa}(v^\mu x_2^\nu - v^\nu x_2^\mu),$$

$$[x_1^\mu, 1] = [x_2^\mu, 1] = 0,\ \ [\bar{x}_1^\mu, \bar{x}_2^\nu] = \overline{[x_1^\mu, x_2^\nu]}$$

Assuming that $[x_1^\mu, x_2^\nu]$ goes to zero as $v^\mu \to 0$ and that it is linear in x_a^μ, we find a unique solution:

$$[x_1^\mu, x_2^\nu] = \frac{i}{\kappa}[v^\mu x_1^\nu - v^\nu x_2^\mu - g^{\mu\nu} g_{\rho\sigma} v^\rho (x_1^\sigma - x_2^\sigma)\,]$$

Braided tensor product algebra

This algebra can be immediately extended to N points, $\mathrm{A}^{\otimes^N_\kappa}$:

$$[x_a^\mu, x_b^\nu] = \frac{i}{\kappa}[v^\mu x_a^\nu - v^\nu x_b^\mu - g^{\mu\nu} g_{\rho\sigma} v^\rho (x_a^\sigma - x_b^\sigma)], \quad a, b = 1, \dots, \mathrm{N}$$

If we impose the Jacobi rule, we find

$$\left[x_a^\mu, [x_b^\nu, x_c^\rho]\right] + \left[x_b^\nu, [x_c^\rho, x_a^\mu]\right] + \left[x_c^\rho, [x_a^\mu, x_b^\nu]\right] =$$
$$= -\frac{g_{\alpha\beta} v^\alpha v^\beta}{\kappa^2}\left(g^{\nu\rho}\left(x_c^\mu - x_b^\mu\right) + g^{\rho\mu}(x_a^\nu - x_c^\nu) + g^{\mu\nu}\left(x_b^\rho - x_a^\rho\right)\right) = 0$$

and this is satisfied only if $g_{\alpha\beta} v^\alpha v^\beta = 0$, i.e. the so-called *lightlike*-κ-Minkowski non-commutativity.

$A^{\otimes N}_{\kappa}$ is a mostly-commutative algebra

$$\left[x_a^\mu, x_b^\nu\right] = \frac{i}{\kappa}\left[v^\mu x_a^\nu - v^\nu x_b^\mu - g^{\mu\nu} g_{\rho\sigma} v^\rho (x_a^\sigma - x_b^\sigma)\right]$$

The coordinate differences $\delta x_{ab}^\mu = x_a^\mu - x_b^\mu$ close an Abelian subalgebra:

$$\left[\delta x_{ab}^\mu, \delta x_{cd}^\nu\right] = 0$$

All the noncommutativity is concentrated on the center-of-mass degrees of freedom

$$x_{cm}^\mu = \frac{1}{N}\Sigma_a x_a^\mu, \quad \left[x_{cm}^\mu, x_{cm}^\nu\right] = \frac{i}{\kappa}\left(v^\mu x_{cm}^\nu - v^\nu x_{cm}^\mu\right)$$

$$y_a^\mu = x_a^\mu - x_{cm}^\mu, \quad \left[x_{cm}^\mu, y_a^\nu\right] = \frac{i}{\kappa}\left(g^{\mu\nu} g_{\rho\sigma} v^\rho y_a^\sigma - v^\nu y_a^\mu\right), \quad \left[y_a^\mu, y_b^\nu\right] = 0$$

A representation of $A^{\otimes^{\kappa}}$

The component of x_{cm}^{μ} along v^{μ}, $x_{cm}^{-} = g_{\mu\nu} v^{\mu} x_{cm}^{\nu}$, commutes with all the y_a^{μ}:

$$\left[x_{cm}^{-}, y_a^{\mu}\right] = 0, \quad \left[x_{cm}^{\mu}, x_{cm}^{-}\right] = \frac{i}{\kappa} v^{\mu} x_{cm}^{-},$$

but the other components are irreducibly noncommutative: the maximal Abelian subalgebra is generated by x_{cm}^{-} and y_a^{μ} (with the constraint $\Sigma_a y_a^{\mu} = 0$).

In 3+1 dimensions, choosing w.l.o.g. $\mathrm{g}^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, $\quad v^{\mu} = (1,1,0,0)$, $x_{cm}^{-} = x_{cm}^{0} - x_{cm}^{1}$ and y_a^{μ} are multiplicative operators with real spectrum and

A representation of $\mathrm{A}^{\otimes^N_\kappa}$

$x^+_{cm} = x^0_{cm} + x^1_{cm},\ x^2_{cm},\ x^3_{cm}$ can be represented as sums of Lorentz matrices and a x^-_{cm}–dilatation:

$$x^2_{cm} = M^{12} - M^{02},\quad x^3_{cm} = M^{13} - M^{03},$$

$$x^+_{cm} = 2M^{10} + 2ix^-_{cm}\frac{\partial}{\partial x^-_{cm}} + i,$$

$M^{\mu\nu} = i\sum_{a=1}^{N-1} y^\mu_a g^{\nu\rho}\frac{\partial}{\partial y^\rho_a} - y^\nu_a g^{\mu\rho}\frac{\partial}{\partial y^\rho_a}$ Generators of rigid Lorentz transformations of N points

κ- Poincaré invariant field theory

QFT requires the concept of N-point functions, i.e. , in the commutative case, Poincaré-invariant distributions which admit a Fourier representation.

