The role of uncertainties in quantum cosmology

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Einsteins General Relativity

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \kappa \, T_{\mu\nu}$$

space-time   matter

$$\kappa = \frac{8\pi G}{c^4}$$

Quantum Gravity ?

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} \quad ? \quad \kappa \, T_{\mu\nu}$$

space-time

QM, QFT

The order of magnitudes desaster

$$E_{Planck} \approx 10^{28} eV$$
$$E_{LHC} \approx 10^{13} eV$$

- Search for semiclassical effects

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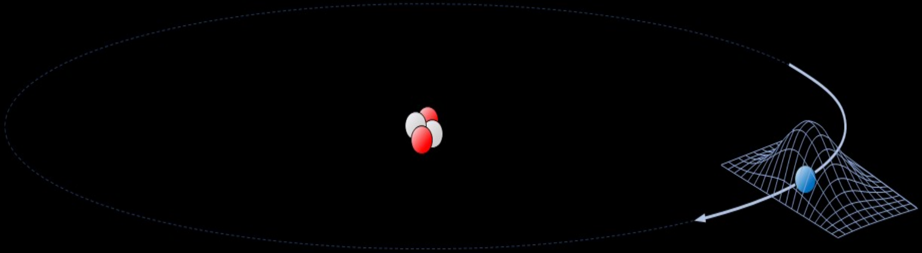
- Look first at the transission from classical to quantum mechanics



The Rydberg Atom

Classical orbits, collapse and revivals

- atomic energy levels:

$$E(n) = \frac{-\mu Z e^4}{2\hbar^2 n^2}$$

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$$T_{rev} = \frac{\pi\hbar}{E''(n_0)}$$

Classical orbits, collapse and revivals

experimentally tested

- atomic energy levels:

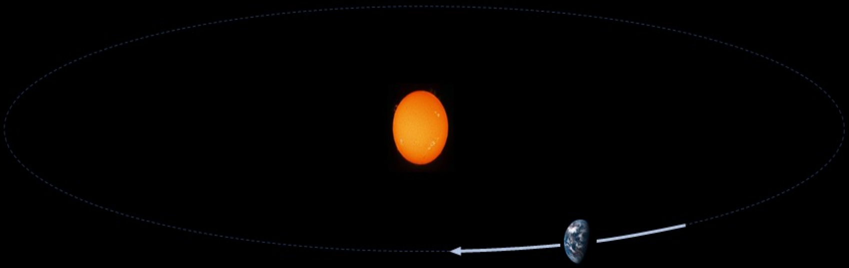
$$E(n) = \frac{-\mu Z e^4}{2\hbar^2 n^2} \qquad \text{potassium}$$

- superposition around $n_0 \Rightarrow$ coherent state $\qquad n_0 = 75$

$$T_{cl} = \frac{2\pi\hbar}{E'(n_0)} \qquad T_{cl} = 57ps$$

$$T_{rev} = \frac{\pi\hbar}{E''(n_0)} \qquad T_{rev} = 5.3ns$$

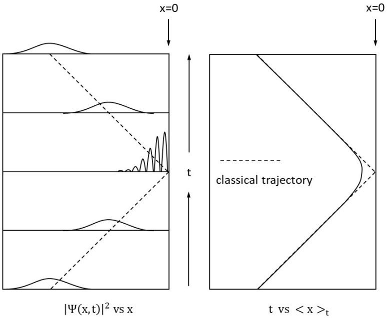
J.A. Yazell, C.R. Stroud: Observation of fractional revivals in the evolution of a Rydberg atomic wave packet, Phys.Rev.A 43 (1991)



The revival time is longer than the age of the solar system!

Quantum Bounce

$$V(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \infty & \text{for} \quad x \geq 0 \end{cases}$$



M.Belloni, M.A. Doncheski, R.W. Robinett: Exact REsults for 'Bouncing' Gaussian Wave packets, Physica Scripta(2005) Vol 71.

What do we learn for possible quantum gravity effects?

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- The transission from classical to quantum theory manifests in a deviation of the expectation values.
- This deviation is inseparably linked to a spreading behaviour.
- On marcoscopic scales even the effects of the combination of Newtonian gravity and quantum mechanics require long time scales.
  $\Rightarrow$ provided by cosmic evolution

Quantum Gravity and Quantum Cosmology

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Structural problems: time vanishes, no positive definite scalar product , no unitary time evolution.

