Spherically symmetric, static spacetimes and their perturbations in the effective field theory of gravity

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with

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Quantum Gravity phenomenology in the Multi-Messenger approach (QG-MM)
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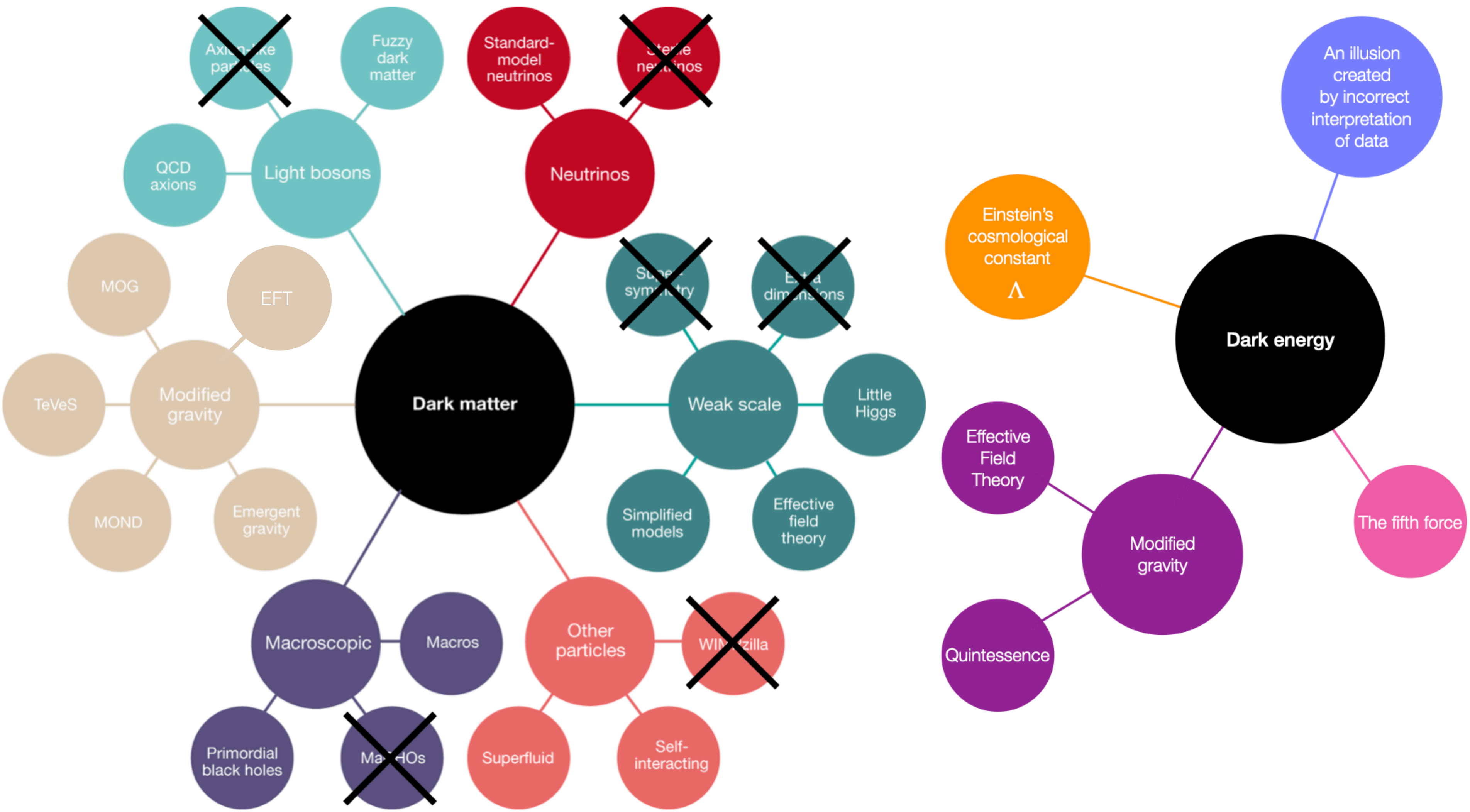
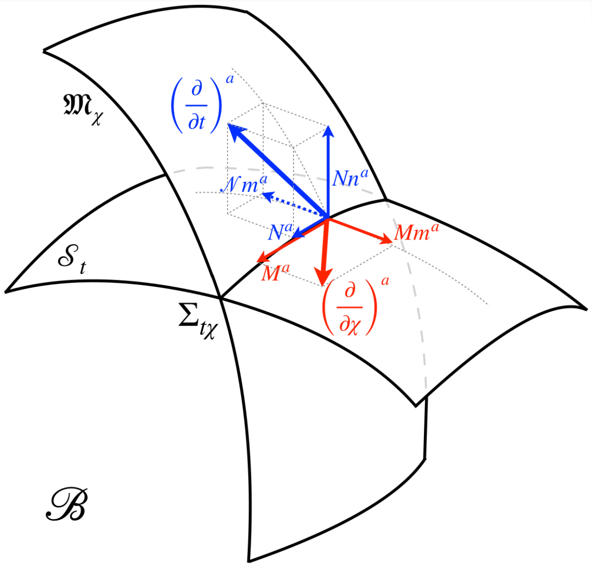
Dark matter and dark energy candidates



Technical details: nonorthogonal double foliation



The basis adapted to the temporal hypersurfaces

$$\tilde{g}_{ab} = -n_a n_b + m_a m_b + g_{ab}$$

From duality relations:

$$\left(\frac{\partial}{\partial t}\right)^a = Nn^a + N^a + \mathcal{N} m^a$$

$$\left(\frac{\partial}{\partial \chi}\right)^a = Mm^a + M^a$$

$$\mathcal{N} = \frac{\mathfrak{s}}{\mathfrak{c}} N$$

$$\mathfrak{s} = \sinh \psi$$

$$\mathfrak{c} = \cosh \psi$$

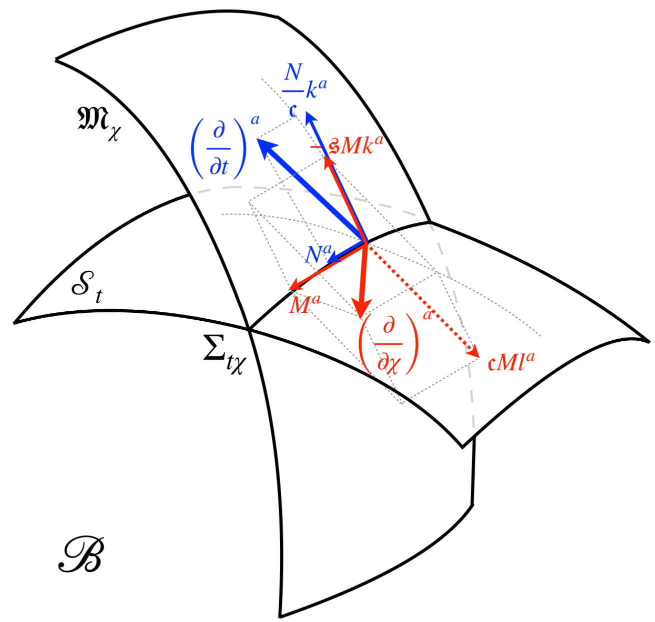
The basis adapted to the spacial hypersurfaces

$$\tilde{g}_{ab} = -k_a k_b + l_a l_b + g_{ab}$$

From duality relations:

$$\left(\frac{\partial}{\partial t}\right)^a = \frac{N}{\mathfrak{c}} k^a + N^a$$

$$\left(\frac{\partial}{\partial \chi}\right)^a = M\left(-\mathfrak{s} k^a + \mathfrak{c} l^a\right) + M^a$$



Embedding variables

The decomposition of the covariant derivatives:

$$\tilde{\nabla}_a n_b = K_{ab} + 2m_{(a}\mathscr{K}_{b)} + m_a m_b \mathscr{K} + n_a m_b \mathscr{L}^* - n_a \mathfrak{a}_b$$

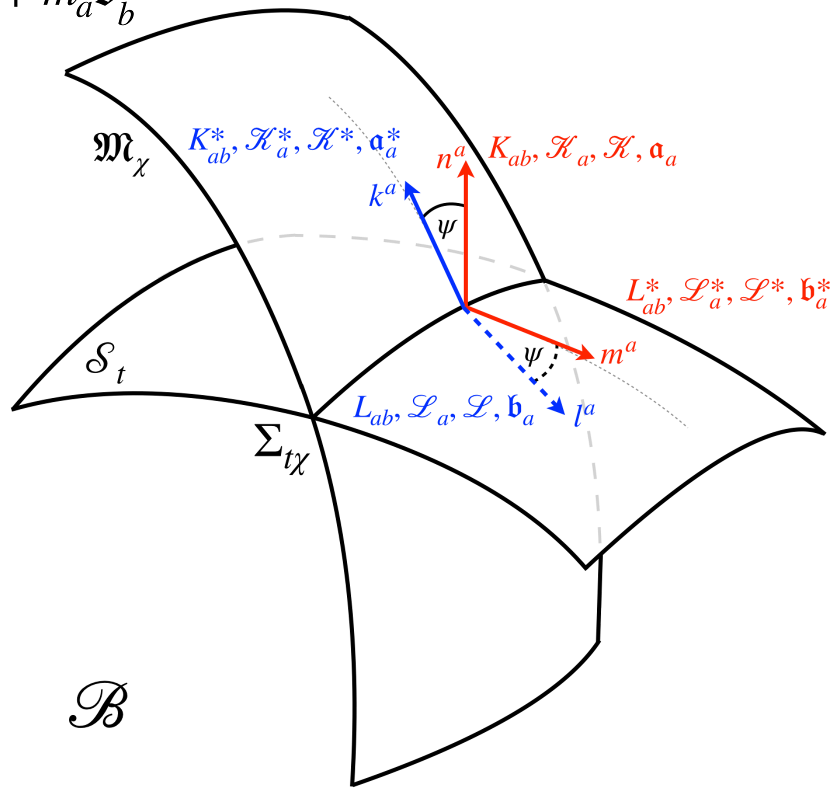
$$\tilde{\nabla}_a l_b = L_{ab} + 2k_{(a}\mathscr{L}_{b)} + k_a k_b \mathscr{L} + l_a k_b \mathscr{K}^* + l_a \mathfrak{b}_b$$

$$\tilde{\nabla}_a k_b = K^*_{ab} + l_a \mathscr{K}^*_b + l_b \mathscr{L}_a + l_a l_b \mathscr{K}^* + k_a l_b \mathscr{L} - k_a \mathfrak{a}^*_b$$

$$\tilde{\nabla}_a m_b = L^*_{ab} + n_a \mathscr{L}^*_b + n_b \mathscr{K}_a + n_a n_b \mathscr{L}^* + m_a n_b \mathscr{K} + m_a \mathfrak{b}^*_b$$

