The Quantum Field Theory of (Quadratic) Gravity

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(joint works with John F. Donoghue)

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Outline:

- Quadratic gravity: General Remarks
- Unitarity, causality and stability
- Some phenomenology
- Scattering amplitudes
- Outlook

Why Quadratic Gravity?

- Fundamental interactions described by the Standard Model: renormalizable quantum field theories!

- All such interactions needed “extra” ingredients to be fully understood (gauge invariance, asymptotic freedom, spontaneous symmetry breaking...).

- Quantum gravity as a renormalizable quantum field theory will also need a particular “variation”.

- Can one find any fundamental obstruction? So far our exploration has provided a negative answer to this question!

- Our exploration: Donoghue and Menezes, PRD 97, 056022 (2018); PRD 97, 126005 (2018); PRD 99, 065017 (2019); PRD 100, 105006 (2019); PRL 123, 171601 (2019); JPPNP 115, 103812 (2020); PRD 104, 045010 (2021); JHEP 11, 010 (2021); Il Nuovo Cimento 45C, 26 (2022). Menezes, JHEP 03, 074 (2022); 2111.11570 [hep-th]; Universe 8, 326 (2022).

The distinctive feature of a renormalizable QFT treatment of gravity

- Loops involving matter fields coupled to the metric yield divergences proportional to the second power of the curvatures.
- The fundamental action must have R^2 terms in order to renormalize the theory.
- Curvatures involve second derivatives of the metric, so that quadratic gravity involves metric propagators which are quartic in the momentum.
- In other words, the “variation” quoted above is related to the presence of *quartic propagators*.

Quadratic gravity: An overview

- Early explorers: Stelle, Fradkin-Tsetlyn, Adler, Zee, Smilga, Tomboulis, Antoniadis, Hasslacher-Mottola, Lee-Wick, Coleman, Boulware-Gross...

- Current explorers: Einhorn-Jones, Salvio-Strumia, Holdom-Ren, Donoghue-Menezes, Mannheim, Anselmi, Odintsov, Shapiro, Accioly, F. F. Faria, Narain-Anishetty...

- Related work: Lu-Perkins-Pope-Stelle, 't Hooft, Grinstein-O'Connell-Wise...

- Action ($\kappa^2 = 32\pi G$):

$$S_{\rm quad} = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{1}{6f_0^2}R^2 - \frac{1}{\xi^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right]$$

- Spin-two part of the propagator: (Parametrization: $g_{\mu\nu} = \eta_{\mu\lambda}(e^h)^\lambda{}_\nu = \eta_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}h_{\mu\lambda}h^\lambda{}_\nu + \ldots$)

Donoghue and GM, PRD 97, 126005 (2018)

$$\begin{aligned} iD_{\mu\nu\alpha\beta} &= i\mathcal{P}^{(2)}_{\mu\nu\alpha\beta}D_2(q) \\ D_2^{-1}(q) &= \frac{q^2+i\epsilon}{\tilde{\kappa}^2(q)} - \frac{q^4}{2\xi^2(\mu)} - \frac{q^4 N_{\rm eff}}{640\pi^2}\ln\left(\frac{-q^2-i\epsilon}{\mu^2}\right) - \frac{q^4 N_q}{1280\pi^2}\ln\left[\frac{(q^2)^2}{\mu^4}\right] \end{aligned}$$

Here $N_q = 199/3$ and $N_{\rm eff}$, is a number that depends on the number of light degrees of freedom with the usual couplings to gravity, $N_{\rm eff} = N_V + \frac{1}{4}N_F + \frac{1}{6}N_S + 21/6$. With the Standard Model fields plus gravity, $N_{\rm eff} = 325/12$.

Lee-Wick theories

- In theories with fundamental curvature-squared terms, the graviton propagator will be quartic in the momentum. This is generally considered to be problematic. With a quartic propagator in free field theory one expects negative norm ghost states, using for example ($\mu^2 > 0$)

$$\frac{1}{q^2 - \frac{q^4}{\mu^2}} = \frac{1}{q^2} - \frac{1}{q^2 - \mu^2}$$

- This is also the case of the so-called Lee-Wick theories (e.g., a higher-derivative QED). Interactions in such theories make the heavy state unstable, with a width which can be calculated in perturbation theory. This feature is a crucial modification as it *removes the ghost from the asymptotic spectrum.*

- Past experience with Lee-Wick theories indicates that they can be stable and unitary, although causality does seem to be violated on microscopic scales of order the width of the resonance.

- The massive states for the Lee-Wick QED model must be heavier than energies probed by the LHC. The associated micro-causality violation would then be associated with a time scale of $\sim\sim 10^{-25}$ seconds. In the gravitational case, the micro-causality violation would be proportional to the Planck time, 10^{-43} seconds.

Stability and energy flow

PRD 100, 105006 (2019)

See also:
- Salvio, PRD (2019)
- Fabris, Pelinson, Salles, Shapiro, JCAP (2012)
- Reis, Shapiro, Shapiro, PRD (2019)
- Alessia Platania, arXiv:2206.04072.

- Fourier transform of the scalar part of the spin-two graviton propagator:

$$iD_2(x) = \int \frac{d^4q}{(2\pi)^4}\; e^{-iq\cdot x} iD_2(q)$$

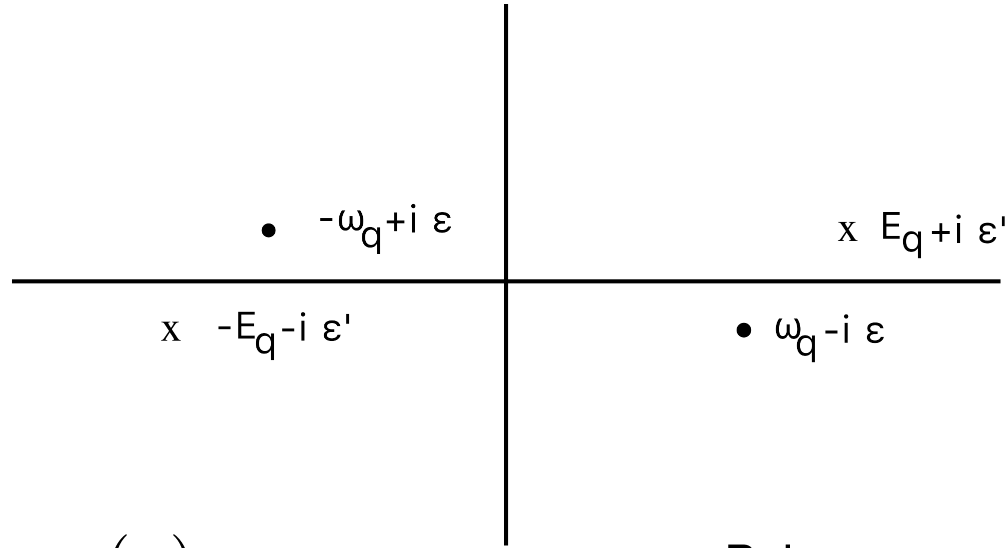
- In the weak coupling limit $\xi \ll 1$

$$D_2(q) = \left[\frac{1}{q^2} - \frac{1}{q^2 - m_r^2 - i\gamma\theta(q^2)}\right]$$

$$m_r^2 = \frac{2\xi^2(\mu)}{\kappa^2} + \frac{\xi^2(\mu)N_{\rm eff}}{320\pi^2\kappa^2}\ln\left(\frac{|m_r^2|}{\mu^2}\right) = \frac{2\xi^2(m_r)}{\kappa^2}$$

$$\gamma = \xi^2 m_r^2 \frac{N_{\rm eff}}{320\pi}.$$

Feynman propagator



Poles:

$$q_0 = \pm(\omega_q - i\epsilon),\ \omega_q = |\vec{q}|$$

$$q_0 \sim \pm\left[E_q + i\frac{\gamma}{2E_q}\right],\ E_q = \sqrt{\vec{q}^2 + m_r^2}$$