1. The Bohmian Strategy

Time from guidance condition of the classical Hamilton-Jacobi Theory:

$$\frac{\partial L}{\partial \dot{q}} = p = \frac{\partial S}{\partial q}\,, \text{where} \quad \Psi = Re^{iS/\hbar}\,.$$

- advantage: Time emerges naturally from the canonical structure of the theory
- disadvantage: still no positive definite scalar product, no usefull notion of uncertainty

2. Matter as Clock

Using one matter variable as "time".

- advantage: Choice of scalar product and self-adjoint time evolution possible
- disadvantage: one canonical variable is taken out and declared as "time".

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- It was possible to define a scalar product and conditions for a self-adjoint time evolution for a flat Friedmann universe filled with a scalar field. $\surd$

Variational formulation of General Relativity - a reminder

$$\delta_{g_{\mu\nu}} \left(\frac{1}{2\kappa} \int d^4x \sqrt{-g}\, R + S_{matter} \right)$$
$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

- Varying the action with respect to the metric.
- Getting Einsteins equations.

Unimodular gravity

$$\delta_{g_{\mu\nu}} \left(\frac{1}{2\kappa} \int d^4x \sqrt{-g}\, R + S_{matter} \right) \Bigg|_{\mathbf{g=-1}} = 0$$

$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} - \Lambda\, g_{\mu\nu}$$

- Varying the action with respect to the metric under the condition $det\, g_{\mu\nu} = g = -1$
- Getting Einsteins equations with an additional term
- Identifying Λ with the cosmological constant

Unimodular Gravity and General Relativity

Any solution of unimodular gravity is a solution of general relativity with a certain Λ and vice versa.

The role of Λ:

constant of nature in general relativity -
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There is only **one universe** to explore!

$\Rightarrow$ Both theories are **practically equivalent** on the classical level.

Canonical unimodular gravity

- the same momentum constraints as in canonical gravity (secondary constraints)
- The Hamiltonian density $\mathcal{H}_0$ is not a constraint
- $\mathcal{H}_0$ is a conserved quantity (result of tertiary constraints)

Quantization of unimodular gravity

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- The theory yields a Schrödinger like equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \int \widehat{\mathcal{H}_0} dx^3 \Psi \,,$$

with $\Psi\left[h_{ab}, t\right]$

Quantization of unimodular gravity

- There is no Hamiltonian constraint!

- The theory yields a Schrödinger like equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \int \widehat{\mathcal{H}_0}dx^3\Psi\,,$$

  with $\Psi\left[h_{ab}, t\right]$

- In the case of reduced models there is not any constraint at all.

The Model:

spacetime:

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2$$

$d\Omega_3^2 \ldots$ 3-dim. flat space

$$det\, g_{\mu\nu} \stackrel{!}{=} -1 \rightarrow N = 1/a^3$$

matter:

Lagrangian of the field ϕ

$$L_{matter} = N\, a^3 \left(\frac{\dot{\phi}^2}{2N^2c^2} - V(\phi) \right)$$

Unimodular Hamiltonian cosmology

Hamiltonian of a spatially flat Friedmann universe with scalar field :

$$H_{uni} = \frac{c^2}{2}\frac{p_\phi^2}{a^6} - \frac{c^2}{4\epsilon}\frac{p_a^2}{a^4} + V(\phi)\,. \qquad (\epsilon = 3c^4/(8\pi G) = 3/\kappa)$$

No Hamiltonian constraint, H_{uni} is a conserved quantity!

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No Hamiltonian constraint, H_{uni} is a conserved quantity!

Choice: $H_{uni} \equiv -\Lambda\epsilon/3\,,$

so that Λ assumes the value of the cosmological constant in general relativity.

Unimodular Hamiltonian operator

- canonical quantization:

$$\hat{p}_a = -i\frac{\hbar}{v_0}\frac{\partial}{\partial a}\,, \quad \hat{p}_\phi = -i\frac{\hbar}{v_0}\frac{\partial}{\partial \phi}\,, \tag{1}$$

- factor ordering that yields a Laplace Beltrami operator

$$\Rightarrow \tag{2}$$

$$\widehat{H} = \frac{\hbar^2 c^2}{4v_0^2\epsilon}\frac{1}{a^5}\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{\hbar^2 c^2}{2v_0^2}\frac{1}{a^6}\frac{\partial^2}{\partial \phi^2} + V(\phi)\,, \tag{3}$$

symmetric with respect to the measure $a^5 da\, d\phi$ (4)