Definitions of the geometric variables:

$K_{ab} = g^c_a g^d_b \tilde{\nabla}_c n_d$	$L_{ab} = g^c_a g^d_b \tilde{\nabla}_c l_d$
$\mathcal{K}_a = g^c_a m^d \tilde{\nabla}_c n_d$	$\mathcal{L}_a = -g^c_a k^d \tilde{\nabla}_c l_d$
$\mathcal{K} = m^d m^c \tilde{\nabla}_c n_d$	$\mathcal{L} = k^d k^c \tilde{\nabla}_c l_d$
$\mathfrak{a}_a = g^d_a n^c \tilde{\nabla}_c n_d$	$\mathfrak{b}_a = g^d_a l^c \tilde{\nabla}_c l_d$
$K^*_{ab} = g^c_a g^d_b \tilde{\nabla}_c k_d$	$L^*_{ab} = g^c_a g^d_b \tilde{\nabla}_c m_d$
$\mathcal{K}^*_a = g^d_a l^c \tilde{\nabla}_c k_d$	$\mathcal{L}^*_a = -g^d_a n^c \tilde{\nabla}_c m_d$
$\mathcal{K}^* = l^d l^c \tilde{\nabla}_c k_d$	$\mathcal{L}^* = n^c n^d \tilde{\nabla}_c m_d$
$\mathfrak{a}^*_a = g^d_a k^c \tilde{\nabla}_c k_d$	$\mathfrak{b}^*_a = g^d_a m^c \tilde{\nabla}_c m_d$



GR in the formalism

Einstein–Hilbert action: $S_{EH} = \int d^4x \sqrt{-\tilde{g}}\tilde{R}$

Decomposition of the:

1) Determinant of the metric: $\sqrt{-\tilde{g}} = NM\sqrt{g}$
2) Ricci-scalar:

(n,m) basis

$$\tilde{R} = R + K_{ab}K^{ab} - L^*_{ab}L^{*ab} + 2\mathscr{K}^a\mathscr{K}_a - K\left(K + 2\mathscr{K}\right)$$
$$+L^*\left(L^* - 2\mathscr{L}^*\right) + 2D^a\left(\ln M\right)D_a\left(\ln N\right)$$
$$-2\tilde{\nabla}_a\left[D^a\left(\ln NM\right) - \left(K + \mathscr{K}\right)n^a + \left(L^* - \mathscr{L}^*\right)m^a\right]$$

(k,l) basis

$$\tilde{R} = R + K^{*ab}K^*_{ab} - L^{ab}L_{ab} + 2\mathscr{L}^a\mathscr{L}_a - K^*\left(K^* + 2\mathscr{K}^*\right)$$
$$+L\left(L - 2\mathscr{L}\right) + 2D^a\left(\ln\frac{N}{\mathfrak{c}}\right)D_a\ln\left(\mathfrak{c}M\right)$$
$$-2\tilde{\nabla}_a\left[D^a\left(\ln NM\right) - \left(K^* + \mathscr{K}^*\right)k^a + \left(L - \mathscr{L}\right)l^a\right]$$

Now, which form we should use?

For starting

1) **Background spherically symmetric, static:**

$$ds^2 = -\bar{N}^2 dt^2 + \bar{M}^2 dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$\bar{k}^a = (-\bar{N},0,0,0); \quad \bar{l}^a = (0,\bar{M},0,0); \quad \bar{N}^a = \bar{M}^a = 0$$

2) **The foliation orthogonal BEFORE the perturbation:** $\bar{\mathcal{N}} = 0$

3) **The relation between the gradient of the scalar and the normal of the spacial hypersurface:**

$$l^a = \frac{\nabla_a \phi}{\sqrt{X}}$$

The decomposed EFT action in the (k,l) basis:

$$S^{EFT} = \int d^4x \sqrt{-\tilde{g}} L^{EFT}\left(N, M, \mathcal{N}, \mathscr{K}^*, \mathfrak{k}, K^*, \varkappa^*, \mathscr{L}, L, \lambda, \zeta; \phi\right)$$

$\mathfrak{k} = \mathscr{L}_a \mathscr{L}^a$; $\varkappa^* = K^{*ab}K^*_{ab}$; $K^* = K^{*a}_a$; $\lambda = L^{ab}L_{ab}$; $L = L^a_a$

Scalars on the background:

Vanishing

$$\bar{\mathfrak{k}} = \bar{\varkappa}^* = \bar{K}^* = 0$$

Nonvanishing

$$\bar{\mathscr{L}} = -\frac{\bar{N}'}{\bar{M}\bar{N}}; \quad \bar{L} = \frac{2}{r\bar{M}}; \quad \bar{\lambda} = \frac{4}{r^2\bar{M}^2};$$

4) **Conformal gauge:**

$$g_{ab} = e^{2\zeta}\bar{g}_{ab}$$

2+1+1 decomposition of the Horndeski Lagrangian

The equations of motion:

$$\bar{L}^{\mathrm{EFT}} + \bar{N} L_N^{\mathrm{EFT}} + \frac{1}{\bar{M}}\left(\frac{2}{r} + \frac{\partial_r \bar{N}}{\bar{N}} + \partial_r\right) L_{\mathcal{L}}^{\mathrm{EFT}} = 0$$

$$\bar{L}^{\mathrm{EFT}} + \bar{M} L_M^{\mathrm{EFT}} + \frac{\partial_r \bar{N}}{\bar{M}\,\bar{N}} L_{\mathcal{L}}^{\mathrm{EFT}} - \frac{2\mathcal{F}}{\bar{M} r} = 0$$

$$\bar{L}^{\mathrm{EFT}} - \frac{2}{r^2} L_R^{\mathrm{EFT}} - \frac{1}{\bar{M}}\left(\frac{2}{r} + \frac{\partial_r \bar{N}}{\bar{N}} + \partial_r\right)\mathcal{F} = 0$$

$$(r^2 \bar{M}\,\bar{N}\,L_{\phi'}^{\mathrm{EFT}})' = r^2 \bar{M}\,\bar{N}\,L_{\phi}^{\mathrm{EFT}}$$

$$' = \partial_r$$

$$L_{\mathcal{G}_B}^{\mathrm{EFT}} = \frac{\partial (L^{\mathrm{EFT}})}{\partial \mathcal{G}_B}$$

$$\mathcal{G}_B = \mathcal{N}, N, M, \mathcal{K}^*, K^*, \mathcal{L}, L, \lambda, R, \phi, \phi'$$

Horndeski Lagrangian:

$$L^{\mathrm{H}} = \sum_{i=2}^{4} L_i^{\mathrm{H}}$$

$$L_2^{\mathrm{H}} = G_2(\phi, X)$$

$$L_3^{\mathrm{H}} = G_3(\phi, X)\,\Box\phi = G_3(\phi, X)\left[\frac{\tilde{\nabla}^a\phi\,\tilde{\nabla}_a X}{2X} + \sqrt{X}\,(L - \mathscr{L})\right]$$

$$L_4^{\mathrm{H}} = G_4\left(R + K^{*ab}K^*_{ab} - K^{*2}\right) + 2\sqrt{X}G_{4\phi}\,(L - \mathscr{L}) - (G_4 - 2XG_{4X}) \times \left[L^{ab}L_{ab} - 2\mathscr{L}^a\mathscr{L}_a + 2K^*\mathscr{K}^* - L^2 + 2L\mathscr{L} - 2D^a \ln(N/\mathfrak{c})\,D_a \ln(\mathfrak{c}M)\right]$$

$$\Box\phi = \tilde{\nabla}_a\tilde{\nabla}^a\phi \qquad X = \tilde{g}^{ab}\partial_a\phi\,\partial_b\phi \qquad G_{4X}(\phi, X) = \frac{\partial G_4(\phi, X)}{\partial X} \qquad G_{4\phi}(\phi, X) = \frac{\partial G_4(\phi, X)}{\partial \phi}$$

Application: k-essence theory with nonminimal coupling

The 2+1+1 nonorthogonal decomposition of the k-essence Lagrangian:

$$L^{EFT} = G_2(\phi, X) + G_4(\phi)\left(R - \lambda + L^2 - 2L\mathscr{L}\right) + 2\sqrt{X}G_{4\phi}(\phi)\left(L - \mathscr{L}\right)$$

The field equations with the choice of $\bar{M}^{-1} = \bar{N}$:

$$1 - \bar{N}^2 - r\left(\bar{N}^2\right)' = \frac{r^2}{\bar{G}_4}\left[-\frac{1}{2}\bar{G}_2 + \left(\frac{2\bar{N}^2}{r} + \frac{\left(\bar{N}^2\right)'}{2} + \bar{N}^2\partial_r\right)\left(\bar{G}_{4\phi}\phi'\right)\right]$$

$$1 - \bar{N}^2 - r\left(\bar{N}^2\right)' = \frac{r^2}{\bar{G}_4}\left[-\frac{1}{2}\bar{G}_2 + \left(\frac{2\bar{N}^2}{r} + \frac{\left(\bar{N}^2\right)'}{2}\right)\bar{G}_{4\phi}\phi' + \bar{N}^2\phi'^2\bar{G}_{2X}\right]$$

$$\frac{\left[r^2\left(\bar{N}^2\right)'\right]'}{2} = \frac{r^2}{\bar{G}_4}\left[\frac{1}{2}\bar{G}_2 - \left(\frac{1}{r}\bar{N}^2 + \left(\bar{N}^2\right)' + \bar{N}^2\partial_r\right)\left(\bar{G}_{4\phi}\phi'\right)\right]$$

$$\left(r^2\bar{N}^2\phi' G_{2X}\right)' - \frac{r^2}{2}\bar{G}_{2\phi} = \left(1 - \bar{N}^2 - r\left(\bar{N}^2\right)' - \frac{\left[r^2\left(\bar{N}^2\right)'\right]'}{2}\right)\bar{G}_{4\phi}$$