- Write the propagator as

$$D_2(t,\vec{x}) = \Theta(t)D_{\rm for}(x) + \Theta(-t)D_{\rm back}(x)$$

$(x_0 = t)$

- Forward propagator

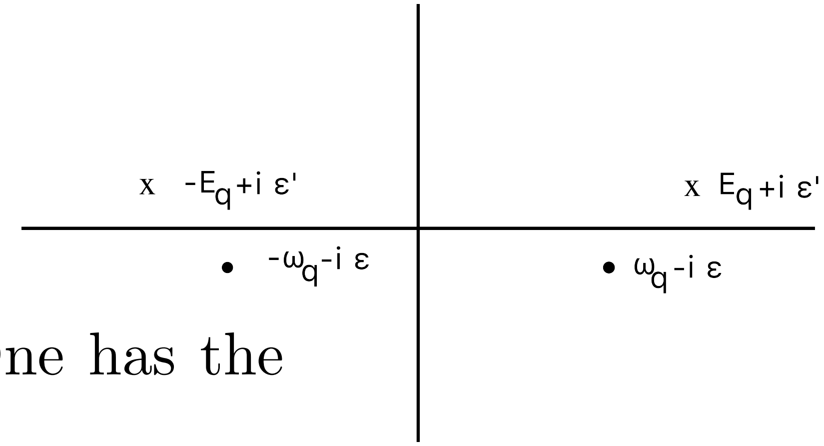
$$D_{\rm for}(t,\vec{x}) = -i\int\frac{d^3q}{(2\pi)^3}\left[\frac{e^{-i(\omega_q t - \vec{q}\cdot\vec{x})}}{2\omega_q} - \frac{e^{i(E_q t - \vec{q}\cdot\vec{x})}}{2(E_q + i\frac{\gamma}{2E_q})}e^{-\frac{\gamma t}{2E_q}}\right]$$

- Propagation backwards in time $(t < 0)$

$$D_{\rm back}(t,\vec{x}) = -i\int\frac{d^3q}{(2\pi)^3}\left[\frac{e^{i(\omega_q t - \vec{q}\cdot\vec{x})}}{2\omega_q} - \frac{e^{-i(E_q t - \vec{q}\cdot\vec{x})}}{2(E_q + i\frac{\gamma}{2E_q})}e^{-\frac{\gamma|t|}{2E_q}}\right]$$

- In both contributions we see exponential decay, and the reversal of the energy flow between the two terms.

Retarded propagator



- Such features can also be seen in the retarded propagator: One has the usual contribution

$$D_{\rm ret}(t>0,\vec{x}) = D^{(0)}_{\rm ret}(t>0,\vec{x})$$

as well as an unusual term for $t<0$

$$D_{\rm ret}(t<0,\vec{x}) \equiv D^{<}_{\rm ret}(t,\vec{x}) = i\int \frac{d^3q}{(2\pi)^3}\left[\frac{e^{-i(E_q t-\vec{q}\cdot\vec{x})}}{2(E_q+i\frac{\gamma}{2E_q})}e^{-\frac{\gamma|t|}{2E_q}} - \frac{e^{i(E_q t-\vec{q}\cdot\vec{x})}}{2(E_q-i\frac{\gamma}{2E_q})}e^{-\frac{\gamma|t|}{2E_q}}\right]$$

- The choice of the latter as a classical Green's function giving the response to an external source corresponds to the propagation of the effect backwards in time:

$$h_{\mu\nu}(t,x) = \int d^3x'\left[\int_{-\infty}^{t} dt' D^{(0)}_{\rm ret}(t-t',x-x') + \int_t^{\infty} dt' D^{<}_{\rm ret}(t-t',x-x')\right] J_{\mu\nu}(t',x')$$

- The energy flow associated with the ghost-like terms in the propagator is different from the usual case. What we normally refer to as "positive energy" is seen to be propagating backwards in time rather than the usual forward propagation – the *Merlin modes*. Microcausality violation on scales of order of the resonance width.

crystalinks.com


BOO!

=

Ostrogradsky instabilities?

Donoghue and GM, PRD 104, 045010 (2021)

- Let us explore the following toy model:

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi\right] - \frac{1}{2M^2}\Box\phi\Box\phi - g\phi\chi^2$$

and use path integrals to study the theory (χ is a normal light field).

- Introduce an auxiliary field η and define a new field $\phi = h - \eta$. The Lagrangian becomes:

$$\begin{aligned}\mathcal{L} &= \left[\frac{1}{2}\partial_\mu h\partial^\mu h - gh\chi^2\right] \\ &- \left[\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}M^2\eta^2 - g\eta\chi^2\right]\end{aligned}$$

- Hence:

$$Z = \int Dh\, e^{iS_h} \int D\eta\, e^{-iS_\eta}$$

Remnant of the original higher-derivative term (i —> -i)

where

$$S_h = \int d^4x \left(\frac{1}{2}\partial_\mu h\partial^\mu h - gh\chi^2\right)$$

and

$$S_\eta = \int d^4x \left(\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}M^2\eta^2 - g\eta\chi^2\right).$$

Avoiding Ostrogradsky

- The second path integral is an acceptable path integral on its own right. It is just the complex conjugate of the usual path integral for a massive particle.
- Interpretation: A time-reversed version of the standard path integral.
- That form of quantization is designed to also produce positive energy for the free particle states for these modes.
- The ghost is a positive energy particle!
- Integrating out the ghost:

$$e^{i\int d^4x d^4y \frac{1}{2} g\chi^2(x) iD_{-F}(x-y) g\chi^2(y)}$$

- Low-energy limit: A shift

$$\frac{g^2}{2M^2}\chi^4$$

  in the coupling $\lambda\chi^4$. It is suppressed by $1/M^2$ and so cannot overwhelm the original coupling for large M (Applequist-Carrazonne theorem).
- The low-energy effective Lagrangian just contains h and χ, and is a normal field theory. Since h is massless, it will have a normal classical theory.

Ghost resonances

- The theories which we are studying have propagators of the form

$$iD(q) = \frac{i}{q^2 + i\epsilon - \frac{q^4}{M^2} + \Sigma(q)}.$$

The pole at $q^2 = 0$ is the stable particle of the theory.

- At one-loop order, the self-energy typically has the form

$$\Sigma(q) = -\frac{\gamma}{\pi}\log\left(\frac{-q^2 - i\epsilon}{\mu^2}\right) = \left[-\frac{\gamma}{\pi}\log\left(\frac{|q^2|}{\mu^2}\right) + i\gamma\theta(q^2)\right]$$

for some calculable quantity γ with dimensions of mass squared.

- A massive resonance for timelike values of q^2 appears. Expanding near that resonance:

$$iD(q)\Big|_{q^2\sim M^2} \sim \frac{-i}{q^2 - M^2 - i\gamma} \quad .$$

Merlin modes!

Observe the minus sign in the numerator – a ghost like resonance. In addition, the width also comes with the opposite sign from normal.

