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Non-unitary quantum dynamics induced by deformed spacetime symmetries

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Based on arXiv:2207.xxxxx with G. Gubitosi and M. Arzano

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Phenomenology: Neutron interferometry and neutral kaons [*J. Ellis et al,Nucl.Phys.B 241, 381 (1984)*], cosmic neutrinos [*J. Christian, Phys.Rev.Lett. 95 (2005)*], optomechanical cavities [*C. Pfister, Nature Communications 7, 13022 (2016)*].

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Deformed symmetries are described with Hopf algebras

Non-linear algebraic sector

$$[X_i, X_j] = f(\boldsymbol{X})$$

Coalgebra becomes relevant

$$\Delta : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H} \text{ (coproduct)}$$

$$S : \mathbb{H} \to \mathbb{H} \text{ (antipode)}$$

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κ-Poincaré algebra in classical basis is undeformed while the relevant coalgebra structures are

$$\begin{aligned} \Delta P_0 &= P_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{\kappa} P_n \Pi_0^{-1} \otimes P^n \ , \ \Delta P_n = P_n \otimes \mathbb{1} + \mathbb{1} \otimes P_n \\ S(P_0) &= -P_0 + \frac{1}{\kappa} \boldsymbol{P}^2 \Pi_0^{-1} \ , \ S(P_i) = -P_i \Pi_0^{-1} \\ \Pi_0 &= \frac{1}{\kappa} P_0 + \sqrt{\mathbb{1} - \frac{1}{\kappa^2} P^\mu P_\mu} \ , \ \Pi_0^{-1} = \frac{-\frac{1}{\kappa} P_0 + \sqrt{\mathbb{1} - \frac{1}{\kappa^2} P^\mu P_\mu}}{\mathbb{1} - \frac{1}{\kappa^2} P^\mu P_\mu} \end{aligned} \tag{1}$$

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The contraction [*A. Ballesteros et al, Phys.Lett.B 805, 135461 (2020)*] is carried out with

$$\boldsymbol{N} \mapsto c^{-1} \boldsymbol{N} \;, \; \boldsymbol{P} \mapsto c^{-1} \boldsymbol{P} \;, \; \kappa \mapsto c^{-2} \kappa \tag{2}$$

with $c \to \infty$.

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with $c \to \infty$.
The algebra becomes the standard Galilei algebra. Non trivial structures on the coalgebra are

$$\Delta P_0 = P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0 + \frac{1}{\kappa} P_n \otimes P^n \ , \ S(P_0) = -P_0 + \frac{1}{\kappa} \boldsymbol{P}^2 \quad (3)$$

Adjoint Action

In standard QM evolution of density operators ρ is given by the adjoint action of time translations generator.

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If coalgebra structures are trivial

$$\mathrm{ad}_A(B) = (id \otimes S)\underbrace{(A \otimes \mathbb{1} + \mathbb{1} \otimes A)}_{\Delta A} \diamond B = (A \otimes \mathbb{1} - \underbrace{\mathbb{1} \otimes A}_{S(A)=-A}) \diamond B = [A,B] \tag{5}$$

Density operator time evolution

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These requirements are met with

$$\boxed{i\,\partial_t \rho = \frac{1}{2}\left\{ \mathrm{ad}_{P_0}(\rho) - \left[\mathrm{ad}_{P_0}(\rho)\right]^\dagger \right\}} \tag{6}$$

leading to

$$\partial_t \rho = -i\,[P_0, \rho] - \frac{1}{2\,\kappa}\left(\rho\,\boldsymbol{P}^2 + \boldsymbol{P}^2\rho - 2\,P_n \rho\, P^n\right) \tag{7}$$

Decoherence time and linear entropy

Free particle solution in momentum basis is

$$\langle \boldsymbol{p}|\rho|\boldsymbol{q}\rangle := \rho_{pq}(t) = \rho_{pq}(0)\exp\left\{-it[E(p)-E(q)] - \frac{t}{2\kappa}(\boldsymbol{p}-\boldsymbol{q})^2\right\} \tag{8}$$

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$$\partial_t S(t) = \partial_t\left(1-\mathrm{tr}\left\{\rho^2\right\}\right) = -2\,\mathrm{tr}\{\rho\,\partial_t\rho\} = \frac{1}{\kappa}\sum_n \mathrm{tr}\left\{O_n\,O_n^\dagger\right\} \geq 0\ ,\ \ O_n := [\rho, P_n] \tag{10}$$

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Using (8) we get

$$S(t) = 1 - \int \mathrm{d}^3p\,\mathrm{d}^3q\,\left|\rho_{pq}(0)\right|^2 e^{-\frac{t}{\kappa}(\boldsymbol{p}-\boldsymbol{q})^2} \overset{t\to\infty}{\sim} 1 - \left(\frac{\pi\kappa}{t}\right)^{\frac{3}{2}}[1-S(0)] \tag{11}$$