Examples: Black holes, horizons, naked singularities

Method: choose $\bar{G}_4$ ➡ $\bar{N}^2 = -2r^2 \int^r \frac{d\sigma}{\sigma^4 \bar{G}_4(\phi(\sigma))} \int^\sigma d\rho \bar{G}_4\Big(\phi(\rho)\Big)$ ➡ $\bar{G}_2$

Special cases		
$\bar{G}_4 = (16\pi G)^{-1}$	$\bar{N}^2 = 1 - \frac{2m}{r} - \Lambda r^2$	$\bar{G}_2 = \frac{6\Lambda}{16\pi G}$ $\Lambda > 0$: Schwarzschild–anti de Sitter $\Lambda < 0$: Schwarzschild–de Sitter $\Lambda = 0$: asymptotically flat
$\bar{G}_4 = \phi = r$	$\bar{N}^2 = \frac{1}{2} + \frac{Q}{r^2} - \Lambda r^2$	$\bar{G}_2(\phi) = -12\Lambda\phi + \frac{1}{\phi}$ Not asymptotically flat even, if $\Lambda = 0$

Horizons: Q~tidal charge

$\Lambda > 0, \quad Q < 0$	$\Lambda > 0, \quad Q > 0$	$\Lambda < 0, \quad Q < 0$	$\Lambda < 0, \quad Q > 0$
$r_{1,2}^2 = \frac{1 \pm \sqrt{1 + 16Q\Lambda}}{4\Lambda}$	$r_2 = \frac{1}{2}\sqrt{\frac{1 + \sqrt{1 + 16Q\Lambda}}{\Lambda}}$	$r_1 = \frac{1}{2}\sqrt{\frac{1 - \sqrt{1 + 16Q\Lambda}}{\Lambda}}$	
$r_2 > r > 0$: homogeneous, not static $r_1 > r > r_2$: spherically symmetric, static $r > r_1$: homogeneous, not static, asymptotically anti de Sitter	$r_2 > r > 0$: spherically symmetric, static $r > r_2$: homogeneous, not static asymptotically anti de Sitter	$r_1 > r > 0$: homogeneous, not static $r > r_1$: spherically symmetric, static, asymptotically de Sitter	naked singularity in the spherically symmetric, static spacetime, asymptotically de Sitter

Black holes, horizons, naked singularities

Method: choose $\bar{G}_4$ → $\bar{N}^2 = -2r^2 \int^r \frac{d\sigma}{\sigma^4 \bar{G}_4(\phi(\sigma))} \int^\sigma d\rho \bar{G}_4\left(\phi(\rho)\right)$ → $\bar{G}_2$

Special cases

$\bar{G}_4 = \phi = r^\alpha$

$\alpha > 1$

$$\bar{N}^2 = \frac{1}{1+\alpha} + \frac{C}{r^{1+\alpha}} - \Lambda r^2$$

$$\bar{G}_2(\phi) = \frac{(\alpha-1)X}{\alpha\phi} + \alpha\phi^{\frac{\alpha-2}{\alpha}} - \left(6 + 5\alpha + \alpha^2\right)\Lambda\phi$$

Not asymptotically flat even, if $\Lambda = 0$

The locations of the horizons is determined by: $-\Lambda r^{3+\alpha} + \frac{1}{1+\alpha} r^{1+\alpha} + C = 0$

Horizons:

$\Lambda > 0, \quad C < 0$

- $r_2 > r > 0$: homogeneous, not static
  $r_1 > r > r_2$: spherically symmetric, static
  $r > r_1$: homogeneous, not static, asymptotically anti de Sitter
- naked singularity in the homogeneous, not static spacetime, asymptotically anti de Sitter

$\Lambda < 0, \quad C > 0$

naked singularity in the spherically symmetric, static spacetime, asymptotically de Sitter

$\Lambda > 0, \quad C > 0$

$r_2 > r > 0$: spherically symmetric, static
$r > r_2$: homogeneous, not static, asymptotically anti de Sitter

$\Lambda < 0, \quad C < 0$

$r_1 > r > 0$: homogeneous, not static
$r > r_1$: spherically symmetric, static, asymptotically de Sitter

Black holes, horizons, naked singularities

Special cases

$\bar{G}_4 = \phi = A\left(1 + Br\right)$

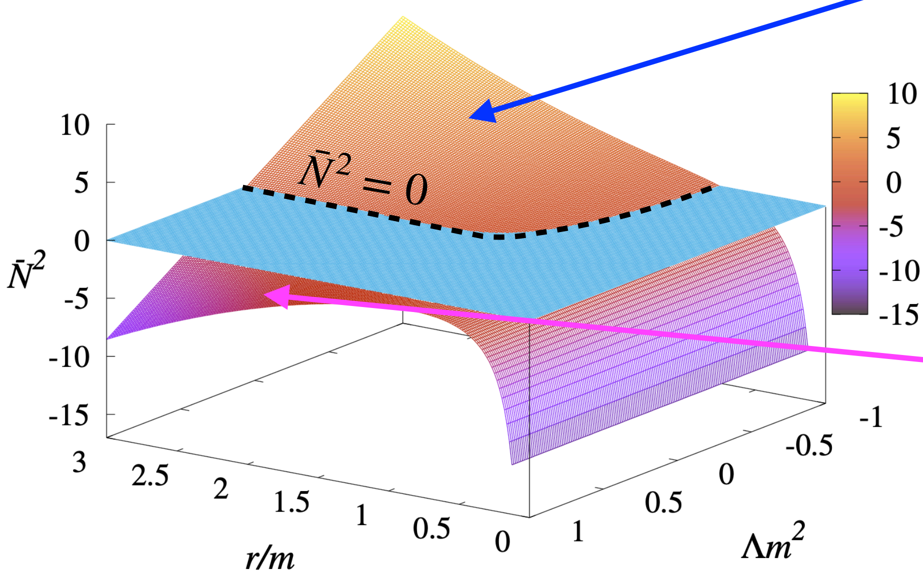
$$\bar{N}^2 = 1 + 3Bm - \frac{2m}{r} - B\left(1 + 6Bm\right)r - \Lambda r^2 - B^2\left(1 + 6Bm\right)r^2 \ln\left|\frac{Br}{1 + Br}\right|$$

$$\bar{G}_2 = -\frac{6mA^2B^3}{\phi - A} - 6\left(2\phi - A\right)\Lambda + AB^2\left(\frac{A}{\phi} - 12\right)\left(1 + 6mB\right) - 6B^2\left(1 + 6mB\right)\left(2\phi - A\right)\ln\left|\frac{\phi - A}{\phi}\right|$$

Not asymptotically flat even, if $\Lambda = 0$

Horizons and singularities:

$Bm = 1$



$\Lambda m^2 < 0$: There is a horizon
Outside: Spherically symmetric, static spacetime
Inside: Homogeneous, not static spacetime with a singularity at $r = 0$

$\Lambda m^2 > 0$:
Homogeneous, not static spacetime without horizon and with a singularity at $r = 0$

Black holes, horizons, naked singularities

Special cases

$\bar{G}_4 = \phi = A\left(1 + Br\right)$

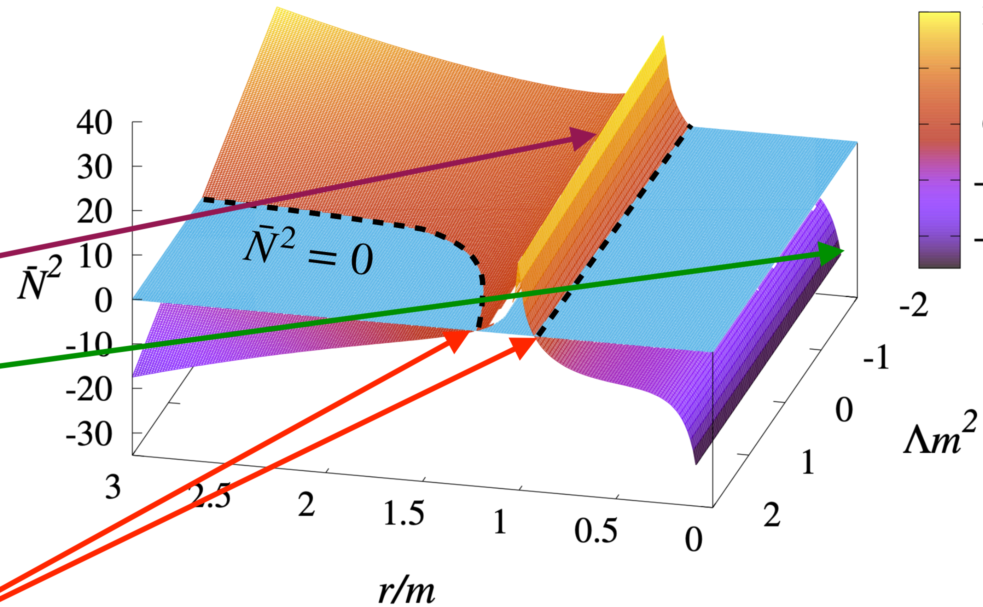
$$\bar{N}^2 = 1 + 3Bm - \frac{2m}{r} - B\left(1+6Bm\right)r - \Lambda r^2 - B^2\left(1+6Bm\right)r^2 \ln\left|\frac{Br}{1+Br}\right|$$

$$\bar{G}_2 = -\frac{6mA^2B^3}{\phi - A} - 6\left(2\phi - A\right)\Lambda + AB^2\left(\frac{A}{\phi} - 12\right)\left(1+6mB\right) - 6B^2\left(1+6mB\right)\left(2\phi - A\right)\ln\left|\frac{\phi - A}{\phi}\right|$$