Formal discussion of unitarity

- Unitarity:

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum_j \langle f|T^\dagger|j\rangle\langle j|T|i\rangle$$

- In processes that involve loop diagrams, the sum over real intermediate states can by accomplished by the Cutkosky cutting rules.
- What we usually do: look first at the free field theory to identify the free particles.
- Turn on interactions: some of these particles become unstable and no longer appear as the asymptotic states of the theory. *The free field limit has mislead us.*
- Should one include such unstable particles in the sums over states required for unitarity? Veltman says *no*!
- Veltman: unitarity is indeed satisfied by the inclusion of *only* the asymptotically stable states. Cuts are not to be taken through the unstable particles, and unstable particles are not to be included in unitarity sums.
- *However*, in the narrow-width approximation, the off-resonance production becomes small and only resonance production is important. In this limit a cut taken through the unstable particle with its width set to zero reproduces the same result as a cut through the decay products.

UNITARITY AND CAUSALITY IN A RENORMALIZABLE FIELD THEORY WITH UNSTABLE PARTICLES

M. VELTMAN *)

- Unitarity works with the stable particles as external states in the unitarity sum.
- The ghost resonance does not occur as an external state.
- Normal resonances and ghost resonances can be described in the same propagator using the coupling to the stable states described by the same $\Sigma(q)$.
- Veltman's work: normal resonances satisfy unitarity to all orders. Hence any discontinuity calculated with normal resonances in the intermediate states, can be converted into a discontinuity with ghost resonances by using:

$$iD(q) = \frac{i}{q^2 - m^2 + \Sigma(q) - q^4/\Lambda^2}$$

- If the normal resonance satisfies the unitarity relation, the ghost resonance will also!

Donoghue and GM, PRD 100, 105006 (2019)

Donoghue and GM, PRD 99, 065017 (2019)

Causality in higher-derivative theories

Donoghue and GM, PRL 123, 171601 (2019) (Editor's suggestion)

Donoghue and GM, Progress in Particle and Nuclear Physics 115, 103812 (2020)

Donoghue and GM, JHEP 11, 010 (2021)

Alessia Platania, arXiv:2206.04072

- Causality in quantum field theory is defined by the vanishing of field commutators for space-like separations.

- However, this does not imply a direction for causal effects. Hidden in our conventions for quantization is a connection to the definition of an arrow of causality.

- Mixing quantization conventions within the same theory, we get a violation of microcausality. In such a theory with mixed conventions the dominant definition of the arrow of causality is determined by the stable states.

- In some quantum gravity theories, such as quadratic gravity and possibly asymptotic safety, such a mixed causality condition occurs.

All gravitational theories display some causal uncertainty due to their spacetime fluctuations!

Donoghue and Menezes, in preparation

PHYSICAL REVIEW D VOLUME 50, NUMBER 6 15 SEPTEMBER 1994

General relativity as an effective field theory: The leading quantum corrections

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(Received 26 May 1994)

I describe the treatment of gravity as a quantum effective field theory. This allows a natural separation of the (known) low energy quantum effects from the (unknown) high energy contributions. Within this framework, gravity is a well-behaved quantum field theory at ordinary energies. In studying the class of quantum corrections at low energy, the dominant effects at large distance can be isolated, as these are due to the propagation of the massless particles (including gravitons) of the theory and are manifested in the nonlocal and/or nonanalytic contributions to vertex functions and propagators. These leading quantum corrections are parameter-free and represent necessary consequences of quantum gravity. The methodology is illustrated by a calculation of the leading quantum corrections to the gravitational interaction of two heavy masses.

- All interaction vertices of GR are energy–dependent and hence organize an EFT energy expansion. The focus is on the IR.
- Light bending (Bjerrum-Bohr, Donoghue, Holstein, Planté, Vanhove):

$$\theta \simeq \frac{4G_{\rm N}M}{b} + \frac{15}{4}\frac{G_{\rm N}^2M^2\pi}{b^2} + \left(8bu^S + 9 - 48\log\frac{b}{2b_0}\right)\frac{\hbar G_{\rm N}^2 M}{\pi b^3} + \dots .$$

Non-universal!

See also: Huber, Brandhuber, De Angelis and Travaglini, arXiv:2006.02375v1 [hep-th]

The lightcone is not a quantum gravity concept

- The causal structure of events is based on the geometric construction of light cones. If these undergo quantum fluctuations, so will the causal structure of the spacetime!
- The phenomenon discussed earlier cannot be described by geodesic motion as different massless species respond differently. It is then not equivalent to a quantum modification to the metric. The trajectory of massless particles is used in flat space to define the lightcone. That cannot be done in this background again due to the species dependence. Moreover, the quantum evolution samples the gravitational field over many points in space, not just along a local geodesic.
- Overall, the non-locality implies that the motion is not purely a geodesic.

Fluctuations in the effective causal structure

JHEP 11 010 (2021)

- When we study the signs of coefficients of local terms in an effective Lagrangian, we observe that the theory at low energies does not have a causality problem.

- However, the effect of Merlin modes at low energy is to produce a negative shift in such coefficients!

$$\left(\eta^{\mu\nu} + c_3(x) C^\mu C^\nu\right)\partial_\mu \partial_\nu \varphi + \cdots = 0$$

- An analysis of the Pauli-Jordan function reveals that it does not vanish for spacelike separations – a consequence of the non-analyticity of scattering amplitudes in the presence of Merlin modes.

Pauli-Jordan function:

$$\begin{aligned} G_{\mu\nu\alpha\beta}(x,x') &= G^{+}_{\mu\nu\alpha\beta}(x,x') - G^{-}_{\mu\nu\alpha\beta}(x,x') \\ &= G^{0}_{\mu\nu\alpha\beta}(x-x') + \frac{\kappa^2}{2\pi}\widetilde{\mathcal{P}}^{(2)}_{\gamma\delta\alpha\beta}(\partial_x) \\ &\times \left\{ \frac{1}{|\mathbf{x}-\mathbf{x}'|}\int \frac{dp^0}{2\pi}\left[F^{\gamma\delta}{}_{\mu\nu}(p^0, p^0\hat{\mathbf{n}})\, e^{ip^0 z} - F^{\gamma\delta}{}_{\mu\nu}(p^0, -p^0\hat{\mathbf{n}})\, e^{-ip^0 z'} + \cdots\right]\right\} \end{aligned}$$

$$z = |\mathbf{x}-\mathbf{x}'| - (t-t')$$

Phenomenology

Vertex displacements (Alvarez, Da Roid, Schat, Szynkman)

- Look for final state emergence.
- Before beam colision.

Early arrival of wave packets (Lee-Wick; Coleman; Grinstein, O'Connell, Wise)

- Wavepacket description of scattering processes.
- Some components arrive at detector early.

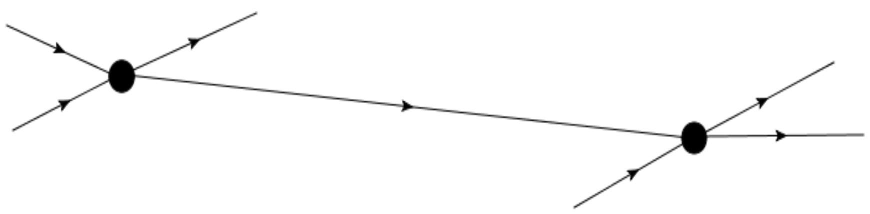
Resonance Wigner time delay reversal

- Normal resonances counterclockwise on Argand diagram. $\Delta t \sim \frac{\partial \delta}{\partial E} \sim > 0$
- Merlin modes are clockwise resonance.