Constraints on mass and localization

"Lifetime" of a state of a quantum system [*J. Hilgevoord APJ 64, 1451 (1996)*]

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$$\frac{(\delta \boldsymbol{p})^2}{\delta E} \lesssim 2\,\kappa \tag{13}$$

Heisenberg uncertainty: $\delta p \gtrsim (2\,\delta x)^{-1}$ thus a bound on the localization is obtained

$$\boxed{(\delta \boldsymbol{x})^2 \gtrsim \frac{1}{8\,\kappa\,\delta E}} \tag{14}$$

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With $E = (2m)^{-1}\boldsymbol{p}^2$, from (13) and for momentum localized states, $\frac{\delta p}{p} \ll 1$

$$\boxed{m \lesssim \kappa} \tag{15}$$

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- This equation describes a decoherence process in the momentum basis; purity is not conserved.
- From the decoherence time, fundamental constraints on the localization and the mass of quantum systems arise.
- With a bottom-up perspective, can this constraints give bounds on the value of κ via table-top experiments?
- Can this framework be extended to field theories? Primordial perturbations decoherence...

Thank you!

Interaction with the Environment

von Neumann measurement:

$$|o_n\rangle \otimes |R\rangle \mapsto |o_n\rangle \otimes |a_n\rangle \Rightarrow |\psi\rangle \otimes |R\rangle \mapsto \sum_n c_n |o_n\rangle \otimes |a_n\rangle \tag{16}$$

$|a_n\rangle \in \mathscr{H}_A$, $|o_n\rangle \in \mathscr{H}_S$.
Two-level system: $|\psi_i\rangle \otimes |E_0\rangle \mapsto |\psi_i\rangle \otimes |E_i\rangle$, $i = 1,2$. Environment-system entanglement emerges dynamically

$$|\psi\rangle \otimes |E_0\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle + |\psi_2\rangle\Big) \otimes |E_0\rangle \mapsto \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle \otimes |E_1\rangle + |\psi_2\rangle \otimes |E_2\rangle\Big) \tag{17}$$

The density operator of the system is

$$\rho_S = \mathrm{tr}_E\{\rho\} = \frac{1}{2}\Big(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_2|\langle E_1|E_2\rangle + |\psi_2\rangle\langle\psi_1|\langle E_2|E_1\rangle\Big) \tag{18}$$

For macroscopic systems $|E_i\rangle = \prod_{\alpha=1}^N \left|e_\alpha^{(i)}\right\rangle$, $\left\langle e_\alpha^{(1)} \middle| e_\beta^{(2)}\right\rangle = \varepsilon \lesssim 1$, thus $\boxed{\langle E_1|E_2\rangle \sim \varepsilon^N \sim 0}$

Example of system-environment interaction

Two-level system $\{\,|0\rangle\,,|1\rangle\,\}$; environment composed of N two-level systems $\{\,|\uparrow\rangle_i\,,|\downarrow\rangle_i\,\}$.

$$\hat{H}_{\text{int}} = \frac{1}{2}\hat{\sigma}_z \otimes \hat{E} \;\;,\;\; \hat{E} := \sum_{i=1}^{N} g_i\,\hat{\sigma}_z^{(i)} \tag{19}$$

$2^N - 1$ energy levels for E given by $|n\rangle = |\uparrow\rangle_i\,|\downarrow\rangle_i\ldots|\uparrow\rangle_i$ with $\varepsilon_n = \sum_{i=1}^{N}(-1)^{n_i} g_i$.

$$e^{-it\hat{H}_{\text{int}}}\,|\psi\rangle = e^{-it\hat{H}_{\text{int}}}\Big(a\,|0\rangle + b\,|1\rangle\Big) \otimes \sum_{i=1}^{2^N-1} c_n\,|n\rangle = a\,|0\rangle\,|\varepsilon_0(t)\rangle + b\,|1\rangle\,|\varepsilon_1(t)\rangle \tag{20}$$

with $|\varepsilon_0(t)\rangle = |\varepsilon_1(-t)\rangle = \sum_{i=1}^{2^N-1} c_n e^{-it\,\frac{\varepsilon_n}{2}}$.
The decoherence parameter

$$r(t) = \langle\varepsilon_1(t)|\varepsilon_0(t)\rangle = \sum_n |c_n|^2 e^{-i\varepsilon_n t} \;\Rightarrow\; \boxed{\langle|r(t)|\rangle \sim 2^{-N}} \tag{21}$$

