



κ -Braided non-commutative field theory

COST CA18108 Second Training School in Belgrade

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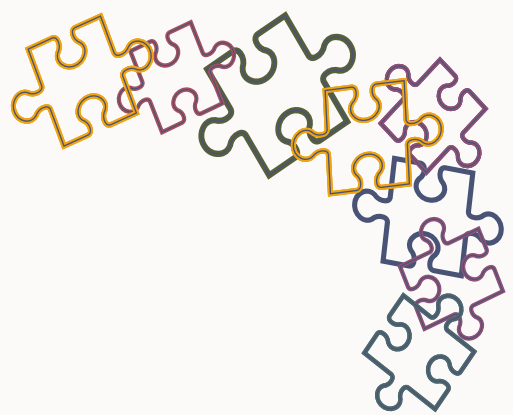
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Burgos

[Lizzi, Mercati, PRD 103 (2021)]

κ - Minkowski



[Celeghini, Giachetti, Sorace, Tarlini, JMP 31 (1990)]

[Lukierski, Nowicki, Ruegg, PLB 264 (1991), 293 (1992)]

[Majid, Ruegg, PLB 334 (1994)] [Zakrzewski, JPA 27 (1993)]

$$[x^\mu, x^\nu] = \frac{i}{\kappa} (v^\mu x^\nu - v^\nu x^\mu), \quad \mu = 0, \dots, d \quad v^\mu \in \mathbb{R}^{d+1},$$

$$(x^\mu)^\dagger = x^\mu, \quad [\kappa] = \ell^{-1} = (\text{length})^{-1} \text{ scale},$$

$x^\mu \in A$ = coordinate algebra = 'non-commutative functions' = scalar fields

κ - Poincaré

Poincaré group described through $C[ISO(3,1)]$

$$\begin{aligned}\Delta[\Lambda^\mu_\nu] &= \Lambda^\mu_\alpha \otimes \Lambda^\alpha_\nu, & \Delta[a^\mu] &= \Lambda^\mu_\nu \otimes a^\nu + a^\mu \otimes 1, \\ S[\Lambda^\mu_\nu] &= (\Lambda^{-1})^\mu_\nu, & S[a^\mu] &= -(\Lambda^{-1})^\mu_\nu a^\nu, & \epsilon[\Lambda^\mu_\nu] &= \delta^\mu_\nu, & \epsilon[a^\mu] &= 0\end{aligned}$$

Non-commutative deformation of $C[ISO(3,1)] \dashrightarrow C_\kappa[ISO(3,1)]$

$$[\Lambda^\mu_\nu, a^\gamma] = \frac{i}{\kappa} [(\Lambda^\mu_\alpha v^\alpha - v^\mu) \Lambda^\gamma_\nu + (\Lambda^\alpha_\nu g_{\alpha\beta} - g_{\nu\beta}) v^\beta g^{\mu\gamma}]$$

$$[\Lambda^\mu_\nu, \Lambda^\alpha_\beta] = 0, \quad [a^\mu, a^\nu] = \frac{i}{\kappa} (v^\mu a^\nu - v^\nu a^\mu),$$

$$\Lambda^\mu_\alpha \Lambda^\nu_\beta g^{\alpha\beta} = g^{\mu\nu}, \quad \Lambda^\mu_\alpha \Lambda^\nu_\beta g_{\mu\nu} = g_{\alpha\beta}$$

Hopf algebra $\forall g_{\mu\nu}: \quad g_{\mu\nu} g^{\nu\gamma} = \delta^\gamma_\mu$

Different models $\leftrightarrow$ $v^\mu v^\nu g_{\mu\nu}$

Relativity principle

Non-commutative Poincaré transformation

Left co-action $\overline{(\cdot)} : A \rightarrow C_\kappa[\text{ISO}(3,1)] \otimes A$

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \quad [x^\mu, \Lambda^\rho_\sigma] = [x^\mu, a^\nu] = 0$$