Horizons and singularities:

$Bm = -1$

$\Lambda m^2 < 0$: There is a horizon
Outside: Spherically symmetric, static spacetime which include an **external naked singularity** (from the logarithmic term)
Inside: Homogeneous, not static spacetime with a **central singularity**



$\Lambda m^2 > 0$: There are **2 horizons**
Outside of the external horizon: Homogeneous, not static spacetime
Between the 2 horizons: Spherically symmetric, static spacetime contains a singularity (from the logarithmic term)
Inside of the interior horizon: Homogeneous, not static spacetime with a **central singularity** at $r = 0$

Black holes, horizons, naked singularities

Special cases

$\bar{G}_4 = \phi = A\,(1 + Br)$

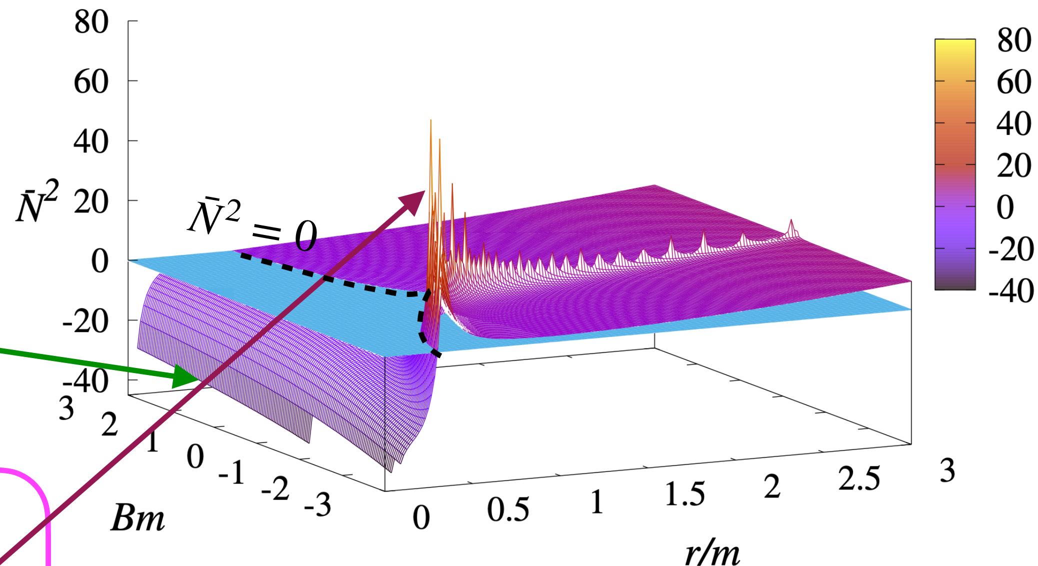
$$\bar{N}^2 = 1 + 3Bm - \frac{2m}{r} - B\,(1 + 6Bm)\,r - \Lambda r^2 - B^2\,(1 + 6Bm)\,r^2 \ln\left|\frac{Br}{1 + Br}\right|$$

$$\bar{G}_2 = -\frac{6mA^2B^3}{\phi - A} - 6\,(2\phi - A)\,\Lambda + AB^2\left(\frac{A}{\phi} - 12\right)(1 + 6mB) - 6B^2\,(1 + 6mB)\,(2\phi - A)\ln\left|\frac{\phi - A}{\phi}\right|$$

Horizons and singularities:

$\Lambda m^2 = -1$

$Bm > 0$: There is a horizon
Outside: Spherically symmetric, static spacetime
Inside: Homogeneous, not static spacetime with a **central singularity** at $r = 0$



$Bm < 0$: There is a horizon
Outside: Spherically symmetric, static spacetime with an **external naked singularity** (from the logarithmic term)
Inside: Homogeneous, not static spacetime with a **central singularity** at $r = 0$

The second order perturbation of the EFT action

Simple form:

$$\delta_2 S^{EFT} = \int d^4x \left(\sqrt{-\bar{g}}\,\delta_2 L^{EFT} + \delta_1\sqrt{-\bar{g}}\,\delta_1 L^{EFT} + \bar{L}^{EFT}\delta_2\sqrt{-\bar{g}}\right)$$

$$= \int d^4x \sqrt{-\bar{g}}\left(\delta_2 L^{EFT}_{\rm full}\right) \ ,$$

A lil' more complex form:

$$\begin{aligned}
\delta_2 L^{EFT}_{\rm full} = & \left[\frac{(\delta N)^2}{\bar{N}^2} + \frac{(\delta M)^2}{\bar{M}^2} + \frac{3}{\bar{M}\bar{N}}\delta M\delta N + \frac{6}{\bar{N}}\zeta\delta N + \frac{6}{\bar{M}}\zeta\delta M + 6\zeta^2\right]\bar{L}^{EFT} \\
& + \left[\frac{(\delta N)^2}{\bar{N}} + 2\zeta\delta N + \frac{\delta M\delta N}{\bar{M}}\right] L^{EFT}_N + \left[\frac{(\delta M)^2}{\bar{M}} + 2\zeta\delta M + \frac{\delta M\delta N}{\bar{N}}\right] L^{EFT}_M \\
& + \left[\frac{\partial_r\bar{N}}{\bar{M}^2\bar{N}}\left(\frac{(\delta M)^2}{\bar{M}} + 2\zeta\delta M - \frac{\bar{M}}{2\bar{N}^2}(\delta\mathcal{N})^2\right) + \left(\frac{\delta N}{\bar{N}^2\bar{M}} + \frac{\delta M}{\bar{N}\bar{M}^2}\right)\partial_r\delta N\right. \\
& + \frac{1}{\bar{N}^3}\delta\mathcal{N}\partial_t\delta N + \frac{1}{\bar{N}^2\bar{M}}\delta\mathcal{N}\partial_t\delta M + \frac{1}{\bar{N}\bar{M}}\delta M^a\bar{D}_a\delta N - \frac{1}{\bar{N}^2}\delta N^a\bar{D}_a\delta\mathcal{N} \\
& \left. + \frac{1}{\bar{N}^2\bar{M}}\delta\mathcal{N}\partial_r\delta\mathcal{N} + \left(\frac{2\zeta\delta N}{\bar{N}\bar{M}} + \frac{(\delta N)^2}{\bar{N}^2\bar{M}} + \frac{\delta M\delta N}{\bar{M}^2\bar{N}}\right)\left(\frac{2}{r} + \frac{\partial_r\bar{N}}{\bar{N}} + \partial_r\right)\right] L^{EFT}_{\mathcal{L}} \\
& + \left[2\zeta\bar{D}^a\delta M_a + \frac{1}{\bar{M}}\bar{g}^{ab}\delta\Gamma^i_{ab}\delta M_i - \frac{2}{\bar{M}^2 r}\left(\frac{(\delta M)^2}{\bar{M}} + 2\zeta\delta M + \frac{\delta M\delta N}{\bar{N}}\right)\right. \\
& \left. - \left(\frac{2\zeta\delta N}{\bar{N}\bar{M}} + \frac{2\zeta\delta M}{\bar{M}^2} + \frac{4\zeta^2}{\bar{M}}\right)\left(\frac{2}{r} + \frac{\partial_r\bar{N}}{\bar{N}} + \partial_r\right)\right]\mathcal{F} \\
& + \left[\frac{4}{r^2}\zeta^2 + 2\zeta\bar{D}_a\bar{D}^a\zeta - 2\left(\bar{D}_a\zeta\right)\left(\bar{D}^a\zeta\right) - \frac{4\zeta\delta M}{\bar{M}r^2} - \frac{4\zeta\delta N}{\bar{N}r^2} - \frac{8}{r^2}\zeta^2\right] L^{EFT}_R \\
& + \left(\frac{\delta N\delta\phi}{\bar{N}} + \frac{\delta M\delta\phi}{\bar{M}} + 2\zeta\delta\phi\right)\left[L^{EFT}_\phi - \frac{\left(r^2\bar{M}\bar{N}L^{EFT}_{\phi'}\right)'}{r^2\bar{M}\bar{N}}\right] \ \cdots
\end{aligned}$$