For gravity, all such effects are Planck scale!

Causality uncertainty

- Wavepackets are an idealization.
- Likewise beam construction from previous scattering (and measurement due to final scattering).
- The timing of scattering will become uncertain!



All gravitational theories display some causal uncertainty due to their space-time fluctuations.

CK duality, BCJ relations and double copy

Bern, Carrasco, Chiodaroli, Johansson and Roiban, 2019

- Gravity amplitudes are notoriously complicated!

- In the mid-80s it was realized that tree-level closed string amplitudes can be written as sums of products of tree-level open-string amplitudes, the Kawai-Lewellen-Tye (KLT) relations. In the limit of infinite string tension, this becomes the field theory statement that the graviton tree amplitudes can be obtained as a sum of products of gluon scattering amplitudes:

$$\text{Gravity} = \text{Gauge Theory}^2.$$

Kawai, Lewellen and Tye, 1986

  This is the so-called *double copy*.

- Tree-level gauge theory amplitudes of gluon scattering could be written in a form where certain kinematic numerators obey the same Jacobi identities as the algebraic color factors of the non-abelian gauge group of the theory. This is called color-kinematics duality. Moreover, if one replaces the color factors in this representation of the amplitude with the kinematic factors of gauge theory, remarkably the result is the gravity tree amplitude! This is the BCJ (Bern, Carrasco, and Johansson) double copy.

Bern, Carrasco and Johansson, 2008

Quadratic gravity amplitudes

See also: Bob Holdom, 2021

GM, *JHEP* 03 (2022) 074

- Work in the Einstein frame (Einstein-Weyl theory):

$$S = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R - \frac{1}{2\xi^2}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}\right]$$

  where $\kappa^2 = 32\pi G$.

- Employ the map:

YM: spontaneously broken!

$$(\text{Higher-derivative YM}) \otimes \text{YM} = \text{Weyl-Einstein}$$

See also: H. Johansson and J. Nohle, 2017; H. Johansson, G. Mogull and F. Teng, 2018

- Spectrum: Besides the graviton, dilaton and axion, we also have additional five physical degrees of freedom associated with a gravitational Merlin, three states associated with a Merlin 2-form field and a Merlin scalar!

- Projection to pure gravity: Simply correlate the helicities in the two gauge-theory copies. This works in a similar fashion for quadratic gravity: In order to work with only gravitational Merlins, we take the symmetric tensor product of gauge-theory Merlins.

3-point amplitudes and propagator

- Propagator:

$$D_{\mu\nu\rho\sigma}(p) = \frac{1}{2p^2}\Big(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma}\Big)-\frac{1}{2(p^2-M^2-iM\Gamma)}\Big(\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\frac{2}{3}\eta_{\mu\nu}\eta_{\rho\sigma}\Big).$$

- 3-particle amplitudes involving only physical gravitons do not display contributions coming from higher-order derivative terms:

$$M_3^{(4)}[1^{h_1},2^{h_2},3^{h_3}] = M^2 M_3^{(2)}[1^{h_1},2^{h_2},3^{h_3}]$$

  and this result generalizes to an arbitrary number of gravitons by using BCFW recursion relations.

- Amplitudes involving a single gravitational Merlin particle vanishes:

$$M_{n+1}^{(4)}(1^{h_1},2^{h_2},\ldots,n^{h_n},\mathbf{k}) = 0$$

- 3-particle amplitude involving two gravitational Merlin particles:

$$\begin{aligned}
M_3(1^{++},\mathbf{2},\mathbf{3}) &= iA_3^{\text{tree, HD}}[1^+,\mathbf{2},\mathbf{3}]A_3^{\text{tree,YM}}[1^+,\mathbf{2},\mathbf{3}] = -2i\frac{\langle r|\mathbf{3}|1\big]^2}{M^4\langle r1\rangle^2}\langle\mathbf{32}\rangle^4\\
M_3(1^{--},\mathbf{2},\mathbf{3}) &= iA_3^{\text{tree, HD}}[1^-,\mathbf{2},\mathbf{3}]A_3^{\text{tree,YM}}[1^-,\mathbf{2},\mathbf{3}] = -2i\frac{\big[r|\mathbf{3}|1\rangle^2}{M^4\big[1r\big]^2}\big[\mathbf{32}\big]^4.
\end{aligned}$$

Tree-level Compton Amplitudes: graviton-Merlin scattering

- Amplitudes involving gravitons and Merlins:

$$\begin{aligned} M_4^{\text{tree}}(\mathbf{2}, 1^{++}, 4^{++}, \mathbf{3}) &= -is_{23} A_4^{\text{tree, HD}}[\mathbf{2}, 1^+, 4^+, \mathbf{3}] A_4^{\text{tree, YM}}[\mathbf{2}, 4^+, 1^+, \mathbf{3}] \\ &= 4i \frac{[14]^4}{s_{23}} \frac{\langle \mathbf{32} \rangle^4}{(s_{12} - M^2)(s_{13} - M^2)} \\ M_4^{\text{tree}}(\mathbf{2}, 1^{--}, 4^{++}, \mathbf{3}) &= -is_{23} A_4^{\text{tree, HD}}[\mathbf{2}, 1^-, 4^+, \mathbf{3}] A_4^{\text{tree, YM}}[\mathbf{2}, 4^+, 1^-, \mathbf{3}] \\ &= 4i \frac{1}{s_{23}(s_{12} - M^2)(s_{13} - M^2)} \Big([4\mathbf{3}] \langle 1\mathbf{2} \rangle + \langle 1\mathbf{3} \rangle [4\mathbf{2}] \Big)^4 \end{aligned}$$

See also: H. Johansson and A. Ochirov, 2019

Triple graviton vertex, brute-force calculation from quadratic gravity (Jordan frame)