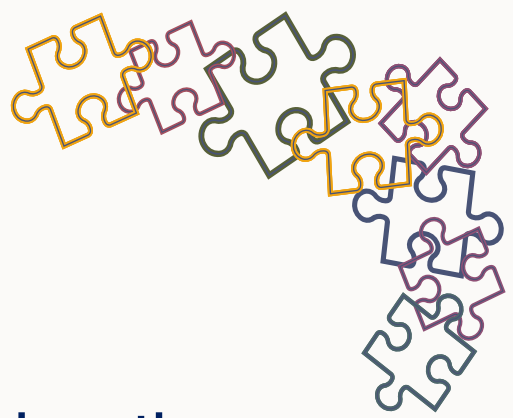
Given the $C_\kappa[\text{ISO}(3,1)]$ algebra

$$[\bar{x}^\mu, \bar{x}^\nu] = \frac{i}{\kappa} (v^\mu \bar{x}^\nu - v^\nu \bar{x}^\mu) = \overline{[x^\mu, x^\nu]}$$

the commutation relations are the same to all observers.



Field theory on κ -Minkowski



Multilocal functions

QFT requires the concept of N-point functions. In the commutative case, given the Abelian algebra of coordinates F , two-point functions are elements of $F \otimes F$, generated by

$$x_1^\mu = x^\mu \otimes 1, \quad x_2^\mu = 1 \otimes x^\mu, \quad 1 = 1 \otimes 1$$

Starting from the non-abelian algebra A , the canonical algebra structure on the tensor product $A \otimes A$ is

$$[x_1^\mu, x_1^\nu] = \frac{i}{\kappa} (v^\mu x_1^\nu - v^\nu x_1^\mu), \quad [x_2^\mu, x_2^\nu] = \frac{i}{\kappa} (v^\mu x_2^\nu - v^\nu x_2^\mu),$$

$$[x_1^\mu, 1] = [x_2^\mu, 1] = 0, \quad [x_1^\mu, x_2^\nu] = 0 \longrightarrow \text{Not covariant!}$$



Braided tensor product algebra



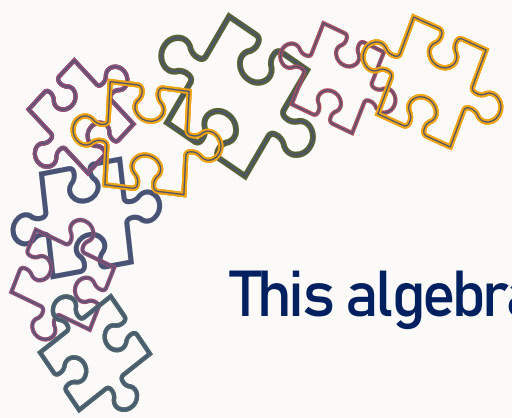
We look for a deformation $A \otimes_{\kappa} A$ of the tensor product algebra, such that:

$$[x_1^{\mu}, x_1^{\nu}] = \frac{i}{\kappa} (v^{\mu} x_1^{\nu} - v^{\nu} x_1^{\mu}), \quad [x_2^{\mu}, x_2^{\nu}] = \frac{i}{\kappa} (v^{\mu} x_2^{\nu} - v^{\nu} x_2^{\mu}),$$

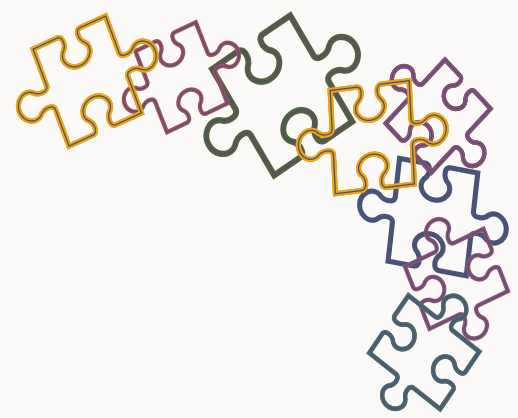
$$[x_1^{\mu}, 1] = [x_2^{\mu}, 1] = 0, \quad [\bar{x}_1^{\mu}, \bar{x}_2^{\nu}] = \overline{[x_1^{\mu}, x_2^{\nu}]}$$