$$\begin{aligned}
& + \frac{L^{EFT}_{\mathfrak{k}}}{2\bar{M}\bar{N}^2}\left[\frac{1}{2\bar{M}}(\partial_t\delta M^a)(\partial_t\delta M_a) - \frac{1}{\bar{M}}(\partial_t\delta M_a)(\partial_r\delta N^a) + (\partial_t\delta M^a)\left(\bar{D}_a\delta\mathcal{N}\right)\right. \\
& \left. + \frac{\bar{g}_{ab}}{2\bar{M}}(\partial_r\delta N^a)\left(\partial_r\delta N^b\right) - \partial_r\delta N^a\bar{D}_a\delta\mathcal{N} + \bar{M}\bar{D}^a\delta\mathcal{N}\bar{D}_a\delta\mathcal{N}\right] \\
& + \frac{L^{EFT}_{\varkappa^*}}{\bar{N}^2}\left[2(\partial_t\zeta)^2 - 2(\partial_t\zeta)\bar{D}^a\delta N_a + \bar{D}_{(a}\delta N_{b)}\bar{D}^{(a}\delta N^{b)}\right] \\
& + \left[\frac{\delta\mathcal{N}}{\bar{N}^3}\partial_t(\delta\mathcal{N}) - \frac{1}{\bar{N}^2\bar{M}}\delta N^a\bar{D}_a\delta N - \frac{\delta M}{\bar{N}\bar{M}^2}\partial_t(\delta M) - \frac{\delta N}{\bar{N}^2\bar{M}}\partial_t(\delta M)\right. \\
& \left. + \left(\frac{\delta N\delta\mathcal{N}}{\bar{N}^2\bar{M}} + \frac{\delta M\delta\mathcal{N}}{\bar{N}\bar{M}^2} + 2\frac{\zeta\delta\mathcal{N}}{\bar{N}\bar{M}}\right)\left(\frac{2}{r} + \partial_r\right)\right] L^{EFT}_{\mathcal{K}^*} + L^{EFT}_{K^*}\left[\frac{2}{\bar{N}}\zeta\bar{D}^a\delta N_a\right. \\
& \left. - \frac{1}{\bar{N}}\bar{g}^{ab}\delta\Gamma^i_{ab}\delta N_i - \frac{\delta N}{\bar{N}^2}\left(2\partial_t\zeta - \bar{D}^a\delta N_a\right) - \frac{2}{r}\left(\frac{\delta N\delta\mathcal{N}}{\bar{N}^2\bar{M}} + \frac{\delta M\delta\mathcal{N}}{\bar{N}\bar{M}^2} + 2\frac{\zeta\delta\mathcal{N}}{\bar{N}\bar{M}}\right)\right] \\
& + \frac{2L^{EFT}_\lambda}{\bar{M}}\left[\frac{1}{\bar{M}}(\partial_r\zeta)^2 - \frac{1}{\bar{M}}\partial_r\zeta\bar{D}^a\delta M_a + \frac{(\delta M)^2}{\bar{M}^3 r^2} + \frac{1}{2\bar{M}}\bar{D}_{(a}\delta M_{b)}\bar{D}^{(a}\delta M^{b)}\right. \\
& \left. + \frac{\delta\mathcal{N}}{\bar{N}^2 r}\left(2\partial_t\zeta - \bar{D}^a\delta N_a - \frac{2\delta\mathcal{N}}{\bar{M}r}\right)\right] + \frac{\mathcal{F}_{\mathcal{F}}}{\bar{M}^2}\left(\delta M\bar{D}^a\delta M_a - 2\partial_r\zeta\delta M\right) \\
& + \frac{\delta\mathcal{N}}{\bar{N}^2}\left[2\partial_t\zeta - \bar{D}^a\delta N_a - \frac{2\delta\mathcal{N}}{\bar{M}r}\right] L^{EFT}_L + \frac{1}{2}L^{EFT}_{NN}(\delta N)^2 + \frac{1}{2}L^{EFT}_{MM}(\delta M)^2 \\
& + \frac{1}{2}L^{EFT}_{\mathcal{N}\mathcal{N}}(\delta\mathcal{N})^2 + L^{EFT}_{NM}\delta N\delta M + L^{EFT}_{N\mathcal{N}}\delta N\delta\mathcal{N} + L^{EFT}_{M\mathcal{N}}\delta M\delta\mathcal{N} \\
& + \frac{1}{2}L^{EFT}_{\phi\phi}\delta\phi\delta\phi + L^{EFT}_{\phi N}\delta\phi\delta N + L^{EFT}_{\phi M}\delta\phi\delta M + L^{EFT}_{\phi\mathcal{N}}\delta\phi\delta\mathcal{N} \ \ldots
\end{aligned}$$

$$\begin{aligned}
& -L^{EFT}_{NR}\left[\frac{4}{r^2}\zeta\delta N + 2\delta N\bar{g}^{ab}\bar{D}_a\bar{D}_b\zeta\right] - L^{EFT}_{\mathcal{N}R}\left[\frac{4}{r^2}\zeta\delta\mathcal{N} + 2\delta\mathcal{N}\bar{D}_a\bar{D}^a\zeta\right] \\
& -L^{EFT}_{MR}\left[\frac{4}{r^2}\zeta\delta M + 2\delta M\bar{g}^{ab}\bar{D}_a\bar{D}_b\zeta\right] - L^{EFT}_{\phi R}\left[\frac{4}{r^2}\zeta\delta\phi + 2\delta\phi\bar{D}_a\bar{D}^a\zeta\right] \\
& + \frac{L^{EFT}_{NK^*}}{\bar{N}}\left(2\delta N\partial_t\zeta - \delta N\bar{D}^a\delta N_a\right) + \frac{L^{EFT}_{\mathcal{N}K^*}}{\bar{N}}\left(2\delta\mathcal{N}\partial_t\zeta - \delta\mathcal{N}\bar{D}^a\delta N_a\right) \\
& + \frac{L^{EFT}_{MK^*}}{\bar{N}}\left(2\delta M\partial_t\zeta - \delta M\bar{D}^a\delta N_a\right) + \frac{L^{EFT}_{\phi K^*}}{\bar{N}}\left(2\delta\phi\partial_t\zeta - \delta\phi\bar{D}^a\delta N_a\right) \\
& + L^{EFT}_{N\mathcal{L}}\left[\frac{\delta N\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta N}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}\delta N\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}(\delta N)^2\right] \\
& + L^{EFT}_{\mathcal{N}\mathcal{L}}\left[\frac{\delta\mathcal{N}\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta\mathcal{N}}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}\delta\mathcal{N}\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}\delta\mathcal{N}\delta N\right] \\
& + L^{EFT}_{M\mathcal{L}}\left[\frac{\delta M\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta M}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}(\delta M)^2 + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}\delta M\delta N\right] \\
& + L^{EFT}_{\phi\mathcal{L}}\left[\frac{\delta\phi\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta\phi}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}\delta\phi\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}\delta\phi\delta N\right] \\
& + \mathcal{F}_N\left[\frac{2\delta N}{\bar{M}}\partial_r\zeta - \frac{\delta N}{\bar{M}}\bar{D}^a\delta M_a - \frac{2\delta N\delta M}{\bar{M}^2 r}\right] + \frac{L^{EFT}_{N\mathcal{K}^*}}{\bar{N}\bar{M}}\delta N\partial_t\delta M \\
& + \mathcal{F}_{\mathcal{N}}\left[\frac{\delta\mathcal{N}}{\bar{M}}\partial_r\zeta - \frac{\delta\mathcal{N}}{\bar{M}}\bar{D}^a\delta M_a - \frac{2\delta\mathcal{N}\delta M}{\bar{M}^2 r}\right] + \frac{L^{EFT}_{\mathcal{N}\mathcal{K}^*}}{\bar{N}\bar{M}}\delta\mathcal{N}\partial_t\delta M \\
& + \mathcal{F}_M\left[\frac{2\delta M}{\bar{M}}\partial_r\zeta - \frac{\delta M}{\bar{M}}\bar{D}^a\delta M_a - \frac{2(\delta M)^2}{\bar{M}^2 r}\right] + \frac{L^{EFT}_{M\mathcal{K}^*}}{\bar{N}\bar{M}}\delta M\partial_t\delta M \\
& + \mathcal{F}_\phi\left[\frac{2\delta\phi}{\bar{M}}\partial_r\zeta - \frac{\delta\phi}{\bar{M}}\bar{D}^a\delta M_a - \frac{2\delta\phi\delta M}{\bar{M}^2 r}\right] + \frac{L^{EFT}_{\phi\mathcal{K}^*}}{\bar{N}\bar{M}}\delta\phi\partial_t\delta M \ \ldots
\end{aligned}$$

Simplifications from symmetry and gouge choices

1) Spherical symmetry:
The Helmholtz-decomposition of Shift-vectors and the 2D metric on a unit sphere is recommended into rotation-free (even), and divergence-free (odd) parts:

$$
\begin{aligned}
\delta N_a &= \bar{D}_a P + E^b_{.a} D_b Q \ , \\
\delta M_a &= \bar{D}_a V + E^b_{.a} D_b W \ , \\
\delta g_{ab} &= \bar{g}_{ab} A + \bar{D}_a \bar{D}_b B + \frac{1}{2} \left(E^c_{.a} D_c D_b + E^c_{.b} D_c D_a \right) C \ .
\end{aligned}
$$

$$E_{ab} = \sqrt{\bar{g}} \varepsilon_{ab} \ , \qquad \varepsilon_{\theta\varphi} = 1$$

Even-type perturbational variables (8db): $N, M, \mathcal{N}, B, A, \delta\phi, P, V$
Odd-type perturbational variables (3db): C, W, Q

2) Gauge fixing:

- Conformal gauge (from the beginning): $B = 0 = C$ → $g_{ab} = (1 + A)\bar{g}_{ab}$ → $g_{ab} = e^{2\zeta}\bar{g}_{ab}$
- What about the 2 dof which is left?

…previously in the literature:

KMS-gauge for Horndeski theory

T. Kobayashi, H. Motohashi, T. Suyama, Phys. Rev. D 85, 084025 (2012) [arXiv:1202.4893 [gr-qc]], Phys. Rev. D 89, 084042 (2014) [arXiv:1402.6740 [gr-qc]].

	Even	Odd
Gauge fixing:	$B = A = P = 0$	$C = 0$
After variable transformations:	ψ, ϕ	Z
	gravitational and scalar modes	gravitational mode

Odd part: final action, stability-analysis?