$$
\begin{aligned}
\mathcal{T}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) &= -\frac{i}{2\kappa^2}\mathcal{E}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) - \frac{i}{12f_0^2}\mathcal{F}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) - \frac{i}{2\xi^2}\mathcal{W}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) \\
&+ \frac{i}{16}\left(\frac{3}{2f_0^2+\xi^2} + \frac{1}{\xi^2}\right)\Big[\eta_{\alpha\beta}\eta_{\gamma\delta}(k^\mu p^\nu + k^\nu p^\mu)(k^2+p^2) - 2(I^{\mu\nu}{}_{\alpha\beta}\eta_{\gamma\delta}p^4 + I^{\mu\nu}{}_{\gamma\delta}\eta_{\alpha\beta}k^4) \\
&+ \eta^{\mu\nu}\eta_{\alpha\beta}\eta_{\gamma\delta}\left(-q_\kappa(k^2p^\kappa + p^2k^\kappa) + k^2p^2\right)\Big] \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) &= \left(I_{\alpha\beta\gamma\delta} - \frac{1}{2}\eta_{\gamma\delta}\eta_{\alpha\beta}\right)\left(k^\mu k^\nu + p^\mu p^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\right) \\
&+ 2q_\lambda q_\sigma\left(I_{\alpha\beta}{}^{\lambda\sigma}I_{\gamma\delta}{}^{\mu\nu} + I_{\gamma\delta}{}^{\lambda\sigma}I_{\alpha\beta}{}^{\mu\nu} - I_{\alpha\beta}{}^{\lambda\mu}I_{\gamma\delta}{}^{\sigma\nu} - I_{\gamma\delta}{}^{\lambda\mu}I_{\alpha\beta}{}^{\sigma\nu}\right) \\
&+ q_\lambda q^\mu\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\lambda\nu} + \eta_{\gamma\delta}I_{\alpha\beta}{}^{\lambda\nu}\right) + q_\lambda q^\nu\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\lambda\mu} + \eta_{\gamma\delta}I_{\alpha\beta}{}^{\lambda\mu}\right) \\
&- q^2\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\mu\nu} + \eta_{\gamma\delta}I_{\alpha\beta}{}^{\mu\nu}\right) - \eta^{\mu\nu}q^\lambda q^\sigma\left(\eta_{\alpha\beta}I_{\gamma\delta\lambda\sigma} + \eta_{\gamma\delta}I_{\alpha\beta\lambda\sigma}\right) \\
&- 2q^\lambda\left(I_{\alpha\beta\lambda\sigma}I_{\gamma\delta}{}^{\sigma\nu}p^\mu + I_{\alpha\beta\lambda\sigma}I_{\gamma\delta}{}^{\sigma\mu}p^\nu + I_{\gamma\delta\lambda\sigma}I_{\alpha\beta}{}^{\sigma\nu}k^\mu + I_{\gamma\delta\lambda\sigma}I_{\alpha\beta}{}^{\sigma\mu}k^\nu\right) \\
&+ q^2\left(I_{\alpha\beta}{}^{\sigma\mu}I_{\gamma\delta\sigma}{}^\nu + I_{\gamma\delta}{}^{\sigma\mu}I_{\alpha\beta\sigma}{}^\nu\right) + \eta^{\mu\nu}q^\lambda q_\sigma\left(I_{\alpha\beta\lambda\rho}I_{\gamma\delta}{}^{\rho\sigma} + I_{\gamma\delta\lambda\rho}I_{\alpha\beta}{}^{\rho\sigma}\right) \\
&+ (k^2+p^2)\left(I_{\alpha\beta}{}^{\sigma\mu}I_{\gamma\delta\sigma}{}^\nu + I_{\alpha\beta}{}^{\sigma\nu}I_{\gamma\delta\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu}\left(I_{\alpha\beta\gamma\delta} - \frac{1}{2}\eta_{\gamma\delta}\eta_{\alpha\beta}\right)\right) - \left(k^2\eta_{\alpha\beta}I_{\gamma\delta}{}^{\mu\nu} + p^2\eta_{\gamma\delta}I_{\alpha\beta}{}^{\mu\nu}\right) \\
&- \frac{1}{2}\eta^{\mu\nu}\eta_{\alpha\beta}\eta_{\gamma\delta}k_\lambda p^\lambda + \eta^{\mu\nu}\eta_{\alpha\beta}I^{\lambda\kappa\gamma\delta}k_\lambda p_\kappa + \eta^{\mu\nu}\eta_{\gamma\delta}I^{\lambda\kappa\alpha\beta}p_\lambda k_\kappa + 2(I^{\mu\nu}{}_{\alpha\beta}p_\gamma p_\delta + I^{\mu\nu}{}_{\gamma\delta}k_\alpha k_\beta) \\
&+ 2(I^{\mu\lambda}{}_{\alpha\beta}p^\nu p_\lambda + I^{\nu\lambda}{}_{\alpha\beta}p^\mu p_\lambda)\eta_{\gamma\delta} + 2(I^{\mu\lambda}{}_{\gamma\delta}k^\nu k_\lambda + I^{\nu\lambda}{}_{\gamma\delta}k^\mu k_\lambda)\eta_{\alpha\beta} + q_\rho(I^{\mu\nu}{}_{\alpha\beta}\eta_{\gamma\delta}p^\rho + I^{\mu\nu}{}_{\gamma\delta}\eta_{\alpha\beta}k^\rho) \\
&- q_\sigma\left[(I^{\sigma\mu}{}_{\alpha\beta}p^\nu + I^{\sigma\nu}{}_{\alpha\beta}p^\mu + \eta^{\mu\nu}I^{\lambda\sigma}{}_{\alpha\beta}p_\lambda)\eta_{\gamma\delta} + (I^{\sigma\mu}{}_{\gamma\delta}k^\nu + I^{\sigma\nu}{}_{\gamma\delta}k^\mu + \eta^{\mu\nu}I^{\lambda\sigma}{}_{\gamma\delta}k_\lambda)\eta_{\alpha\beta}\right] \\
&- \eta^{\mu\nu}\left(I_{\lambda\kappa\alpha\beta}I^{\lambda\sigma}{}_{\gamma\delta}k^\kappa p_\sigma + I_{\lambda\kappa\gamma\delta}I^{\lambda\sigma}{}_{\alpha\beta}p^\kappa k_\sigma\right) - I^{\mu\lambda}{}_{\alpha\beta}I^{\nu\kappa}{}_{\gamma\delta}p_\lambda p_\kappa - I^{\mu\lambda}{}_{\gamma\delta}I^{\nu\kappa}{}_{\alpha\beta}k_\lambda k_\kappa - I^{\nu\lambda}{}_{\alpha\beta}I^{\mu\kappa}{}_{\gamma\delta}p_\lambda p_\kappa - I^{\nu\lambda}{}_{\gamma\delta}I^{\mu\kappa}{}_{\alpha\beta}k_\lambda k_\kappa \\
&- 2\left(I^{\mu\lambda}{}_{\alpha\beta}p^\nu p^\kappa + I^{\nu\lambda}{}_{\alpha\beta}p^\mu p^\kappa\right)I_{\lambda\kappa\gamma\delta} - 2\left(I^{\mu\lambda}{}_{\gamma\delta}k^\nu k^\kappa + I^{\nu\lambda}{}_{\gamma\delta}k^\mu k^\kappa\right)I_{\lambda\kappa\alpha\beta} \\
