Perturbation theory and the large-charge expansion

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[*"Quantum gravity phenomenology in the multi-messenger approach*"](#)

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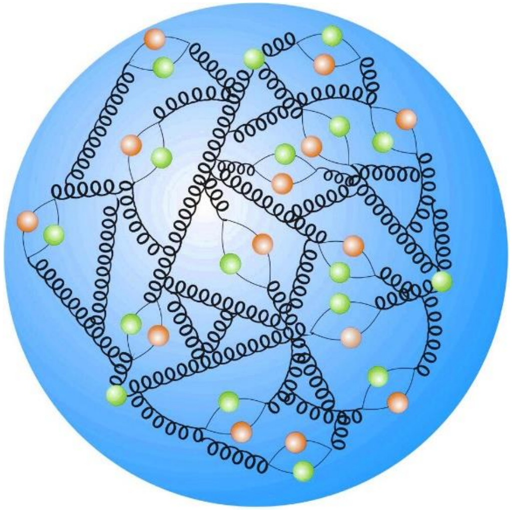
Solve QFT



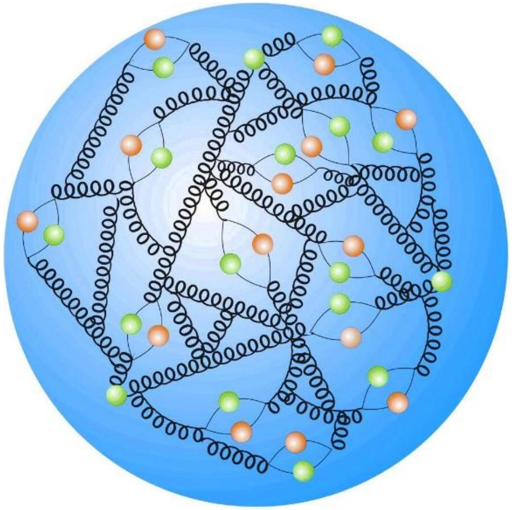
Strongly-coupled QFTs
Non-perturbative physics

Weakly-coupled QFTs
Perturbative expansion:
(Feynman diagrams)

$\left\{ \begin{array}{l} \textbf{Many-loops} \\ \\ \textbf{Many-legs} \end{array} \right.$

Rapid growth of the number of Feynman diagrams, with the number of loops/external legs

Solve QFT



Strongly-coupled QFTs
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Perturbative expansion:
(Feynman diagrams)

Many-loops

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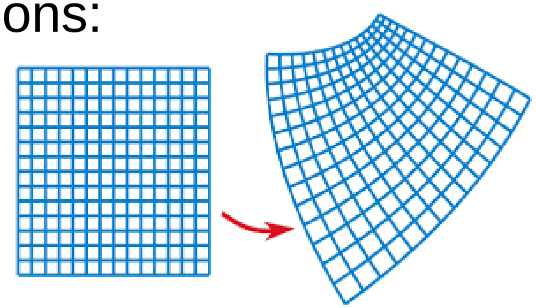
Rapid growth of the number of Feynman diagrams, with the number of loops/external legs

Conformal field theories

CFT = QFT invariant under conformal transformations: transformations which locally preserve angles.



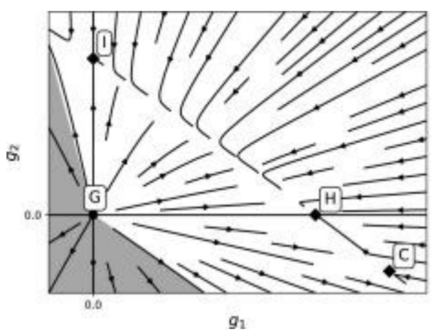
CFTs are defined by their CFT data:

$$\{\mathcal{O}_i \;, \{\Delta_i, C_{ij}^k\}\}$$

Primary operators — *Scaling dimensions* — *OPE coefficients*

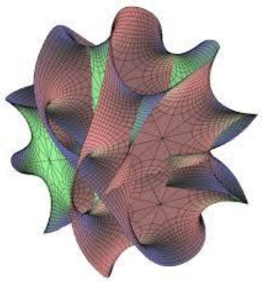
$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle = \frac{C_{ij}^k}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|y-z|^{\Delta_j+\Delta_k-\Delta_i}|z-x|^{\Delta_k+\Delta_i-\Delta_j}}$$

Extrema of the RG flow

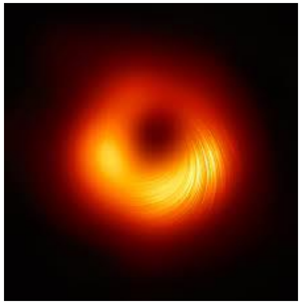


Critical phenomena

String theory



Quantum gravity



Solve CFT

- CFT (QFT) simplifies in certain limits when a small/large parameter exists. (Perturbative expansion)
- Our large parameter(s): conserved charge(s) of the internal symmetry group of the CFT:

LARGE-CHARGE EXPANSION FOR CFT DATA

Diagrammatics

Conventional Feynman diagram expansion (in the number of loops):

$$\mathcal{O} = \sum c_i(Q, N, N_f, \ldots) g^i$$

Tree-level diagrams dominates

Large-N (number of colors) expansion in gauge theories

$$\mathcal{O} = \sum d_i(Q, N_f, \mathcal{A}, \ldots) \frac{1}{N^i}\,, \qquad \mathcal{A} \equiv gN$$

Planar diagrams dominates

Large-Nf (number of flavors) expansion

$$\mathcal{O} = \sum b_i(Q, N, \mathcal{A}, \ldots) \frac{1}{N_f^i}\,, \qquad \mathcal{A} \equiv gN_f$$

Bubble diagrams dominates

Large-charge expansion

$$\mathcal{O} = \sum a_i(N, N_f, \mathcal{A}, \ldots) \frac{1}{Q^i}\,, \qquad \mathcal{A} \equiv gQ$$

Quantum VS Classical

Quantum physics *"classicalizes"* in the presence of large quantum numbers.

Hydrogen atom with infinite mass of the proton at fixed magnetic quantum number m:

QUANTUM ground state energy:

$$E_0^{\mathrm{QM}}(m) = -\frac{M_e \alpha^2}{2(m+1)^2}$$

CLASSICAL ground state energy:

$$E_0^{cl}(m) = -\frac{M_e \alpha^2}{2m^2}$$

$$\boxed{\lim_{m\to\infty} \left(E_0^{\mathrm{QM}}(m) - E_0^{\mathrm{cl}}(m)\right) = 0}$$

LARGE-CHARGE EXPANSION =
SEMICLASSICAL EXPANSION

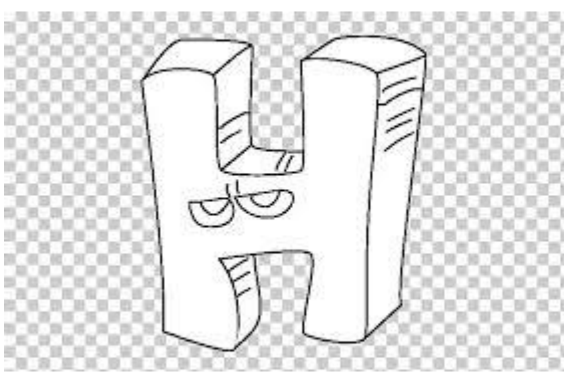
Charging the O(N) model

We study the O(N) scalar theory in d= 4-ε dimensions where it features an infrared Wilson-Fisher fixed point

$$\mathcal{S} = \int d^d x \left(\frac{(\partial \phi_i)^2}{2} + \frac{(4\pi)^2 g_0}{4!} (\phi_i \phi_i)^2 \right) \qquad g^*(\epsilon) = \frac{3\epsilon}{8+N} + \mathcal{O}\left(\epsilon^2\right)$$

$\epsilon = \mathbf{0}$ and $\mathbf{N} = \mathbf{4}$

Standard Model Higgs

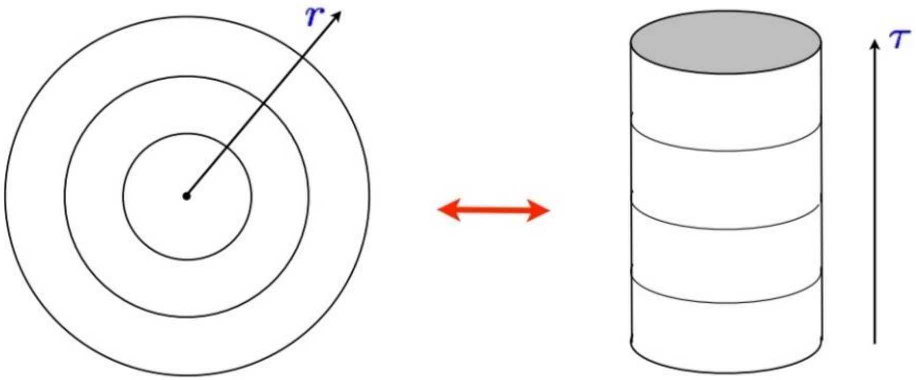


$\epsilon \to \mathbf{1}$

Superfluid He^4, Magnets, Superconductors, ..

Map to the cylinder

$$\mathbb{R}^d \to \mathbb{R}\times S^{d-1}$$



The eigenvalues of the dilation charge (the scaling dimensions) become the energy spectrum on the cylinder (state-operator correspondence)

$$\Delta = E$$

We compute the scaling dimension of operators with total charge Q and the minimal scaling dimension.

i.e. we compute the ground state energy on the cylinder.