Coordinate transformation:

$$A = a^3/3 \qquad B = \frac{3}{\sqrt{2\epsilon}}\phi \quad \Rightarrow$$

$$\widehat{H} = \frac{\hbar^2 c^2}{v_0^2 4\epsilon}\left\{\frac{1}{A}\frac{\partial}{\partial A}A\frac{\partial}{\partial A} - \frac{1}{A^2}\frac{\partial^2}{\partial B^2}\right\},$$

measure: $\quad A\,dA\,dB$

Lightcone coordinates

$$u = Ae^{-B} \qquad v = Ae^{B}\,,$$

$$\widehat{H} = \frac{\hbar^2 c^2}{v_0^2\,\epsilon} \frac{\partial^2}{\partial u \partial v} V\left(\frac{u}{v}\right)$$

measure $du\, dv$

$u \in (0, \infty), v \in (0, \infty)$

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Classical Hamiltonian in light cone coordinates:

$$H = -\frac{c^2}{\epsilon} p_u\, p_v + V\left(\frac{u}{v}\right)\,.$$

Schrödinger equation of unimodular quantum cosmology

$$\frac{\hbar^2 c^2}{v_0^2 \epsilon} \frac{\partial^2}{\partial u \partial v} \psi + V\left(\frac{u}{v}\right) \psi = i \frac{\hbar}{v_0} \frac{\partial}{\partial t} \psi$$

Conventional probability interpretation for unitary time evolution possible !

Condition for the unitary time evolution

requirement on the wavefunction:

$$\frac{d}{dt}\langle\psi|\widehat{H}^n|\psi\rangle = 0 \quad \text{for} \quad n = 2, 3, \ldots .$$

sufficient condition:

$$\psi(0, v, t) = C(t)f_1(v) \qquad \psi(u, 0, t) = C(t)f_2(u)\,,$$

where $f_1(x), f_2(x)$ are real functions with $f_1(0) = \pm f_2(0)$ and $C(t)$ is arbitrary.

Constructing solutions for an arbitrary scalar field

Search for eigenstates:

$$\frac{\partial^2}{\partial u \partial v}\psi_\Lambda(u,v) + V\left(\frac{u}{v}\right) = -\frac{\Lambda\epsilon}{3}\psi_\Lambda(u,v)$$

with the boundary conditions

$$\psi_\Lambda(0,x) = f_1(x) \qquad \psi_\Lambda(x,0) = f_2(x),$$

where $f_1(x), f_2(x)$ are real functions.
We construct wavepacket solutions by superposition

$$\psi(u,v,\tau) = \int_0^\infty e^{i\tau\,\epsilon\frac{\Lambda}{3}}\psi_\Lambda(u,v)F\left(\Lambda\right)d\Lambda\,,$$

$(\tau = t\hbar c^2/\epsilon)$.

We obtain for the time evolution at the edges

$$\psi(0, v, \tau) = C(\tau) f_1(v) \qquad \psi(u, 0, \tau) = C(\tau) f_2(u)$$

$$\text{where} \qquad C(\tau) = \int_0^\infty e^{i\epsilon\tau\frac{\Lambda}{3}} F(\Lambda)\, d\Lambda\,.$$

- **The solutions meet the condition for the unitary time evolution!**
- **For late times: asymptotic boundary conditions**

$$\lim_{\tau\to\infty} \psi(u, 0, \tau) = \lim_{\tau\to\infty} \psi(0, v, \tau) = 0\,.$$

Time evolution in the asymptotic future

The evolution of the expectation values of an observable $\widehat{O}$ is given by

$$\frac{d}{dt}\langle\psi|\widehat{O}|\psi\rangle = -\frac{i}{\hbar}\left(\langle\psi|\widehat{O}\widehat{H}|\psi\rangle - \langle\widehat{H}\psi|\widehat{O}\psi\rangle\right) . \tag{5}$$

This implies the Heisenberg equation

$$\frac{d}{dt}\langle\psi|\widehat{O}|\psi\rangle = -\frac{i}{\hbar}\left\langle\psi\left[\widehat{O},\widehat{H}\right]\psi\right\rangle ,$$

if and only only if

$$\langle\widehat{H}\psi|\widehat{O}\psi\rangle \stackrel{!}{=} \langle\psi|\widehat{H}\widehat{O}|\psi\rangle .$$

This condition reads for our Hamiltonian

$$\int_0^\infty \psi^*(0,v,t)\frac{\partial}{\partial v}\widehat{O}\psi(0,v,t)dv - \int_0^\infty\left(\frac{\partial}{\partial u}\psi^*(u,t)\right)\widehat{O}\psi(u,t)du .$$

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$\rightarrow$ This condition is fulfilled in the limit $t \rightarrow \infty$ if the wavefunction obeys the asymptotic boundary condition.