&+ q_\lambda\Big(I^{\nu\lambda}{}_{\alpha\beta}I^{\mu\kappa}{}_{\gamma\delta}p_\kappa + I^{\mu\lambda}{}_{\alpha\beta}I^{\nu\kappa}{}_{\gamma\delta}p_\kappa - 2I^{\mu\nu}{}_{\alpha\beta}I^{\lambda\kappa}{}_{\gamma\delta}p_\kappa + 2\eta^{\mu\nu}I^{\sigma\lambda}{}_{\alpha\beta}I_{\sigma\kappa\gamma\delta}p^\kappa \\
&+ I^{\nu\lambda}{}_{\gamma\delta}I^{\mu\kappa}{}_{\alpha\beta}k_\kappa + I^{\mu\lambda}{}_{\gamma\delta}I^{\nu\kappa}{}_{\alpha\beta}k_\kappa - 2I^{\mu\nu}{}_{\gamma\delta}I^{\lambda\kappa}{}_{\alpha\beta}k_\kappa + 2\eta^{\mu\nu}I^{\sigma\lambda}{}_{\gamma\delta}I_{\sigma\kappa\alpha\beta}k^\kappa\Big) \qquad (64)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) &= 2\Big[\eta^{\mu\nu}q_\lambda q_\kappa + I^{\mu\nu}{}_{\lambda\kappa}q^2 - q^\sigma\left(I^\mu{}_{\sigma\lambda\kappa}q^\nu + I^\nu{}_{\sigma\lambda\kappa}q^\mu\right)\Big]\Big[I^{\lambda\kappa}{}_{\alpha\beta}\left(p^2\eta_{\gamma\delta} - p_\gamma p_\delta\right) + I^{\lambda\kappa}{}_{\gamma\delta}\left(k^2\eta_{\alpha\beta} - k_\alpha k_\beta\right)\Big] \\
&- 2\Big(\eta^{\mu\nu}q^2 - q^\mu q^\nu\Big)\left\{\bar{\mathcal{R}}_{\alpha\beta\gamma\delta} + \frac{3}{2}\left[I_{\lambda\kappa\alpha\beta}I^{\kappa\sigma}{}_{\gamma\delta}(k^\lambda p_\sigma + p^\lambda p_\sigma) + I_{\lambda\kappa\gamma\delta}I^{\kappa\sigma}{}_{\alpha\beta}(p^\lambda k_\sigma + k^\lambda k_\sigma)\right]\right\} \qquad (65)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}^{\mu\nu}{}_{\alpha\beta\gamma\delta}(k,q) &= \Big[\eta^{\mu\nu}q_\lambda q_\kappa + I^{\mu\nu}{}_{\lambda\kappa}q^2 - q^\sigma\Big(I^\mu{}_{\sigma\lambda\kappa}q^\nu + I^\nu{}_{\sigma\lambda\kappa}q^\mu\Big)\Big]\bar{\mathcal{R}}^{\lambda\kappa}{}_{\alpha\beta\gamma\delta}(k,q) \\
&- \frac{2}{3}\Big(\eta^{\mu\nu}q^2 - q^\mu q^\nu\Big)\bar{\mathcal{R}}_{\alpha\beta\gamma\delta}(k,q) - 2\mathcal{P}^{\mu\nu\sigma\tau}I_\sigma{}^\lambda{}_{\gamma\delta}I_{\tau\lambda\alpha\beta}p^2k^2 \\
&+ \Big(\mathcal{P}^{\mu\nu}{}_{\sigma\lambda} - \frac{1}{4}\eta^{\mu\nu}\eta_{\sigma\lambda}\Big)q^\sigma\Big(p^\lambda k^2 + k^\lambda p^2\Big)I_{\alpha\beta\gamma\delta} - \Big(p^\mu p^\nu k^2 + k^\mu k^\nu p^2\Big)I_{\alpha\beta\gamma\delta} \\
&- \frac{1}{2}q^2\Big[I_\lambda{}^\nu{}_{\gamma\delta}I^{\lambda\mu}{}_{\alpha\beta}k^2 + I_\lambda{}^\nu{}_{\alpha\beta}I^{\lambda\mu}{}_{\gamma\delta}p^2 + (\mu\leftrightarrow\nu)\Big] \\
&+ q_\rho\Big[k^2\Big(I_\lambda{}^\nu{}_{\gamma\delta}I^{\lambda\mu}{}_{\alpha\beta}p^\rho + p^\mu(I^{\nu\lambda}{}_{\gamma\delta}I^\rho{}_{\lambda\alpha\beta} - I^{\rho\lambda}{}_{\gamma\delta}I^\nu{}_{\lambda\alpha\beta}) + \frac{1}{2}q^\mu\Big(I^{\rho\lambda}{}_{\gamma\delta}I^\nu{}_{\lambda\alpha\beta} - I^{\nu\lambda}{}_{\gamma\delta}I^\rho{}_{\lambda\alpha\beta}\Big)\Big) \\
&+ p^2\Big(I_\lambda{}^\nu{}_{\alpha\beta}I^{\lambda\mu}{}_{\gamma\delta}k^\rho + k^\mu(I^{\nu\lambda}{}_{\alpha\beta}I^\rho{}_{\lambda\gamma\delta} - I^{\rho\lambda}{}_{\alpha\beta}I^\nu{}_{\lambda\gamma\delta}) + \frac{1}{2}q^\mu\Big(I^{\rho\lambda}{}_{\alpha\beta}I^\nu{}_{\lambda\gamma\delta} - I^{\nu\lambda}{}_{\alpha\beta}I^\rho{}_{\lambda\gamma\delta}\Big)\Big) + (\mu\leftrightarrow\nu)\Big] \\
&+ q_\lambda q_\sigma\Big[k^2\Big(I^{\lambda\nu}{}_{\gamma\delta}I^{\mu\sigma}{}_{\alpha\beta} - \frac{1}{2}\Big(I^{\mu\nu}{}_{\gamma\delta}I^{\sigma\lambda}{}_{\alpha\beta} + I^{\mu\nu}{}_{\alpha\beta}I^{\lambda\sigma}{}_{\gamma\delta}\Big)\Big) + p^2\Big(I^{\lambda\nu}{}_{\alpha\beta}I^{\mu\sigma}{}_{\gamma\delta} - \frac{1}{2}\Big(I^{\mu\nu}{}_{\alpha\beta}I^{\sigma\lambda}{}_{\gamma\delta} + I^{\mu\nu}{}_{\gamma\delta}I^{\lambda\sigma}{}_{\alpha\beta}\Big)\Big) \\
&+ \frac{1}{2}q^\sigma\Big(k_\kappa\Big(I^{\mu\nu}{}_{\gamma\delta}I^{\kappa\lambda}{}_{\alpha\beta} - I^{\lambda\nu}{}_{\gamma\delta}I^{\mu\kappa}{}_{\alpha\beta}\Big) + p_\kappa\Big(I^{\mu\nu}{}_{\alpha\beta}I^{\kappa\lambda}{}_{\gamma\delta} - I^{\lambda\nu}{}_{\alpha\beta}I^{\mu\kappa}{}_{\gamma\delta}\Big)\Big) \\
&+ \frac{1}{2}q^\nu\Big(k_\kappa\Big(I^{\lambda\sigma}{}_{\gamma\delta}I^{\mu\kappa}{}_{\alpha\beta} - I^{\mu\lambda}{}_{\gamma\delta}I^{\kappa\sigma}{}_{\alpha\beta}\Big) + p_\kappa\Big(I^{\lambda\sigma}{}_{\alpha\beta}I^{\mu\kappa}{}_{\gamma\delta} - I^{\mu\lambda}{}_{\alpha\beta}I^{\kappa\sigma}{}_{\gamma\delta}\Big)\Big) + (\mu\leftrightarrow\nu)\Big] \\
&+ \frac{1}{2}\Big(I^{\sigma\kappa\mu\nu}q^\tau q^\lambda + I^{\lambda\tau\mu\nu}q^\kappa q^\sigma - I^{\lambda\kappa\mu\nu}q^\tau q^\sigma - I^{\sigma\tau\mu\nu}q^\kappa q^\lambda\Big)\Big(I_{\sigma\tau\alpha\beta}\eta_{\gamma\delta}k_\kappa p_\lambda + I_{\sigma\tau\gamma\delta}\eta_{\alpha\beta}p_\kappa k_\lambda \\