LARGE-CHARGE EXPANSION =
FINITE DENSITY QFT

What are we computing?

We compute the scaling dimension of operators with total charge Q and the minimal scaling dimension.

These operators transform according to the Q-indices traceless symmetric O(N) representations.

$$\underbrace{\square\square\square\square\square\cdots\square}_{Q}$$

$$Q = 1 \to \phi_i \qquad\qquad Q = 2 \to \phi_i \phi_j - \frac{\delta_{ij}}{N}\phi^2$$

Physically, these operators represent anisotropic perturbations in O(N)-invariant systems. Their scaling dimension define a set of crossover (critical) exponents measuring the stability of the system (e.g. magnets) against anisotropic perturbations (e.g. crystal structure).

Computation

To get the ground state energy on the cylinder we consider the matrix element of the evolution operator between charge-Q states.

$$\langle Q|e^{-H(\tau_f-\tau_i)}|Q\rangle = \frac{1}{\mathcal{Z}}\int D\phi D\bar{\phi}\; e^{-QS_{\text{eff}}[\phi,\bar{\phi},gQ]} \underset{\tau_f-\tau_i\to\infty}{=} e^{-E(\tau_f-\tau_i)}$$

$$\mathcal{S}_{\text{eff}} = \mathcal{S} + \mu Q + \frac{1}{8}\int d^dx\,(d-2)^2\phi\bar{\phi}$$

Charge-fixing *Conformal coupling*

Q counts loops.

Computing the path integral semiclassically, we have

$$\mathbf{\Delta_Q = \sum_{k=-1}^{\infty}\frac{1}{Q^k}\Delta_k(\mathcal{A})} \qquad \mathcal{A}\equiv \mathbf{gQ}$$

Every Δ_k resums an infinite series of Feynman diagrams.

Leading order: Δ_{-1}

$$\mathcal{S} = \mathcal{S}(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 \mathcal{S}''(\phi_0) + \ldots$$

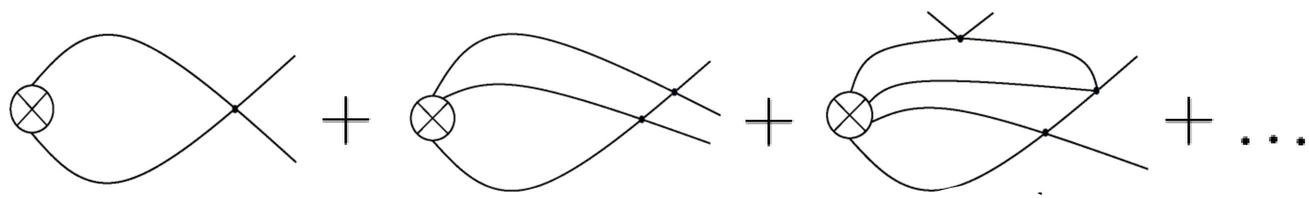
$$\mathbf{\Delta_Q = \sum_{k=-1}^{\infty} \frac{1}{Q^k} \Delta_k(\mathcal{A})} \qquad \mathcal{A} \equiv \mathbf{gQ}$$

Given by the effective action evaluated on the classical solution of the EOM

$$\frac{4\Delta_{-1}(\mathcal{A})}{\mathcal{A}} = \frac{3^{\frac{2}{3}} x^{\frac{1}{3}}}{3^{\frac{1}{3}} + x^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left(3^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}{x^{\frac{1}{3}}} \qquad x \equiv 6\mathcal{A} + \sqrt{-3 + 36\mathcal{A}^2}$$

This classical result resums an infinite number of Feynman diagrams!

$$Q\Delta_{-1} =$$



$$\sim gQ^2 \qquad \sim g^2 Q^3 \qquad \sim g^3 Q^4$$

Q counts the number of external legs.

g counts the number of vertices.

Many-loops – Many-legs

Next-to-leading order: Δ_0

$$\mathcal{S} = \mathcal{S}(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 \mathcal{S}''(\phi_0) + \ldots$$

At NLO we have to compute a quadratic (Gaussian) path integral.

Δ_0 is given by the fluctuation determinant around the classical trajectory

$$\Delta_0 = \frac{R}{2} \sum_{\ell=0}^{\infty} n_\ell \left[\omega_+(\ell) + \omega_-(\ell) + \left(\frac{N}{2} - 1 \right) (\omega_{++}(\ell) + \omega_{--}(\ell)) \right]$$

ℓ labels the eigenvalues of the momentum which have degeneracy n_ℓ.

$\omega_+ \ , \omega_-, \omega_{++}, \omega_{--}$ → Dispersion relations of the spectrum.