Assuming that $[x_1^{\mu}, x_2^{\nu}]$ goes to zero as $v^{\mu} \rightarrow 0$ and that it is linear in x_a^{μ} , we find a unique solution:

$$[x_1^{\mu}, x_2^{\nu}] = \frac{i}{\kappa} [v^{\mu} x_1^{\nu} - v^{\nu} x_2^{\mu} - g^{\mu\nu} g_{\rho\sigma} v^{\rho} (x_1^{\sigma} - x_2^{\sigma})]$$



Braided tensor product algebra



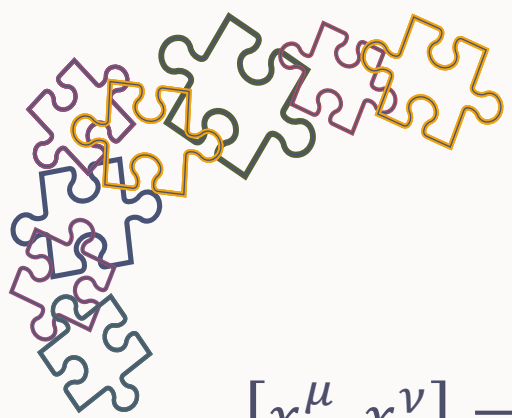
This algebra can be immediately extended to N points, $A^{\otimes_{\kappa} N}$:

$$[x_a^{\mu}, x_b^{\nu}] = \frac{i}{\kappa} [v^{\mu} x_a^{\nu} - v^{\nu} x_b^{\mu} - g^{\mu\nu} g_{\rho\sigma} v^{\rho} (x_a^{\sigma} - x_b^{\sigma})], \quad a, b = 1, \dots, N$$

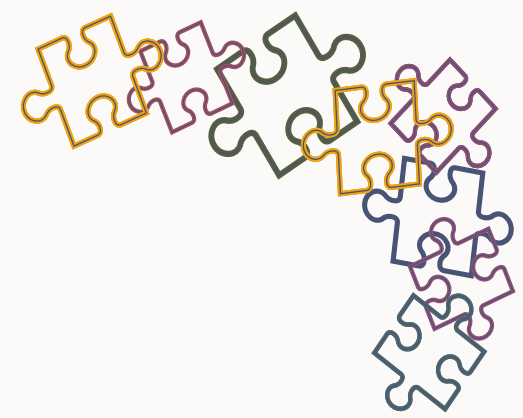
If we impose the Jacobi rule, we find

$$\begin{aligned} & [x_a^{\mu}, [x_b^{\nu}, x_c^{\rho}]] + [x_b^{\nu}, [x_c^{\rho}, x_a^{\mu}]] + [x_c^{\rho}, [x_a^{\mu}, x_b^{\nu}]] = \\ & = -\frac{g_{\alpha\beta} v^{\alpha} v^{\beta}}{\kappa^2} \left(g^{\nu\rho} (x_c^{\mu} - x_b^{\mu}) + g^{\rho\mu} (x_a^{\nu} - x_c^{\nu}) + g^{\mu\nu} (x_b^{\rho} - x_a^{\rho}) \right) = 0 \end{aligned}$$

and this is satisfied only if $g_{\alpha\beta} v^{\alpha} v^{\beta} = 0$, i.e. the so-called *lightlike*- κ -Minkowski non-commutativity.