Methods:

1) Introduction of a Lagrange multiplier and a new variable:

$$\boxed{Y^a} = \bar{D}^a Z \qquad \boxed{Z} = \dot{W} - Q' + \frac{2}{r} Q$$

2) The use of spherical harmonics and their orthogonality:

$$F(t,r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F_{lm}(t,r)\, Y_l^m(\theta,\varphi) = \sum_{l=0}^{\infty} F_l(t,r,\theta,\varphi)$$

$$F, G = Q, W, Z$$

$$\int d\varphi d\theta \sin\theta Y_l^m(\theta,\varphi)\, Y_{l'}^{*m'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'} \longrightarrow \int d\varphi d\theta \sin\theta F G = \sum_l \int d\varphi d\theta \sin\theta F_l G_l$$

The odd-type twice perturbed EFT action:

$$\delta_2 S_{\text{odd}}^{EFT} = \sum_{l=2}^{\infty} \frac{l(l+1)}{(l+2)(l-1)} \int d^4x \sqrt{-\tilde{g}} \left[-\frac{a_1^2}{a_3} \dot{Z}_l^2 - \frac{a_1^2}{a_2} (Z_l')^2 - a_1 \left(\bar{D} Z_l\right)^2 - U_H Z_l^2 \right] + \delta_2 S_{\text{odd},l=0}^{EFT} + \delta_2 S_{\text{odd},l=1}^{EFT}$$

$$U_H = -a_1 \partial_r \left[\frac{1}{a_2 \sqrt{-\tilde{g}}} \partial_r \left(a_1 \sqrt{-\tilde{g}} \right) \right] - \frac{2 a_1}{r^2}$$

$$\dot{} = \partial_t$$

$$(\bar{D} F)^2 = \bar{D}_a F \bar{D}^a F$$

The avoidance of ghosts: $U_H(r) > 0$ and $a_3 < 0$

$$a_1 = \frac{L_{\mathfrak{k}}^{EFT}}{4 \bar{M}^2 \bar{N}^2}$$

$$a_2 = \frac{L_{\varkappa^*}^{EFT}}{2 \bar{N}^2}$$

$$a_3 = \frac{L_{\lambda}^{EFT}}{2 \bar{M}^2}$$

The avoidance of Laplace instabilities: the square of sound speed (s?) has to be positive

A. D. Felice, S. Tsujikawa, JCAP 1202:007, [arXiv:1110.3878 [gr-qc]](2012).

Even part: gauge fixing?,final action(s)?, stability-analysis?

The even-type EFT action in…

…KMS-gauge: $B = P = A = 0$

$$
\begin{aligned}
\delta_2 L^{EFT}_{\text{even}} = & \left[\frac{(\delta N)^2}{\bar{N}^2} + \frac{(\delta M)^2}{\bar{M}^2} + \frac{3}{\bar{M}\bar{N}}\delta M \delta N\right] \bar{L}^{EFT} + \left[\frac{(\delta N)^2}{\bar{N}} + \frac{\delta M \delta N}{\bar{M}}\right] L^{EFT}_{N} \\
& + \left[\frac{(\delta M)^2}{\bar{M}} + \frac{\delta M \delta N}{\bar{N}}\right] L^{EFT}_{M} - \frac{2}{\bar{M} r}\left(\frac{(\delta M)^2}{\bar{M}^2} + \frac{\delta M \delta N}{\bar{N}\bar{M}} + \frac{(\delta \mathcal{N})^2}{\bar{N}^2}\right)\mathcal{F} \\
& + \left[\frac{\partial_r \bar{N}}{\bar{M}^2 \bar{N}}\left(\frac{(\delta M)^2}{\bar{M}} - \frac{\bar{M}}{2\bar{N}^2}(\delta\mathcal{N})^2\right) + \left(\frac{\delta N}{\bar{N}^2\bar{M}} + \frac{\delta M}{\bar{N}\bar{M}^2}\right)\partial_r \delta N\right. \\
& + \frac{1}{\bar{N}^3}\delta\mathcal{N}\partial_t \delta N + \frac{1}{\bar{N}^2\bar{M}}\delta\mathcal{N}\partial_t\delta M + \frac{1}{\bar{N}\bar{M}}\bar{D}^a V \bar{D}_a \delta N \\
& \left. + \frac{1}{\bar{N}^2\bar{M}}\delta\mathcal{N}\partial_r\delta\mathcal{N} + \left(\frac{(\delta N)^2}{\bar{N}^2\bar{M}} + \frac{\delta M \delta N}{\bar{M}^2\bar{N}}\right)\left(\frac{2}{r} + \frac{\partial_r\bar{N}}{\bar{N}} + \partial_r\right)\right] L^{EFT}_{\mathcal{L}} \\
& + \left(\frac{\delta N \delta\phi}{\bar{N}} + \frac{\delta M \delta \phi}{\bar{M}}\right)\left[L^{EFT}_{\phi} - \frac{\left(r^2 \bar{M}\bar{N} L^{EFT}_{\phi'}\right)'}{r^2\bar{M}\bar{N}}\right] \\
& + \frac{L^{EFT}_{\mathfrak{k}}}{2\bar{M}\bar{N}^2}\left[\frac{1}{2\bar{M}}\left(\bar{D}^a\dot{V}\right)\left(\bar{D}_a\dot{V}\right) + \left(\partial_t\bar{D}^a V\right)\left(\bar{D}_a\delta\mathcal{N}\right) + \bar{M}\bar{D}^a\delta\mathcal{N}\bar{D}_a\delta\mathcal{N}\right] \\
& + \left[\frac{\delta\mathcal{N}}{\bar{N}^3}\partial_t(\delta\mathcal{N}) - \frac{\delta M}{\bar{N}\bar{M}^2}\partial_t(\delta M) - \frac{\delta N}{\bar{N}^2\bar{M}}\partial_t(\delta M)\right. \\
& \left. + \left(\frac{\delta N \delta\mathcal{N}}{\bar{N}^2\bar{M}} + \frac{\delta M\delta\mathcal{N}}{\bar{N}\bar{M}^2}\right)\left(\frac{2}{r} + \partial_r\right)\right] L^{EFT}_{\mathcal{K}^*} - \frac{2}{r}\left(\frac{\delta N\delta\mathcal{N}}{\bar{N}^2\bar{M}} + \frac{\delta M \delta\mathcal{N}}{\bar{N}\bar{M}^2}\right) L^{EFT}_{K^*} \\
& + \frac{2L^{EFT}_{\lambda}}{\bar{M}}\left[\frac{(\delta M)^2}{\bar{M}^3 r^2} - \frac{1}{2\bar{M}}\bar{D}_a\bar{D}^a V\bar{D}_b\bar{D}^b V + \frac{1}{2\bar{M}r^2}\left(1 - \frac{1}{\bar{M}^2} + \frac{r\bar{M}'}{\bar{M}^3} - \frac{\bar{N}' r}{\bar{N}\bar{M}^2}\right)\bar{D}_a V \bar{D}^a V\right] \\
& + \frac{1}{2}L^{EFT}_{NN}(\delta N)^2 + \frac{1}{2}L^{EFT}_{MM}(\delta M)^2 + \frac{1}{2}L^{EFT}_{\mathcal{N}\mathcal{N}}(\delta\mathcal{N})^2 + L^{EFT}_{NM}\delta N\delta M \\
& + L^{EFT}_{N\mathcal{N}}\delta N\delta\mathcal{N} + \frac{1}{2}L^{EFT}_{\phi\phi}\delta\phi\delta\phi + L^{EFT}_{\phi N}\delta\phi\delta N + L^{EFT}_{\phi M}\delta\phi\delta M \\
& + L^{EFT}_{\phi\mathcal{N}}\delta\phi\delta\mathcal{N} + L^{EFT}_{M\mathcal{N}}\delta M\delta\mathcal{N} + \frac{\mathcal{F}_{\mathcal{F}}}{\bar{M}^2}\delta M\bar{D}^a\bar{D}_a V \\
& + L^{EFT}_{N\mathcal{L}}\left[\frac{\delta N\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta N}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}\delta N\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}(\delta N)^2\right] \\
& + L^{EFT}_{M\mathcal{L}}\left[\frac{\delta M\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta M}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}(\delta M)^2 + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}\delta M\delta N\right] \\
& + L^{EFT}_{\mathcal{N}\mathcal{L}}\left[\frac{\delta\mathcal{N}\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta\mathcal{N}}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}\delta\mathcal{N}\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}\delta\mathcal{N}\delta N\right] \\
& + L^{EFT}_{\phi\mathcal{L}}\left[\frac{\delta\phi\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta\phi}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}\delta\phi\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}\delta\phi\delta N\right] \\
& + L^{EFT}_{\mathcal{K}^*\mathcal{L}}\left[\frac{(\partial_t\delta M)\partial_t\delta\mathcal{N}}{\bar{N}^3\bar{M}} - \frac{\partial_t\delta M}{\bar{N}^2\bar{M}^2}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}^3}(\partial_t\delta M)\delta M + \frac{\partial_r\bar{N}}{\bar{N}^3\bar{M}^2}(\partial_t\delta M)\delta N\right] \\
& - \mathcal{F}_N\left[\frac{\delta N}{\bar{M}}\bar{D}^a\bar{D}_a V + \frac{2\delta N\delta M}{\bar{M}^2 r}\right] + L^{EFT}_{M\mathcal{K}^*}\frac{\delta M}{\bar{N}\bar{M}}\partial_t\delta M \\
& - \mathcal{F}_M\left[\frac{\delta M}{\bar{M}}\bar{D}^a\bar{D}_a V + \frac{2(\delta M)^2}{\bar{M}^2 r}\right] + \frac{L^{EFT}_{\mathcal{N}\mathcal{K}^*}}{\bar{N}\bar{M}}\delta\mathcal{N}\partial_t\delta M \\
& - \mathcal{F}_{\mathcal{N}}\left[\frac{\delta\mathcal{N}}{\bar{M}}\bar{D}^a\bar{D}_a V + \frac{2\delta\mathcal{N}\delta M}{\bar{M}^2 r}\right] + L^{EFT}_{N\mathcal{K}^*}\frac{\delta N}{\bar{N}\bar{M}}\partial_t\delta M \\
& - \mathcal{F}_{\phi}\left[\frac{\delta\phi}{\bar{M}}\bar{D}^a\bar{D}_a V + \frac{2\delta\phi\delta M}{\bar{M}^2 r}\right] + \frac{L^{EFT}_{\phi\mathcal{K}^*}}{\bar{N}\bar{M}}\delta\phi\partial_t\delta M \\
& - \mathcal{F}_{\mathcal{K}^*}\left[\frac{\partial_t\delta M}{\bar{N}\bar{M}^2}\bar{D}^a\bar{D}_a V + \frac{2\delta M\partial_t\delta M}{\bar{N}\bar{M}^3 r}\right] + \frac{L^{EFT}_{\mathcal{K}^*\mathcal{K}^*}}{2\bar{N}^2\bar{M}^2}(\partial_t\delta M)^2 \\
& + \frac{L^{EFT}_{\mathcal{L}\mathcal{L}}}{2}\left[\frac{(\partial_t\delta\mathcal{N})^2}{\bar{N}^4} - \frac{2\partial_t\delta\mathcal{N}}{\bar{N}^3\bar{M}}\partial_r\delta N + \frac{2\partial_r\bar{N}}{\bar{N}^3\bar{M}^2}(\partial_t\delta\mathcal{N})\delta M\right. \\
& + \frac{2\partial_r\bar{N}}{\bar{N}^4\bar{M}}(\partial_t\delta\mathcal{N})\delta N + \frac{1}{\bar{N}^2\bar{M}^2}(\partial_r\delta N)^2 - \frac{2\partial_r\bar{N}}{\bar{N}^2\bar{M}^3}(\partial_r\delta N)\delta M \\
& \left. - \frac{2\partial_r\bar{N}}{\bar{N}^3\bar{M}^2}(\partial_r\delta N)\delta N + \frac{\left(\partial_r\bar{N}\right)^2}{\bar{N}^2\bar{M}^4}(\delta M)^2 + \frac{2\left(\partial_r\bar{N}\right)^2}{\bar{N}^3\bar{M}^3}\delta N\delta M + \frac{\left(\partial_r\bar{N}\right)^2}{\bar{N}^4\bar{M}^2}(\delta N)^2\right] \\
& + \frac{\mathcal{F}_2}{\bar{M}^2}\left[\frac{1}{2}\bar{D}^a\bar{D}_a V\bar{D}^b\bar{D}_b V + \frac{2}{\bar{M}r}\delta M\bar{D}^a\bar{D}_a V + \frac{2}{\bar{M}^2 r^2}(\delta M)^2\right] \\
& + \frac{\mathcal{F}_{\mathcal{L}}}{\bar{N}\bar{M}}\left[-\frac{\partial_t\delta\mathcal{N}}{\bar{N}}\bar{D}^a\bar{D}_a V + \frac{\bar{D}^a\bar{D}_a V}{\bar{M}}\partial_r\delta N - \frac{\partial_r\bar{N}}{\bar{M}^2}\left(\bar{D}^a\bar{D}_a V\right)\delta M\right. \\
& \left. - \frac{\partial_r\bar{N}}{\bar{N}\bar{M}}\left(\bar{D}^a\bar{D}_a V\right)\delta N - \frac{2\delta M\partial_t\delta\mathcal{N}}{\bar{N}\bar{M}r} + \frac{2\delta M}{\bar{M}^2 r}\partial_r\delta N - \frac{2\partial_r\bar{N}}{\bar{M}^3 r}(\delta M)^2 - \frac{2\partial_r\bar{N}}{\bar{N}\bar{M}^2 r}\delta N\delta M\right]
\end{aligned}
$$