Recurrence time τ_{rec} exists since N is always finite. $g_i = g\;\forall\, i$ and $\sum_{i=1}^{2^N-1} c_n\,|n\rangle = \bigotimes_{i=1}^{N} \frac{1}{\sqrt{2}}\Big(|\downarrow\rangle_i + |\uparrow\rangle_i\Big)$ give $r(t) = \big[\cos(g\,t)\big]^N$ with $\tau_{\text{rec}} = \frac{\pi}{g}$. Highly improbable, typically $\tau_{\text{rec}} \propto N!$

Environmental superselection

Interaction with the environment selects the preferred basis.

$$|\psi_\pm\rangle \otimes |E_0\rangle \mapsto \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle \otimes |E_1\rangle \pm |\psi_2\rangle \otimes |E_2\rangle\Big) \tag{22}$$

$|\psi_\pm\rangle$ get entangled with the environment.
Superselected states are the ones that get least entangled with the environment ($|\psi_i\rangle$).
Typically $\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{\text{int}}$.
Quantum measurement limit: $\hat{H} \approx \hat{H}_{\text{int}}$. Typically $\hat{H}_{\text{int}} = \hat{S} \otimes \hat{E}$, thus eigenstates of $\hat{S}$ get selected

$$e^{-it\hat{S}\otimes\hat{E}}\,|s_i\rangle \otimes |E_0\rangle = |s_i\rangle \otimes e^{-it\lambda_i \hat{E}}\,|E_0\rangle := |s_i\rangle \otimes |E_i(t)\rangle \tag{23}$$

Quantum limit: $\hat{H} \approx \hat{H}_S$. Constants of motion of S (energy) get selected.

Lindblad equation

In general $\hat{\rho}_S(t) = \mathrm{tr}_E\left\{\hat{U}(t)\hat{\rho}_{S,E}(0)\hat{U}^\dagger(t)\right\}$. Master equations give approximations

$$\partial_t\hat{\rho}_S(t) = -i\left[\hat{H}_S,\hat{\rho}_S\right] + \hat{D}\left[\hat{\rho}_S(t)\right] \quad (24)$$

Lindblad equation: most general master equation preserving the positivity of $\hat{\rho}_S$.

$$\boxed{\partial_t\hat{\rho} = -i\left[\hat{H}_S,\hat{\rho}_S\right] - \frac{1}{2}g^{mn}\left(\hat{Q}_m\hat{Q}_n\,\hat{\rho} + \hat{\rho}\,\hat{Q}_m\hat{Q}_n - 2\,\hat{Q}_m\,\hat{\rho}\,\hat{Q}_n\right)} \quad (25)$$

with $g^{mn}\in\mathbb{R}$ and $u_m\,u_n\,g^{mn}\geq 0\ ,\ \forall u$.

Details of the contraction

The contraction

$$\boldsymbol{N} \mapsto c^{-1}\boldsymbol{N} \;,\; \boldsymbol{P} \mapsto c^{-1}\boldsymbol{P} \;,\; \kappa \mapsto c^{-2}\kappa \tag{26}$$

gives

$$\Pi_0 = \frac{1}{c^2\kappa}P_0 + \sqrt{\mathbb{1} - \frac{1}{c^4\kappa^2}\left(P_0^2 - c^2\boldsymbol{P}^2\right)} \to \mathbb{1} \;,\; \Pi_0^{-1} = \frac{-\frac{1}{c^2\kappa}P_0 + \sqrt{\mathbb{1} - \frac{1}{c^4\kappa^2}\left(P_0^2 - c^2\boldsymbol{P}^2\right)}}{\mathbb{1} - \frac{1}{c^4\kappa^2}\left(P_0^2 - c^2\boldsymbol{P}^2\right)} \to \mathbb{1} \tag{27}$$

thus

$$\begin{aligned}\Delta P_0 &= P_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{c^2\kappa} cP_n\Pi_0^{-1} \otimes cP^n \to P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0 + \frac{1}{\kappa}P_n \otimes P^n \\ S(P_0) &= -P_0 + \frac{1}{c^2\kappa}c^2\boldsymbol{P}^2\Pi_0^{-1} \to -P_0 + \frac{1}{\kappa}\boldsymbol{P}^2 \;,\quad cS(P_i) = -cP_i\Pi_0^{-1} \Rightarrow S(P_i) = -P_i\end{aligned} \tag{28}$$

Representation on Hilbert Space

To study the effect of these deformations we need to represent the algebra on the algebra of operators of quantum systems $\mathscr{A}_{\mathscr{H}}$.
This is done with the standard $G \mapsto -iR(G)$ procedure.
For the coalgebra this means

$$\Delta P_0 = P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0 - \frac{i}{\kappa} P_n \otimes P^n \ , \ S(P_0) = -P_0 - \frac{i}{\kappa} \boldsymbol{P}^2 \qquad (29)$$