$A^{\otimes N}_\kappa$ is a mostly-commutative algebra



$$[x_a^\mu, x_b^\nu] = \frac{i}{\kappa} [v^\mu x_a^\nu - v^\nu x_b^\mu - g^{\mu\nu} g_{\rho\sigma} v^\rho (x_a^\sigma - x_b^\sigma)]$$

The coordinate differences $\delta x_{ab}^\mu = x_a^\mu - x_b^\mu$ close an Abelian subalgebra:

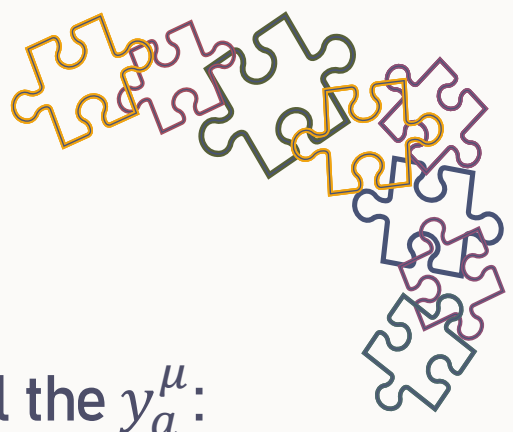
$$[\delta x_{ab}^\mu, \delta x_{cd}^\nu] = 0$$

All the noncommutativity is concentrated on the center-of-mass degrees of freedom

$$x_{cm}^\mu = \frac{1}{N} \sum_a x_a^\mu, \quad [x_{cm}^\mu, x_{cm}^\nu] = \frac{i}{\kappa} (v^\mu x_{cm}^\nu - v^\nu x_{cm}^\mu)$$
$$y_a^\mu = x_a^\mu - x_{cm}^\mu, \quad [x_{cm}^\mu, y_a^\nu] = \frac{i}{\kappa} (g^{\mu\nu} g_{\rho\sigma} v^\rho y_a^\sigma - v^\nu y_a^\mu), \quad [y_a^\mu, y_b^\nu] = 0$$



A representation of $A^{\otimes \kappa}$

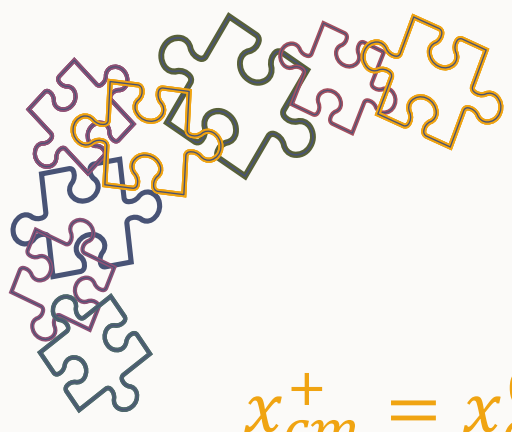


The component of x_{cm}^μ along v^μ , $x_{cm}^- = g_{\mu\nu} v^\mu x_{cm}^\nu$, commutes with all the y_a^μ :

$$[x_{cm}^-, y_a^\mu] = 0, \quad [x_{cm}^\mu, x_{cm}^-] = \frac{i}{\kappa} v^\mu x_{cm}^-,$$

but the other components are irreducibly noncommutative: the maximal Abelian subalgebra is generated by x_{cm}^- and y_a^μ (with the constraint $\Sigma_a y_a^\mu = 0$).

In 3+1 dimensions, choosing w.l.o.g. $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, $v^\mu = (1, 1, 0, 0)$, $x_{cm}^- = x_{cm}^0 - x_{cm}^1$ and y_a^μ are multiplicative operators with real spectrum and



A representation of $A \otimes_k^N$



$x_{cm}^+ = x_{cm}^0 + x_{cm}^1$, x_{cm}^2 , x_{cm}^3 can be represented as sums of Lorentz matrices and a x_{cm}^- -dilatation:

$$x_{cm}^2 = M^{12} - M^{02}, \quad x_{cm}^3 = M^{13} - M^{03},$$

$$x_{cm}^+ = 2M^{10} + 2ix_{cm}^- \frac{\partial}{\partial x_{cm}^-} + i,$$

$$M^{\mu\nu} = i \sum_{a=1}^{N-1} y_a^\mu g^{\nu\rho} \frac{\partial}{\partial y_a^\rho} - y_a^\nu g^{\mu\rho} \frac{\partial}{\partial y_a^\rho} \quad \text{Generators of rigid Lorentz transformations of } N \text{ points}$$