In the non-commutative case, we can choose an ordering prescription for the x_a^μ and try to write a function of N points as

$$f\left(x_a^\mu\right) = \int d^4k^1 \dots d^4k^N \quad \tilde{f}\left(k_\mu^a\right) \quad e^{ik_\mu^1 x_1^\mu} \dots e^{ik_\mu^N x_N^\mu}$$

and we can prove that κ-Poincaré invariance implies that $f\left(x_a^\mu\right)$ only depends upon y_a^μ.

N-point functions are commutative!

1+1D κ- deformed field theory

1-point ordered plane waves $e_a[k_\mu] = e^{ik_- x_a^-} e^{ik_+ x_a^+}$, $k_\pm \in \mathbb{R}$, form a Lie group under noncommutative multiplication.

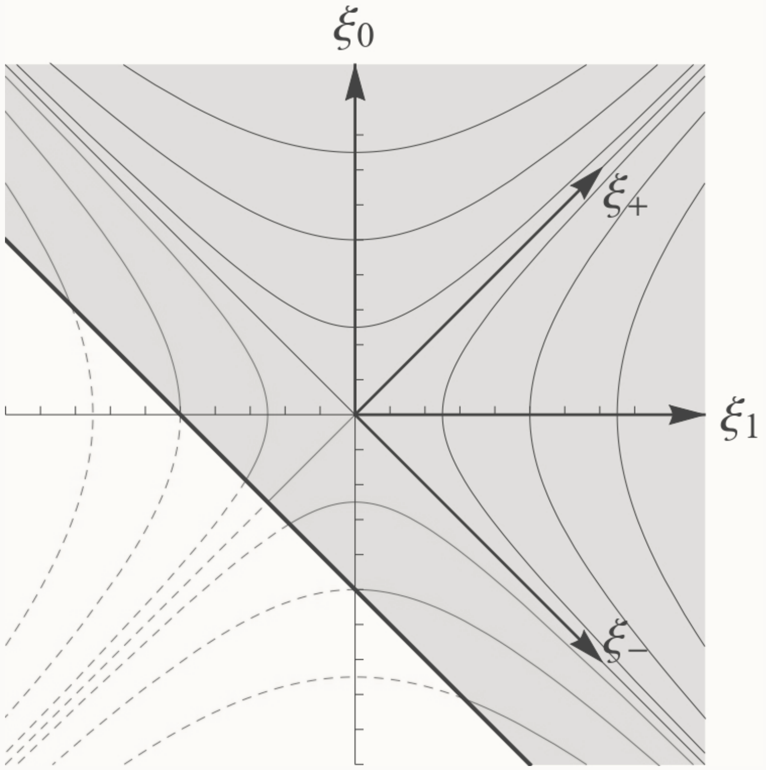
The group manifold is a half Minkowski space

→ Lorentz breaking?

Nonlinear action of the κ -Poincaré group on momentum space:

$$\bar{e}_a[k] = e^{ik_- \bar{x}_a^-} e^{ik_+ \bar{x}_a^+} = e_a[\lambda(k,\omega)] e^{ik_- a^-} e^{ik_+ a^+}$$

$$\lambda(k,\omega) = \left(e^{-\omega} k_-, \tfrac{1}{2}\ln\left[1 + e^{\omega}\left(e^{\frac{2k_+}{\kappa}} - 1\right)\right]\right)$$