Boosting perturbation theory

By expanding the Δ_k's in the limit of small 't Hooft-like coupling A=gQ, we obtain the conventional perturbative expansion

$$\Delta_Q = Q + \left(\frac{Q^2}{8+N} - \frac{(N+10)}{2(8+N)}Q\right)\epsilon$$
$$-\left[\frac{2}{(8+N)^2}Q^3 + \frac{(N-22)(N+6)}{2(8+N)^3}Q^2 + \frac{184+N(14-3N)}{4(8+N)^3}Q\right]\epsilon^2$$
$$+\left[\frac{8}{(8+N)^3}Q^4 + \frac{-456-64N+N^2+2(8+N)(14+N)\zeta(3)}{(8+N)^4}Q^3\right.$$
$$-\frac{-31136-8272N-276N^2+56N^3+N^4+24(N+6)(N+8)(N+26)\zeta(3)}{4(N+8)^5}Q^2$$
$$\left.+\frac{-65664-8064N+4912N^2+1116N^3+48N^4-N^5+64(N+8)(178+N(37+N))\zeta(3)}{16(N+8)^5}Q\right]\epsilon^3$$
$$+\left[c_5Q^5 + c_4Q^4 + c_3Q^3 + c_2Q^2 + c_1Q\right]\epsilon^4 + \left[e_6Q^6 + e_5Q^5 + e_4Q^4 + e_3Q^3 + e_2Q^2 + e_1Q\right]\epsilon^5 + \ldots$$

Red terms: Δ_{-1}

Blue terms: Δ_0

Complete 4-loop (ε^4) scaling dimension obtained by combining our results with the known perturbative results for Q= 1, 2, 4.

Infinite number of checks for future diagrammatic computations:

4-loop: I. Jack & D.R.T. Jones, Phys.Rev.D 103 (2021) 8, 085013
5-loop: Q. Jin & Y. Li, 2205.02535 [hep-th] (2022)
6-loop: A. Bednyakov & A. Pikelner, 2208.04612 [hep-th] (2022)

What can we do?

- Efficient CFT data computation.

  G. Badel, G. Cuomo, A. Monin, R. Rattazzi, JHEP 11 (2019) 110
  O. Antipin, JB, F. Sannino, C. Zhang, Z.W. Wang, Phys.Rev.D 102 (2020) 4, 045011

- Non-trivial tests of dualities in QFT.

  O. Antipin, JB, F. Sannino, C. Zhang, Z.W. Wang, Phys.Rev.D 104 (2021) 085002

- Access the large-order behavior of perturbation theory.

  N. Dondi, I. Kalogerakis, D. Orlando, S. Reffert, JHEP 05 (2021) 035
  O. Antipin, JB, F. Sannino, M. Torres, JHEP 06 (2022) 041

- Gain insight on the structure of QFT (e.g. Convex Charge Conjecture: $\Delta_{Q_1+Q_2} \geq \Delta_{Q_1} + \Delta_{Q_2}$).

  O. Aharony & E. Palti, Phys.Rev.D 104 (2021) 12, 126005
  O. Antipin, JB, F. Sannino, C. Zhang, Z.W. Wang, JHEP 12 (2021) 204

- Understand non-perturbative properties of multi-boson production in the Standard Model

- …….

Semiclassical expansion

$$\mathcal{L} = \partial\bar{\phi}\partial\phi + \lambda_0 \left(\bar{\phi}\phi\right)^2$$

The operator Φ^Q carries U(1) charge Q.

$$< \bar{\phi}^Q(x_f)\phi^Q(x_i) > = Q^Q \frac{1}{\mathcal{Z}} \int D\phi D\bar{\phi}\, \bar{\phi}^Q(x_f)\phi^Q(x_i) e^{-Q\mathcal{S}}$$

$$\phi \to \phi\sqrt{Q}$$

We bring the field insertions into the exponent, obtaining

$$< \bar{\phi}^Q(x_f)\phi^Q(x_i) > = Q^Q \frac{1}{\mathcal{Z}} \int D\phi D\bar{\phi}\; e^{-Q\left[\int \partial\bar{\phi}\partial\phi + \frac{Q\lambda_0}{4}(\bar{\phi}\phi)^2 - \ln\bar{\phi}(x_f) - \ln\phi(x_i)\right]}$$

For large Q the path integral is dominated by the extrema of

$$\mathcal{S}_{eff} \equiv \int d^d x \left[\partial\bar{\phi}\partial\phi + \frac{Q\lambda_0}{4}\left(\bar{\phi}\phi\right)^2 - \ln\bar{\phi}(x_f) - \ln\phi(x_i)\right]$$

We can evaluate the integral via a saddle-point expansion
1/Q counts loops and is our expansion parameter.