κ - Poincaré invariant field theory



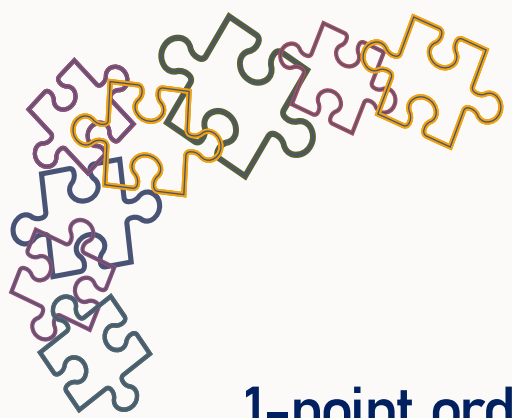
QFT requires the concept of N-point functions, i.e. , in the commutative case, Poincaré-invariant distributions which admit a Fourier representation.

In the non-commutative case, we can choose an ordering prescription for the x_a^μ and try to write a function of N points as

$$f(x_a^\mu) = \int d^4k^1 \dots d^4k^N \tilde{f}(k_\mu^a) e^{ik_\mu^1 x_1^\mu} \dots e^{ik_\mu^N x_N^\mu}$$

and we can prove that κ -Poincaré invariance implies that $f(x_a^\mu)$ only depends upon y_a^μ .

N-point functions are commutative!



1+1D κ - deformed field theory



1-point ordered plane waves $e_a[k_\mu] = e^{ik_-x_a^-} e^{ik_+x_a^+}$, $k_\pm \in \mathbb{R}$, form a Lie group under noncommutative multiplication.

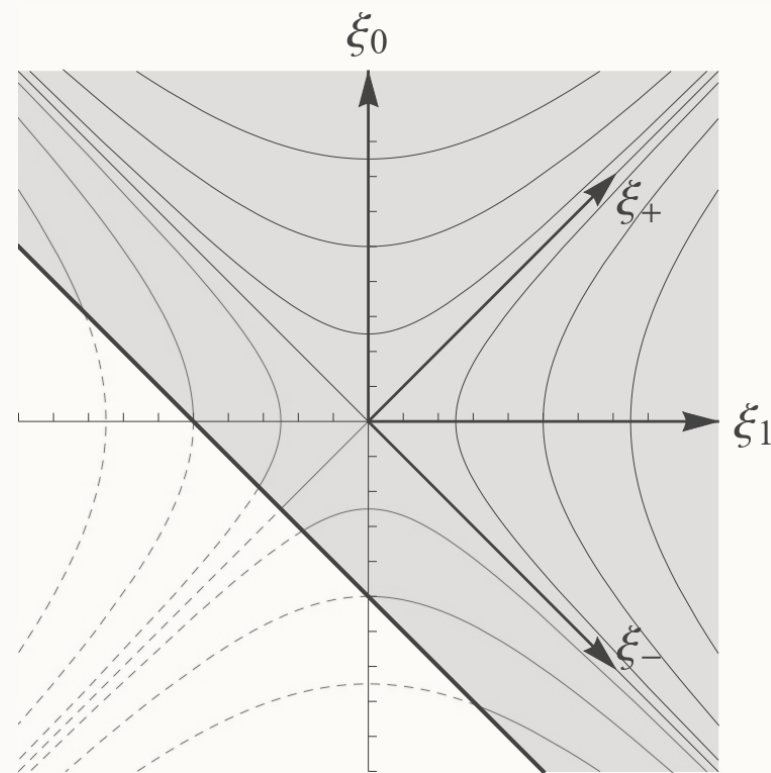
The group manifold is a **half** Minkowski space

└ Lorentz breaking?