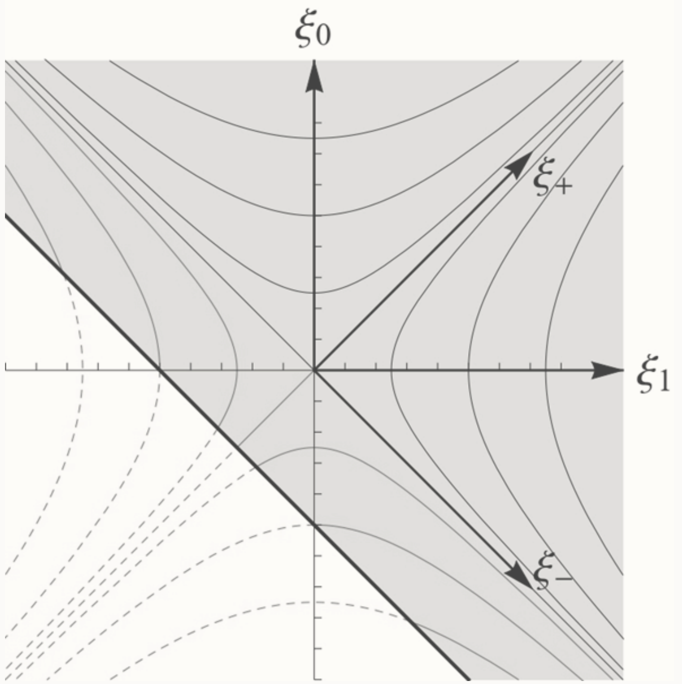


The missing jigsaw pieces

For $k_+ < 0, \quad e^{\omega} > \frac{1}{1-e^{\frac{2k_+}{\kappa}}}$,

$$\lambda(k,\omega)_+ = \frac{1}{2}\ln\left[e^{\omega}\left(1-e^{\frac{2k_+}{\kappa}}\right)-1\right] \boxed{+\frac{i\pi}{2}+n\pi i}$$

[Arzano, Bevilacqua, Kowalski-Glikman, Rosati, Unger, PRD 103 (2021)]



$$e_a[k] \rightarrow \epsilon_a[p] \boxed{e^{-n\pi x_a^+}} = e^{ip_- x_a^-} e^{ip_+ x_a^+} e^{-\frac{\pi}{2}x_a^+} e^{-n\pi x_a^+}$$

$= (-1)^n$ in our representation of $A^{\otimes N}_{\kappa}$: always appears to an even power in N-point functions

Conclusions

$$\phi(x_a) = \int_{\mathbb{R}_+} dk_+ \frac{e^{2k_+}}{e^{2k_+} - 1} \left(a(k_+) e_a(k_+) + e^{-2k_+} b^*(k_+) e_a^\dagger(k_+) \right) +$$

$$\int_{\mathbb{R}_+} dk_+ \frac{e^{2k_+}}{e^{2k_+} + 1} \left(\alpha(k_+) \epsilon_a(k_+) + e^{-2k_+} \beta^*(k_+) \epsilon_a^\dagger(k_+) \right)$$

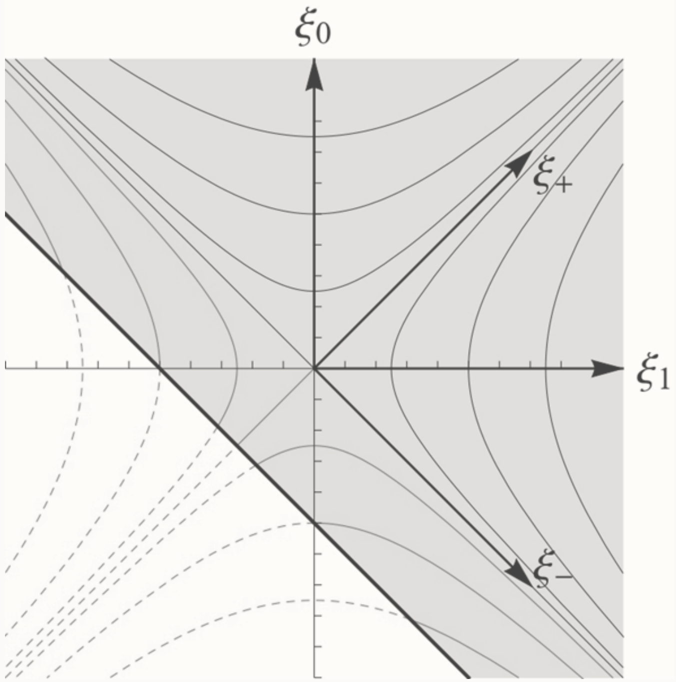
- All 2-point functions are identical to the commutative ones;
- $\left[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)\right] = i\Delta_{PJ}(x_1 - x_2), \quad \left[\hat{\phi}(x_1), \hat{\phi}(x_2)\right] = \left[\hat{\phi}^\dagger(x_1), \hat{\phi}^\dagger(x_2)\right] = 0;$
- consistent and κ-Poincaré invariant deformed h.o. algebra.
- Future perspectives: N-point functions and interacting fields.

Thank you

Geometry of momentum space

Left-invariant metric:

$$ds^2 = d\xi_- d\xi_+,$$
$$\xi = \left(k_-, \frac{e^{2k_+} - 1}{2}\right) \rightarrow \xi_+ > -\frac{1}{2}$$



Right-invariant metric:

$$ds^2 = d\chi_- d\chi_+,$$
$$\chi = \left(-e^{2k_+} k_-, \frac{e^{-2k_+} - 1}{2}\right) \rightarrow \chi_+ > -\frac{1}{2}$$

κ - Lorentz transformation

$$\xi_\pm = e^{\pm\omega} \xi_\pm$$
$$\chi_\pm = e^{\mp\omega} \chi_\pm$$