Ehrenfest theorem for unimodular quantum cosmology

Characteristic evolution equations for the asymptotic future:

$$\frac{d}{dt}\langle p_u \rangle = -\left\langle \frac{\partial V}{\partial v} \right\rangle \qquad \frac{d}{dt}\langle p_v \rangle = -\left\langle \frac{\partial V}{\partial u} \right\rangle$$

$$\frac{d}{dt}\langle u \rangle = -\mu \langle p_v \rangle \qquad \frac{d}{dt}\langle v \rangle = -\mu \langle p_u \rangle$$

$\mu = c^2/\epsilon = 2\pi G/(3c^2)$

for the scale factor $A^2 = uv = \frac{a^6}{9}$:

$$\frac{d^2}{dt^2}\langle A^2 \rangle = \frac{c^2}{\epsilon} \left\langle -2\,\widehat{H} + 2V \right\rangle \tag{6}$$

The expectation values obey the classical evolution equations, provided the uncertainties remain small enough!

The massless scalar field

The **massless scalar field** with $V(\phi) = 0$ is equivalent to the perfect fluid **stiff matter** model ($w = 1$): $p = \rho c^2 = \dot{\phi}^2/(2c^2)$.

Equation for Eigenstates

$$\frac{\partial^2}{\partial u \partial v}\psi_\Lambda(u, v) = -\frac{\Lambda\epsilon}{3}\psi_\Lambda(u, v).$$

Solutions for $\Lambda > 0$

$$\psi_\Lambda(u, v) = \int_0^v J_0\left[2\sqrt{\frac{\Lambda\,\epsilon u}{3}}\sqrt{v - x}\right]\,\frac{\partial f_1}{\partial x}dx + \int_0^u J_0\left[2\sqrt{\frac{\Lambda\,\epsilon v}{3}}\sqrt{u - x}\right]\,\frac{\partial f_2}{\partial x}dx + J_0\left[2\sqrt{\frac{\Lambda\,\epsilon}{3}vu}\right]\,f_1(0),$$

$f_1(x), f_2(x)$: real functions, $f_1(0) = f_2(0)$

Results for the massless scalar field

- Explicit solutions were found for $\Lambda > 0$
- For late times the expectation values fulfil the classical evolution equations

$$\lim_{t \to \infty} \frac{d^2}{dt^2} \langle A^2 \rangle = \frac{2}{3} \Lambda c^2$$

- uncertainties are growing for late times

$$\Delta(A^2) \sim t^2$$

The uncertainty dynamics for an arbitrary scalar field

- Consider an expansion of the terms depending on $V\left(\frac{u}{v}\right)$ about the (classical) expectation values
- take no higher order terms than $(\Delta u)^2, (\Delta v)^2, \Delta(u, v)$
- get a non-autonomous system of 10 linear equations for the uncertainties.

$$\frac{d}{dt}(\Delta u)^2 = -2\mu(\Delta(u, p_v)$$
$$\frac{d}{dt}(\Delta v)^2 = -2\mu(\Delta(v, p_u))$$
$$\dots$$

It contains the time-dependent functions

$$V_{11}(t) = \frac{\partial^2 V}{\partial v^2}, \quad V_{22}(t) = \frac{\partial^2 V}{\partial u^2} \quad V_{12}(t) = \frac{\partial^2 V}{\partial v \partial u}$$

which are taken at the classical values $u(t), v(t)$.

Analysis of the limit $t \to \infty$

$$\frac{d}{dt}\overrightarrow{\Delta} = \mathcal{M} \cdot \overrightarrow{\Delta}$$

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Define

$$\mathcal{M}_0 \equiv \mathcal{M}\Big|_{\mathbf{V}=\mathbf{0}}, \quad \mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$$

$\mathcal{M}_0$ charcaterizes the uncertainty dynamics of the massless scalar field: known to be unstable.

Analysis of the limit $t \to \infty$

$tr\,[\mathcal{M}] = 0 \Rightarrow det\,[Y(t)] = const$
$Y(t)\ldots$ fundamental solution

This means
$Y(t)$ stable implies $Y(t)$ uniformly stable

Proof by contradiction

Assume $\int_{t_0}^{\infty} |\mathcal{M}_1(t)| < \infty$ and the system $\mathcal{M}$ is stable

$\Rightarrow \mathcal{M}_0 = \mathcal{M} - \mathcal{M}_1$ is stable

contradicts the result for the massless scalar field

The system is unstable for integrable potentials.

Consequences and open questions

- Dynamics of fluctuation depends on the matter content.
- strong indication for generic spreading behaviour
- Could increasing quantum fluctuations contribute to dark matter?
- What is the netto effect of increasing space-time fluctuations?