&+ I_{\sigma\tau\alpha\beta}\eta_{\gamma\delta}p_\kappa p_\lambda + I_{\sigma\tau\gamma\delta}\eta_{\alpha\beta}k_\kappa k_\lambda\Big) \qquad (66)
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{\mathcal{R}}^{\lambda\kappa}{}_{\alpha\beta\gamma\delta}(k,q) &= -I_{\alpha\beta\gamma\delta}q^\lambda q^\kappa + q^\sigma\Big[I_{\sigma\tau\gamma\delta}\Big(\frac{1}{2}I^{\tau\kappa}{}_{\alpha\beta}k^\lambda - I^{\lambda\kappa}{}_{\alpha\beta}k^\tau\Big) + I_{\sigma\tau\alpha\beta}\Big(\frac{1}{2}I^{\tau\kappa}{}_{\gamma\delta}p^\lambda - I^{\lambda\kappa}{}_{\gamma\delta}p^\tau\Big)\Big] \\
&+ \Big(I^{\tau\lambda}{}_{\gamma\delta}p_\sigma + I^\tau{}_{\sigma\gamma\delta}p^\lambda - I^\lambda{}_{\sigma\gamma\delta}p^\tau\Big)\Big(I^{\sigma\kappa}{}_{\alpha\beta}k_\tau + I^\sigma{}_{\tau\alpha\beta}k^\kappa - I_\tau{}^\kappa{}_{\alpha\beta}k^\sigma\Big) + k^2I^\kappa{}_{\tau\gamma\delta}I^{\lambda\tau}{}_{\alpha\beta} + p^2I^\kappa{}_{\tau\alpha\beta}I^{\lambda\tau}{}_{\gamma\delta} \\
&- 2k_\sigma k_\tau\left(\frac{1}{3}I^{\lambda\kappa}{}_{\gamma\delta}I^{\sigma\tau}{}_{\alpha\beta} + \frac{1}{2}I^{\kappa\sigma}{}_{\gamma\delta}I^{\tau\lambda}{}_{\alpha\beta}\right) - 2p_\sigma p_\tau\left(\frac{1}{3}I^{\lambda\kappa}{}_{\alpha\beta}I^{\sigma\tau}{}_{\gamma\delta} + \frac{1}{2}I^{\kappa\sigma}{}_{\alpha\beta}I^{\tau\lambda}{}_{\gamma\delta}\right) \\
&- q^\lambda\Big(I^{\kappa\sigma}{}_{\gamma\delta}I_{\tau\sigma\alpha\beta}k^\tau + I^{\kappa\sigma}{}_{\alpha\beta}I_{\tau\sigma\gamma\delta}p^\tau\Big) \\
&+ \frac{1}{2}\left(k^\kappa k^\lambda\eta_{\alpha\beta} - k^\kappa k_\rho I^{\rho\lambda}{}_{\alpha\beta} - k^\lambda k_\rho I^{\rho\kappa}{}_{\alpha\beta} + k^2I^{\kappa\lambda}{}_{\alpha\beta}\right)\eta_{\gamma\delta} + \frac{1}{2}\left(p^\kappa p^\lambda\eta_{\gamma\delta} - p^\kappa p_\rho I^{\rho\lambda}{}_{\gamma\delta} - p^\lambda p_\rho I^{\rho\kappa}{}_{\gamma\delta} + p^2I^{\kappa\lambda}{}_{\gamma\delta}\right)\eta_{\alpha\beta} \\
&- \frac{1}{2}(k^\kappa I^\lambda{}_{\rho\alpha\beta} + p^\lambda I^\kappa{}_{\rho\alpha\beta} - k_\rho I^{\kappa\lambda}{}_{\alpha\beta})p^\rho\eta_{\gamma\delta} - \frac{1}{2}(p^\kappa I^\lambda{}_{\rho\gamma\delta} + p^\lambda I^\kappa{}_{\rho\gamma\delta} - p_\rho I^{\kappa\lambda}{}_{\gamma\delta})k^\rho\eta_{\alpha\beta} \\
&+ \frac{2}{3}I^{\kappa\lambda}{}_{\alpha\beta}p^2\eta_{\gamma\delta} + \frac{2}{3}I^{\kappa\lambda}{}_{\gamma\delta}k^2\eta_{\alpha\beta} + I_{\alpha\beta\rho}{}^{(\lambda}p^{\kappa)}p^\rho\eta_{\gamma\delta} + I_{\gamma\delta\rho}{}^{(\lambda}k^{\kappa)}k^\rho\eta_{\alpha\beta} \\
&+ \frac{1}{4}\Big[\eta_{\alpha\beta}\eta_{\gamma\delta}(k^\lambda p^\kappa + p^\lambda k^\kappa) - 2(I_{\alpha\beta\rho}{}^{(\kappa}p^{\lambda)}k^\rho\eta_{\gamma\delta} + I_{\alpha\beta\rho}{}^{(\kappa}p^{\lambda)}p^\rho\eta_{\gamma\delta} + I_{\gamma\delta\rho}{}^{(\kappa}k^{\lambda)}p^\rho\eta_{\alpha\beta} + I_{\gamma\delta\rho}{}^{(\kappa}k^{\lambda)}k^\rho\eta_{\alpha\beta})\Big] \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathcal{R}}_{\alpha\beta\gamma\delta}(k,q) &= -q^2I_{\alpha\beta\gamma\delta} + \Big(I^\lambda{}_{\tau\gamma\delta}p_\sigma + I^\lambda{}_{\sigma\gamma\delta}p_\tau - I_{\tau\sigma\gamma\delta}p^\lambda\Big)\Big(I_\lambda{}^\tau{}_{\alpha\beta}k^\sigma + I^{\tau\sigma}{}_{\alpha\beta}k_\lambda - I_\lambda{}^\sigma{}_{\alpha\beta}k^\tau\Big) \\
&+ I^\lambda{}_{\sigma\gamma\delta}k_\tau\Big(I^\tau{}_{\lambda\alpha\beta}k^\sigma + I^{\tau\sigma}{}_{\alpha\beta}k_\lambda - I^\sigma{}_{\lambda\alpha\beta}k^\tau\Big) + I^\lambda{}_{\sigma\alpha\beta}p_\tau\Big(I^\tau{}_{\lambda\gamma\delta}p^\sigma + I^{\tau\sigma}{}_{\gamma\delta}p_\lambda - I^\sigma{}_{\lambda\gamma\delta}p^\tau\Big) \\
&+ \frac{1}{2}q^\lambda\Big(I_{\lambda\sigma\gamma\delta}I^{\kappa\sigma}{}_{\alpha\beta}k_\kappa + I_{\lambda\sigma\alpha\beta}I^{\kappa\sigma}{}_{\gamma\delta}p_\kappa\Big) \\
&+ \eta_{\alpha\beta}\eta_{\gamma\delta}k^\lambda p_\lambda - 2\left[I^\kappa{}_{\lambda\alpha\beta}\eta_{\gamma\delta}(k_\kappa p^\lambda + p_\kappa p^\lambda) + I^\kappa{}_{\lambda\gamma\delta}\eta_{\alpha\beta}(p_\kappa k^\lambda + k_\kappa k^\lambda)\right] \\
&+ \eta_{\alpha\beta}\left(p^2\eta_{\gamma\delta} - p^\lambda p^\kappa I_{\lambda\kappa\gamma\delta}\right) + \eta_{\gamma\delta}\left(k^2\eta_{\alpha\beta} - k^\lambda k^\kappa I_{\lambda\kappa\alpha\beta}\right). \qquad (68)
\end{aligned}
$$