Nonlinear action of the κ -Poincaré group on momentum space:

$$\bar{e}_a[k] = e^{ik_- \bar{x}_a^-} e^{ik_+ \bar{x}_a^+} = e_a[\lambda(k, \omega)] e^{ik_- a^-} e^{ik_+ a^+}$$

$$\lambda(k, \omega) = \left(e^{-\omega} k_-, \frac{1}{2} \ln \left[1 + e^\omega \left(e^{\frac{2k_+}{\kappa}} - 1 \right) \right] \right)$$



The missing jigsaw pieces

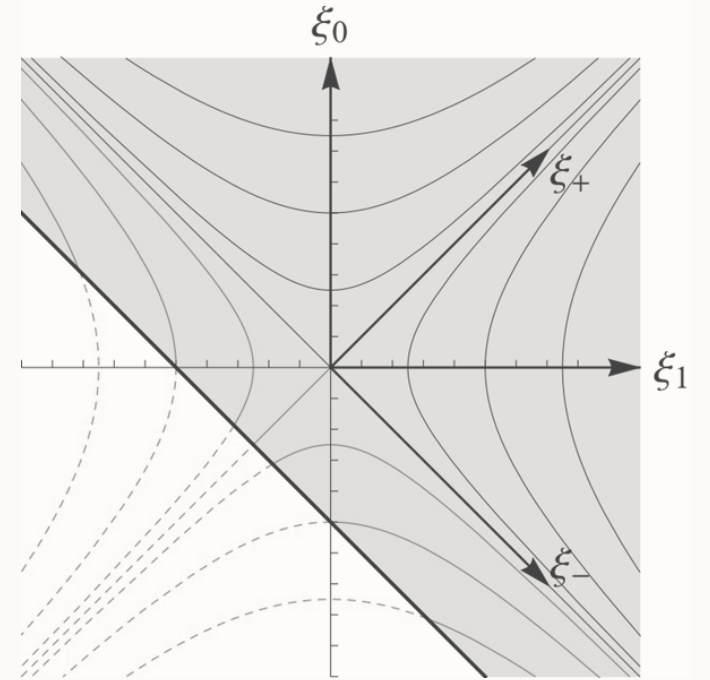
For $k_+ < 0$, $e^\omega > \frac{1}{1 - e^{\frac{2k_+}{\kappa}}}$,

$$\lambda(k, \omega)_+ = \frac{1}{2} \ln \left[e^\omega \left(1 - e^{\frac{2k_+}{\kappa}} \right) - 1 \right] \boxed{+ \frac{i\pi}{2} + n\pi i}$$

[Arzano, Bevilacqua, Kowalski-Glikman, Rosati, Unger,
PRD 103 (2021)]

$$e_a[k] \rightarrow \epsilon_a[p] \boxed{e^{-n\pi x_a^+}} = e^{ip_- x_a^-} e^{ip_+ x_a^+} e^{-\frac{\pi}{2} x_a^+} e^{-n\pi x_a^+}$$

$\downarrow$
 $= (-1)^n$ in our representation of $A^{\otimes N_\kappa}$: always appears to an even power in N-point functions