C. Gergely, Z. Keresztes, L. Á. Gergely, Phys. Rev. D 99, 104071 (2019) [arXiv:1905.00039 [gr-qc]],

…GKG-gauge: $B = P = \delta\phi = 0$

$$
\begin{aligned}
\delta_2 L^{EFT}_{\text{even}} = & \left[\frac{(\delta N)^2}{\bar{N}^2} + \frac{(\delta M)^2}{\bar{M}^2} + \frac{3}{\bar{M}\bar{N}}\delta M\delta N + \frac{6}{\bar{N}}\zeta\delta N + \frac{6}{\bar{M}}\zeta\delta M + 6\zeta^2\right]\bar{L}^{EFT} \\
& + \left[\frac{(\delta N)^2}{\bar{N}} + 2\zeta\delta N + \frac{\delta M\delta N}{\bar{M}}\right] L^{EFT}_N + \left[\frac{(\delta M)^2}{\bar{M}} + 2\zeta\delta M + \frac{\delta M\delta N}{\bar{N}}\right] L^{EFT}_M \\
& + \left[\frac{\partial_r\bar{N}}{\bar{M}^2\bar{N}}\left(\frac{(\delta M)^2}{\bar{M}} + 2\zeta\delta M - \frac{\bar{M}}{2\bar{N}^2}(\delta\mathcal{N})^2\right) + \left(\frac{\delta N}{\bar{N}^2\bar{M}} + \frac{\delta M}{\bar{N}\bar{M}^2}\right)\partial_r\delta N\right. \\
& + \frac{1}{\bar{N}^3}\delta\mathcal{N}\partial_t\delta N + \frac{1}{\bar{N}^2\bar{M}}\delta\mathcal{N}\partial_t\delta M + \frac{1}{\bar{N}^2\bar{M}}\delta\mathcal{N}\partial_r\delta\mathcal{N} \\
& \left. + \left(\frac{2\zeta\delta N}{\bar{N}\bar{M}} + \frac{(\delta N)^2}{\bar{N}^2\bar{M}} + \frac{\delta M\delta N}{\bar{M}^2\bar{N}}\right)\left(\frac{2}{r} + \frac{\partial_r\bar{N}}{\bar{N}} + \partial_r\right)\right] L^{EFT}_{\mathcal{L}} \\
& + \left[2\zeta\bar{D}^a\bar{D}_a V - \frac{2}{\bar{M}^2 r}\left(\frac{(\delta M)^2}{\bar{M}} + 2\zeta\delta M + \frac{\delta M\delta N}{\bar{N}}\right)\right. \\
& \left. + \frac{2\delta\mathcal{N}}{\bar{N}^2}\left(\partial_t\zeta - \frac{\delta\mathcal{N}}{\bar{M}r}\right) - \left(\frac{2\zeta\delta N}{\bar{N}\bar{M}} + \frac{2\zeta\delta M}{\bar{M}^2} + \frac{4\zeta^2}{\bar{M}}\right)\left(\frac{2}{r} + \frac{\partial_r\bar{N}}{\bar{N}} + \partial_r\right)\right]\mathcal{F} \\
& + \left[\frac{4}{r^2}\zeta^2 + 2\zeta\bar{D}_a\bar{D}^a\zeta - 2\left(\bar{D}_a\zeta\right)\left(\bar{D}^a\zeta\right) - \frac{4\zeta\delta M}{\bar{M}r^2} - \frac{4\zeta\delta N}{\bar{N}r^2} - \frac{8}{r^2}\zeta^2\right] L^{EFT}_R \\
& + \frac{L^{EFT}_{\mathfrak{k}}}{2\bar{M}\bar{N}^2}\left[\frac{1}{2\bar{M}}\left(\partial_t\bar{D}^a V\right)\left(\partial_t\bar{D}_a V\right) + \left(\partial_t\bar{D}^a V\right)\left(\bar{D}_a\delta\mathcal{N}\right) + \bar{M}\bar{D}^a\delta\mathcal{N}\bar{D}_a\delta\mathcal{N}\right] \\
& + \left[\frac{\delta\mathcal{N}}{\bar{N}^3}\partial_t(\delta\mathcal{N}) - \frac{\delta M}{\bar{N}\bar{M}^2}\partial_t(\delta M) - \frac{\delta N}{\bar{N}^2\bar{M}}\partial_t(\delta M)\right. \\
& \left. + \left(\frac{\delta N\delta\mathcal{N}}{\bar{N}^2\bar{M}} + \frac{\delta M\delta\mathcal{N}}{\bar{N}\bar{M}^2} + \frac{2\zeta\delta\mathcal{N}}{\bar{N}\bar{M}}\right)\left(\frac{2}{r} + \partial_r\right)\right] L^{EFT}_{\mathcal{K}^*} \\
& + \frac{1}{2}L^{EFT}_{NN}(\delta N)^2 + \frac{1}{2}L^{EFT}_{MM}(\delta M)^2 + L^{EFT}_{M\mathcal{N}}\delta M\delta\mathcal{N} + \frac{2L^{EFT}_{\varkappa^*}}{\bar{N}^2}(\partial_t\zeta)^2 \\
& + \frac{1}{2}L^{EFT}_{\mathcal{N}\mathcal{N}}(\delta\mathcal{N})^2 + L^{EFT}_{NM}\delta N\delta M + L^{EFT}_{N\mathcal{N}}\delta N\delta\mathcal{N} + \frac{2L^{EFT}_{NK^*}}{\bar{N}}\delta N\partial_t\zeta + \frac{2L^{EFT}_{MK^*}}{\bar{N}}\delta M\partial_t\zeta \\
& + \frac{2L^{EFT}_{\mathcal{N}K^*}}{\bar{N}}\delta\mathcal{N}\partial_t\zeta + \frac{2L^{EFT}_{\mathcal{K}^*K^*}}{\bar{N}^2\bar{M}}(\partial_t\zeta)(\partial_t\delta M) + \frac{2L^{EFT}_{K^*K^*}}{\bar{N}^2}(\partial_t\zeta)^2 \\
& + \frac{L^{EFT}_{\mathcal{N}\mathcal{K}^*}}{\bar{N}\bar{M}}\delta\mathcal{N}\partial_t\delta M + \frac{L^{EFT}_{M\mathcal{K}^*}}{\bar{N}\bar{M}}\delta M\partial_t\delta M + \frac{L^{EFT}_{\mathcal{K}^*\mathcal{K}^*}}{2\bar{N}^2\bar{M}^2}(\partial_t\delta M)^2 \\
& - 2L^{EFT}_{MR}\left[\frac{2}{r^2}\zeta\delta M + \delta M\bar{D}_a\bar{D}^a\zeta\right] - \frac{4L^{EFT}_{K^*R}}{\bar{N}}\left[\frac{2}{r^2}\zeta\partial_t\zeta + (\partial_t\zeta)\bar{D}_a\bar{D}^a\zeta\right] \\
& - 2L^{EFT}_{NR}\left[\frac{2}{r^2}\zeta\delta N + \delta N\bar{D}_a\bar{D}^a\zeta\right] - \frac{2L^{EFT}_{\mathcal{K}^*R}}{\bar{N}\bar{M}}\left[\frac{2}{r^2}\zeta\partial_t\delta M + (\partial_t\delta M)\bar{D}_a\bar{D}^a\zeta\right] \\
& - 2L^{EFT}_{\mathcal{N}R}\left[\frac{2}{r^2}\zeta\delta\mathcal{N} + \delta\mathcal{N}\bar{D}_a\bar{D}^a\zeta\right] + 2L^{EFT}_{RR}\left[\frac{4}{r^4}\zeta^2 + \frac{4}{r^2}\zeta\bar{D}_a\bar{D}^a\zeta + \left(\bar{D}_a\bar{D}^a\zeta\right)\left(\bar{D}_b\bar{D}^b\zeta\right)\right] \\
& + L^{EFT}_{N\mathcal{L}}\left[\frac{\delta N\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta N}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}\delta N\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}(\delta N)^2\right] \\
& + L^{EFT}_{M\mathcal{L}}\left[\frac{\delta M\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta M}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}(\delta M)^2 + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}\delta M\delta N\right] \\
& + L^{EFT}_{\mathcal{N}\mathcal{L}}\left[\frac{\delta\mathcal{N}\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\delta\mathcal{N}}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}\delta\mathcal{N}\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}\delta\mathcal{N}\delta N\right] \\