Tree-level Compton Amplitudes: scalar-Merlin scattering

- To evaluate Compton scattering amplitude involving two gravitational Merlin particles and two matter particles, we apply the BCJ double-copy prescription:

$$M(1_s, 2, 3, 4_s) = i \sum_k \frac{n_k^{(s_1)} \tilde{n}_k^{(s_2)}}{s_k}$$

where $s = s_1 + s_2$, $2, 3$ are graviton or Merlin particles, $\tilde{n}_k$ are numerators belonging to the spontaneously broken gauge theory of the double copy described earlier and s_k are inverse propagators (they could be massive).

- For the scalar case:

$$\begin{aligned} M_4(\ell_A, \mathbf{1}, \mathbf{2}, \ell_B) &= \left[\frac{2}{M^2}\langle\mathbf{1}|\ell_A|\mathbf{1}]\langle\mathbf{2}|\ell_B|\mathbf{2}] + \frac{\langle\mathbf{12}\rangle[\mathbf{21}]}{M^2}(2(\mathbf{1}\cdot\ell_A) + M^2)\right]^2 \frac{i}{s-m^2} \\ &+ \left[\frac{2}{M^2}\langle\mathbf{1}|\ell_B|\mathbf{1}]\langle\mathbf{2}|\ell_A|\mathbf{2}] + \frac{\langle\mathbf{12}\rangle[\mathbf{21}]}{M^2}(2(\mathbf{2}\cdot\ell_A) + M^2)\right]^2 \frac{i}{u-m^2} \\ &- \frac{4i}{M^4}\Big(\langle\mathbf{12}\rangle[\mathbf{21}](\mathbf{1}-\mathbf{2})\cdot\ell_A + \langle\mathbf{1}|\ell_A|\mathbf{1}]\langle\mathbf{2}|\ell_B|\mathbf{2}] - \langle\mathbf{2}|\ell_A|\mathbf{2}]\langle\mathbf{1}|\ell_B|\mathbf{1}]\Big)^2\left(\frac{1}{t} - \frac{2}{t-M^2}\right) \end{aligned}$$

Loops and Generalized Unitarity

Bern, Dixon, Dunbar and Kosower, 1994, 1995

- The strategy of calculating gluon amplitudes of standard Yang-Mills theory from Grassmann integrations of $\mathcal{N} = 4$ SYM does not work at loop level. Now the gluon amplitudes differ in both theories. However, the tree-level method will still be useful since we are going to use tree-level amplitudes to reconstruct loop-level amplitudes. This is the so-called generalized unitarity method

- The knowledge of tree amplitudes can be recycled into information about loop integrands. The operation of taking loop propagators on-shell is called a unitarity cut. It originates from the unitarity constraint of the S-matrix. To see how, recall the unitarity demands that generalized optical theorem holds, that is, for an arbitrary process $a \to b$ one has that

$$i\mathcal{A}(a \to b) - i\mathcal{A}^*(b \to a) = -\sum_f \int d\Pi_f \mathcal{A}^*(b \to f)\mathcal{A}(a \to f)(2\pi)^4\delta^4(a - f)$$

and there is an overall delta function assuring energy-momentum conservation.

Golden goal: Understand deeply the analytic properties of the S-matrix!

How to implement the unitarity method in the presence of unstable particles

GM, arXiv:2111.11570 [hep-th]

- In order to implement the technique in a straightforward way, one must ensure that external momentum configurations of an amplitude allows the unstable propagator to become resonant. In this case the cut unstable propagator will have the correct cut structure to guarantee that unitarity is satisfied.

- On the other hand, there is also other situation that the method can be applied without further issues: In the narrow-width approximation! Near the resonance, we can treat the resonant particle as being on-shell. This means that in this limit a cut taken through the unstable particle with its width set to zero reproduces the same result as a cut through the decay products.

- In other words, for unstable particles the present practice of the unitarity method is valid if the assumption of a resonant unstable propagator is warrant. This can happen depending on external momentum configurations or else one should verify whether the narrow-width approximation holds in the particular case under studied.

Color-ordered one-loop amplitude associated with the process $g^+g^+ \to g^+g^+$

GM, *JHEP* 03 (2022) 074

Unitarity method produces

$$A_4^{1-\mathrm{loop}}(1^+,2^+,3^+,4^+) = -\frac{2i}{(4\pi)^{2-\epsilon}}\frac{s_{12}s_{23}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left[I_4[\mu^4]+8\mathcal{I}_4[(M^2+\mu^2)^2]\right]$$

One-loop amplitude for the graviton scattering process $h^{++}h^{++} \to h^{++}h^{++}$

Unitarity method produces

$$\begin{aligned} M_4^{1-\mathrm{loop}}(1^{++},2^{++},3^{++},4^{++}) \quad &= \quad \frac{2i}{(4\pi)^{2-\epsilon}}\left[\frac{s_{12}s_{23}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right]^2\Big[I_{1234}[\mu^8]+I_{1243}[\mu^8]+I_{1324}[\mu^8] \\ &+ \quad 128\Big(\mathcal{I}_{1234}[(M^2+\mu^2)^4]+\mathcal{I}_{1243}[(M^2+\mu^2)^4]+\mathcal{I}_{1324}[(M^2+\mu^2)^4]\Big)\Big] \end{aligned}$$

Leading Singularities

- Generalized unitarity explores discontinuities and these can be realized as contour integrals. Every time a residue is computed one explores a higher codimension singularity. The maximal number of residues at L-loop order in four dimensions is $4L$. Taking $4L$ residues gives rise to the highest codimension singularity and its discontinuity is known as the *leading singularity*.

- Leading singularities are computed using only compact contours and are therefore finite. Also, just as unitarity cuts, leading singularities only involve physical states and are gauge invariant!

- Leading singularities, which are computed as multidimensional residues, generically have support outside the physical region of integration. Therefore they are not naturally located on any of the regions mentioned above.

But they can also be useful to study observables associated with classical scattering!

See: Cachazo and Guevara, 2017; Guevara, 2017; Guevara, Ochirov and Vines, 2019; GM and Sergola, arXiv:2205.11701

Leading singularities in higher-derivative theories

GM, Universe 8, 326 (2022)

- Since leading singularities are generalizations of unitarity cuts, the presence of unstable Merlin modes in higher-derivative theories could engender issues.

- In any case, leading singularities are still well defined and accordingly they are able to capture relevant information on the analytic structure of amplitudes in such higher-derivative theories.

- Scattering of identical matter particles $\phi\phi \to \phi\phi$ interacting gravitationally:

$$\begin{aligned} \mathrm{LS}_h^{(s)} &= \sum_{h_1,h_3=++,--} \oint_\Gamma \frac{d^4\ell}{(2\pi)^4(\ell^2-m^2)} \frac{1}{(\ell+p_1')^2(\ell-p_1)^2} \\ &\times \quad M_3(\mathbf{1}',\boldsymbol{\ell},-\ell_1^{h_1})M_3(-\boldsymbol{\ell},\mathbf{1},-\ell_3^{h_3})M_4(\mathbf{2},\ell_3^{-h_3},\ell_1^{-h_1},\mathbf{2}') \end{aligned}$$

for gravitons running in the loop, whereas for Merlins we find that

$$\begin{aligned} \mathrm{LS}_M^{(s)} &= \oint_{-\Gamma} \frac{d^4\ell}{(2\pi)^4(\ell^2-m^2)} \frac{-1}{[(\ell+p_1')^2-M^2]} \frac{-1}{[(\ell-p_1)^2-M^2]} \\ &\times \quad M_3(\mathbf{1}',\boldsymbol{\ell},-\boldsymbol{\ell}_1)M_3(-\boldsymbol{\ell},\mathbf{1},-\boldsymbol{\ell}_3)M_4(\mathbf{2},\boldsymbol{\ell}_3,\boldsymbol{\ell}_1,\mathbf{2}') \end{aligned}$$

where m is the mass of the scalar particles.

Outlook

Quadratic gravity is a renormalizable quantum field theory that makes sense!

Essential features:

- Massless graviton identified through pole in propagator.
- Ghost resonance is unstable – it does not appear in spectrum.
- Formal quantization schemes exist but not needed.
- Stable under perturbations.
- Unitarity with only stable asymptotic states.
- LW contour as shortcut via narrow width approximation.
- Causality uncertainty near Planck scale.