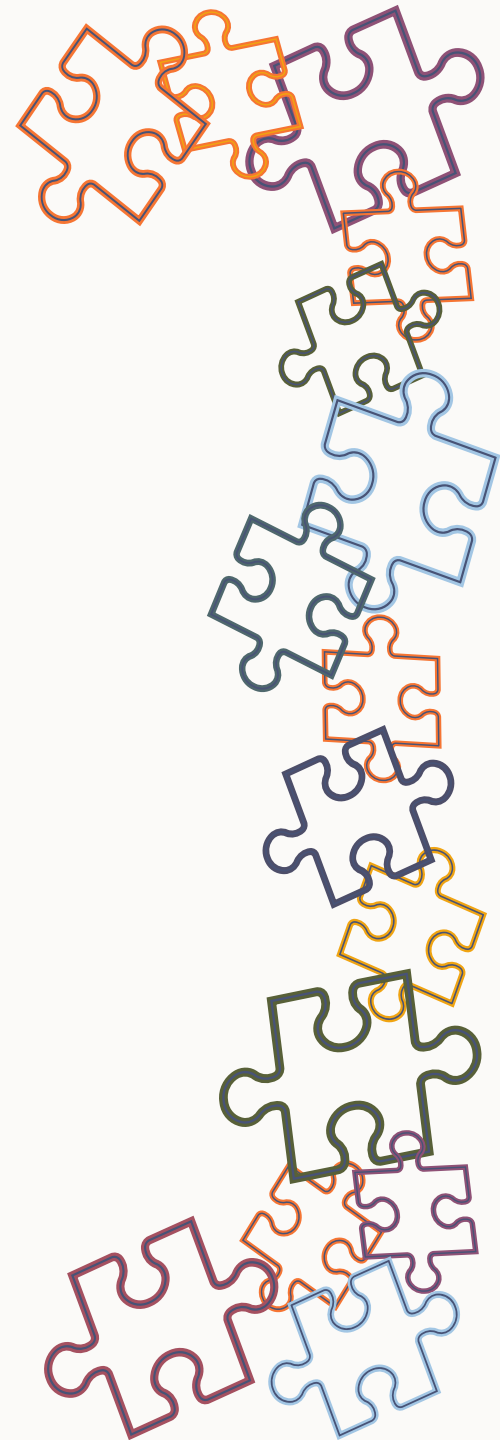
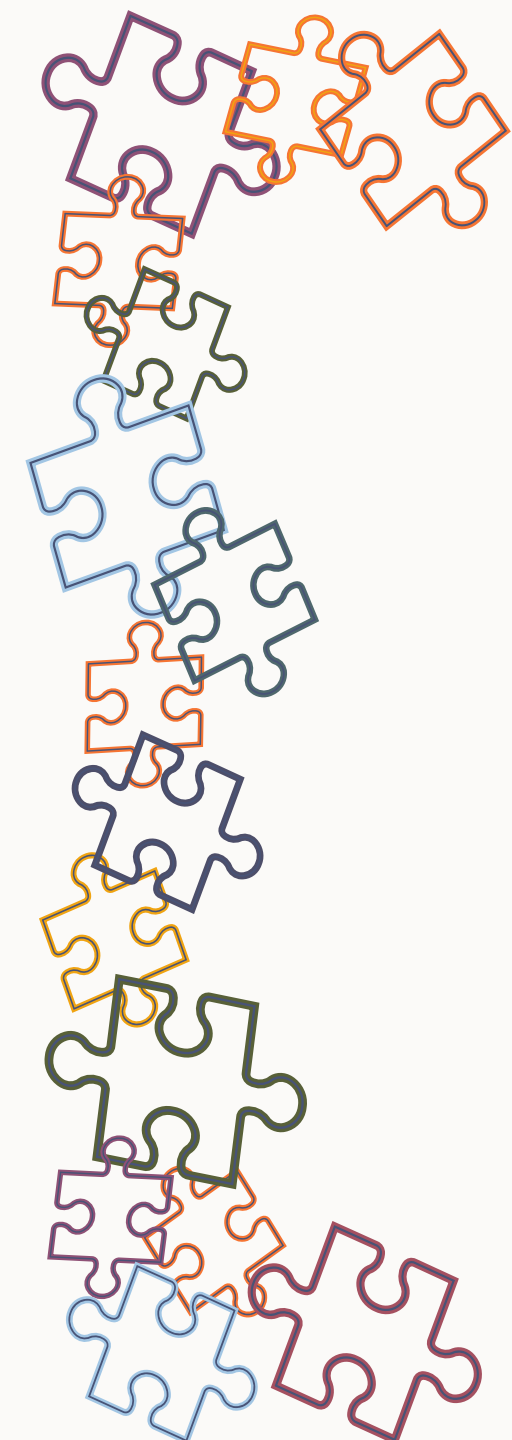