& + \frac{2L^{EFT}_{K^*\mathcal{L}}}{\bar{N}}\left[\frac{\partial_t\zeta\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\partial_t\zeta}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}(\partial_t\zeta)\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}(\partial_t\zeta)\delta N\right] \\
& + \frac{L^{EFT}_{\mathcal{K}^*\mathcal{L}}}{\bar{N}\bar{M}}\left[\frac{\partial_t\delta M\partial_t\delta\mathcal{N}}{\bar{N}^2} - \frac{\partial_t\delta M}{\bar{N}\bar{M}}\partial_r\delta N + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}^2}(\partial_t\delta M)\delta M + \frac{\partial_r\bar{N}}{\bar{N}^2\bar{M}}(\partial_t\delta M)\delta N\right] \\
& - 2L^{EFT}_{K^*}\left[\frac{\delta N\partial_t\zeta}{\bar{N}^2} + \frac{\delta N\delta\mathcal{N}}{\bar{N}^2\bar{M}r} + \frac{\delta M\delta\mathcal{N}}{\bar{N}\bar{M}^2 r} + \frac{2\zeta\delta\mathcal{N}}{\bar{N}\bar{M}r}\right] \\
& + \mathcal{F}_N\left[\frac{2}{\bar{M}}\delta N\partial_r\zeta - \frac{1}{\bar{M}}\delta N\bar{D}^a\bar{D}_a V - \frac{2}{\bar{M}^2 r}\delta N\delta M\right] \\
& + \mathcal{F}_M\left[\frac{2}{\bar{M}}\delta M\partial_r\zeta - \frac{1}{\bar{M}}\delta M\bar{D}^a\bar{D}_a V - \frac{2}{\bar{M}^2 r}(\delta M)^2\right] \\
& + \mathcal{F}_{\mathcal{N}}\left[\frac{2}{\bar{M}}\delta\mathcal{N}\partial_r\zeta - \frac{1}{\bar{M}}\delta\mathcal{N}\bar{D}^a\bar{D}_a V - \frac{2}{\bar{M}^2 r}\delta\mathcal{N}\delta M\right] \\
& + \frac{\mathcal{F}_{\mathcal{K}^*}}{\bar{N}\bar{M}}\left[\frac{2}{\bar{M}}\partial_r\zeta\partial_t\delta M - \frac{1}{\bar{M}}\partial_t\delta M\bar{D}^a\bar{D}_a V - \frac{2}{\bar{M}^2 r}\delta M\partial_t\delta M\right] \\
& + \frac{2\mathcal{F}_{K^*}}{\bar{N}}\left[\frac{2}{\bar{M}}\partial_r\zeta\partial_t\zeta - \frac{1}{\bar{M}}\partial_t\zeta\bar{D}^a\bar{D}_a V - \frac{2}{\bar{M}^2 r}\delta M\partial_t\zeta\right] \\
& - \frac{\mathcal{F}_{\mathcal{F}}}{\bar{M}}\left[\frac{2}{\bar{M}}\partial_r\zeta\delta M - \frac{1}{\bar{M}}\delta M\bar{D}^a\bar{D}_a V\right] \\
& + \frac{\mathcal{F}_2}{\bar{M}}\left[\frac{2}{\bar{M}}(\partial_r\zeta)^2 - \frac{2}{\bar{M}}\partial_r\zeta\bar{D}^a\bar{D}_a V - \frac{4}{\bar{M}^2 r}\delta M\partial_r\zeta\right. \\
& \left. + \frac{1}{2\bar{M}}\left(\bar{D}^a\bar{D}_a V\right)\left(\bar{D}^b\bar{D}_b V\right) + \frac{2}{\bar{M}^2 r}\delta M\bar{D}^a\bar{D}_a V + \frac{2}{\bar{M}^3 r^2}(\delta M)^2\right] \\
& + \frac{L^{EFT}_{\lambda}}{\bar{M}}\left[\frac{2}{\bar{M}}(\partial_r\zeta)^2 - \frac{2}{\bar{M}}\partial_r\zeta\bar{D}^a\bar{D}_a V + \frac{2(\delta M)^2}{\bar{M}^3 r^2} + \frac{1}{\bar{M}}\bar{D}_{(a}\bar{D}_{b)}V\bar{D}^{(a}\bar{D}^{b)}V\right] \\
& + \frac{\mathcal{F}_{\mathcal{L}}}{\bar{N}\bar{M}}\left[\frac{2}{\bar{N}}\partial_r\zeta\partial_t\delta\mathcal{N} - \frac{2}{\bar{M}}\partial_r\zeta\partial_r\delta N + \frac{2\partial_r\bar{N}}{\bar{M}^2}\delta M\partial_r\zeta + \frac{2\partial_r\bar{N}}{\bar{N}\bar{M}}\delta N\partial_r\zeta\right. \\
& - \frac{1}{\bar{N}}\partial_t\delta\mathcal{N}\bar{D}^a\bar{D}_a V + \frac{1}{\bar{M}}\partial_r\delta N\bar{D}^a\bar{D}_a V - \frac{\partial_r\bar{N}}{\bar{M}^2}\delta M\bar{D}^a\bar{D}_a V - \frac{\partial_r\bar{N}}{\bar{N}\bar{M}}\delta N\bar{D}^a\bar{D}_a V \\
& \left. - \frac{2}{\bar{N}\bar{M}r}\delta M\partial_t\delta\mathcal{N} + \frac{2}{\bar{M}^2 r}\delta M\partial_r\delta N - \frac{2\partial_r\bar{N}}{\bar{M}^3 r}(\delta M)^2 - \frac{2\partial_r\bar{N}}{\bar{N}\bar{M}^2 r}\delta N\delta M\right] \\
& + \frac{L^{EFT}_{\mathcal{L}\mathcal{L}}}{2}\left[\frac{1}{\bar{N}^4}(\partial_t\delta\mathcal{N})^2 + \frac{1}{\bar{N}^2\bar{M}^2}(\partial_r\delta N)^2 - \frac{2}{\bar{N}^3\bar{M}}\partial_t\delta\mathcal{N}\partial_r\delta N\right. \\
& + \frac{2\partial_r\bar{N}}{\bar{N}^4\bar{M}}\delta N\partial_t\delta\mathcal{N} + \frac{2\partial_r\bar{N}}{\bar{N}^3\bar{M}^2}\delta M\partial_t\delta\mathcal{N} - \frac{2\partial_r\bar{N}}{\bar{N}^2\bar{M}^3}\delta M\partial_r\delta N - \frac{2\partial_r\bar{N}}{\bar{N}^3\bar{M}^2}\delta N\partial_r\delta N \\
& \left. + \frac{\left(\partial_r\bar{N}\right)^2}{\bar{N}^2\bar{M}^4}(\delta M)^2 + \frac{2\left(\partial_r\bar{N}\right)^2}{\bar{N}^3\bar{M}^3}\delta N\delta M + \frac{\left(\partial_r\bar{N}\right)^2}{\bar{N}^4\bar{M}^2}(\delta N)^2\right] \\
& - \frac{2L^{EFT}_{\mathcal{L}R}}{\bar{N}}\left[\frac{2}{\bar{N}r^2}\zeta\partial_t\delta\mathcal{N} - \frac{2}{\bar{M}r^2}\zeta\partial_r\delta N + \frac{2\partial_r\bar{N}}{\bar{M}^2 r^2}\zeta\delta M + \frac{2\partial_r\bar{N}}{\bar{N}\bar{M}r^2}\zeta\delta N\right. \\
& \left. + \frac{1}{\bar{N}}\partial_t\delta\mathcal{N}\bar{D}_a\bar{D}^a\zeta - \frac{1}{\bar{M}}\partial_r\delta N\bar{D}_a\bar{D}^a\zeta + \frac{\partial_r\bar{N}}{\bar{M}^2}\delta M\bar{D}_a\bar{D}^a\zeta + \frac{\partial_r\bar{N}}{\bar{N}\bar{M}}\delta N\bar{D}_a\bar{D}^a\zeta\right] \\
& - 2\mathcal{F}_R\left[\frac{4}{\bar{M}r^2}\zeta\partial_r\zeta - \frac{2}{\bar{M}r^2}\zeta\bar{D}^a\bar{D}_a V - \frac{4}{\bar{M}^2 r^3}\zeta\delta M + \frac{2}{\bar{M}}\partial_r\zeta\bar{D}_a\bar{D}^a\zeta\right. \\
& \left. - \frac{1}{\bar{M}}\left(\bar{D}^a\bar{D}_a\zeta\right)\left(\bar{D}^b\bar{D}_b V\right) - \frac{2}{\bar{M}^2 r}\delta M\bar{D}_a\bar{D}^a\zeta\right]
\end{aligned}
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Thank you for your attention!