Conclusions

$$\phi(x_a) = \int_{\mathbb{R}_+} dk_+ \frac{e^{2k_+}}{e^{2k_+} - 1} \left(a(k_+) e_a(k_+) + e^{-2k_+} b^*(k_+) e_a^\dagger(k_+) \right) + \\ \int_{\mathbb{R}_+} dk_+ \frac{e^{2k_+}}{e^{2k_+} + 1} \left(\alpha(k_+) \epsilon_a(k_+) + e^{-2k_+} \beta^*(k_+) \epsilon_a^\dagger(k_+) \right)$$

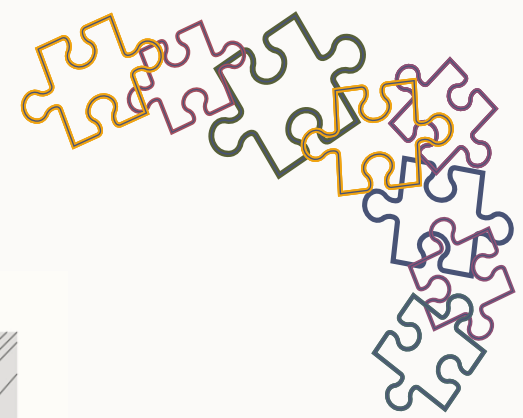
- All 2-point functions are identical to the commutative ones;
- $[\hat{\phi}(x_1), \hat{\phi}^\dagger(x_2)] = i\Delta_{PJ}(x_1 - x_2)$, $[\hat{\phi}(x_1), \hat{\phi}(x_2)] = [\hat{\phi}^\dagger(x_1), \hat{\phi}^\dagger(x_2)] = 0$;
- consistent and κ -Poincaré invariant deformed h.o. algebra.
- Future perspectives: N-point functions and interacting fields.

Thank you





Geometry of momentum space



Left-invariant metric:

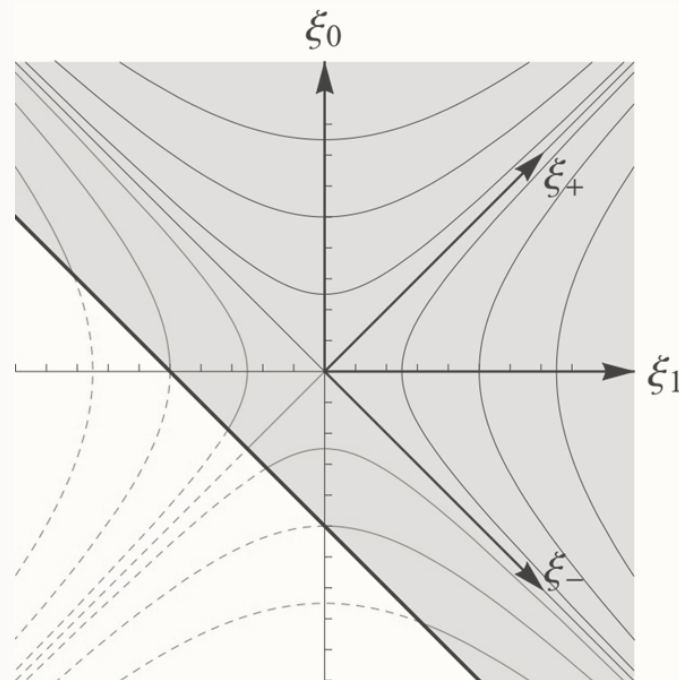
$$ds^2 = d\xi_- d\xi_+,$$

$$\xi = \left(k_-, \frac{e^{2k_+} - 1}{2} \right) \rightarrow \xi_+ > -\frac{1}{2}$$

Right-invariant metric:

$$ds^2 = d\chi_- d\chi_+,$$

$$\chi = \left(-e^{2k_+} k_-, \frac{e^{-2k_+} - 1}{2} \right) \rightarrow \chi_+ > -\frac{1}{2}$$



κ - Lorentz transformation

$$\xi_{\pm} = e^{\pm\omega} \xi_{\pm}$$

$$\chi_{\pm} = e^{\mp\omega} \chi_{\pm}$$