Conserved charges from dynamics in non commutative spaces

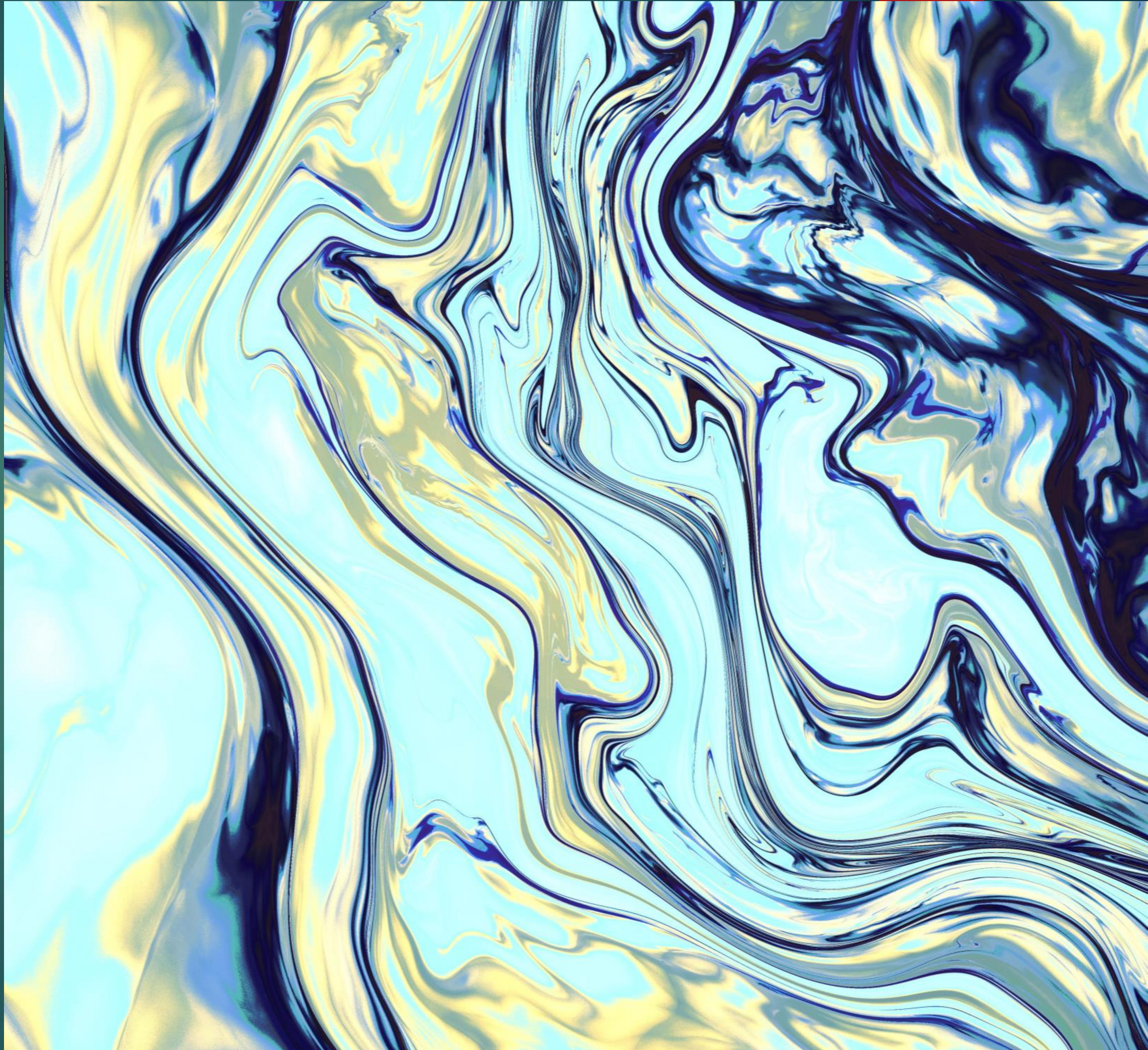
DOMENICO FRATTULILLO

UNIVERSITÀ DI NAPOLI FEDERICO II

COST CA18108 2° Training school – Belgrade

In collaboration with:

G. Amelino-Camelia, G. Fabiano



Non commutative space-times

In many quantum gravity models non commutative space-times are considered where an inverse energy scale is introduced $\ell \simeq l_p$.

In the last years people have mainly studied the kinematics inspired from the non-commutative framework.

We will investigate particle dynamics on these non-commutative spaces, gaining some insights on the conserved charges for multiparticle systems.

Hopf-Algebra Symmetries for empty 1+1D κ-Minkowski

Coproducts (bicross-product basis)

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 \qquad \Delta P_1 = P_1 \otimes 1 + e^{-\ell P_0} \otimes P_1 \qquad \Delta N = N \otimes 1 + e^{-\ell P_0} \otimes N$$

Commutators

$$[P_0, P_1] = 0 \qquad [N, P_0] = iP_1 \quad [N, P_1] = \frac{i}{2\ell}\left(1 - e^{-2\ell P_0}\right) - \frac{i\ell}{2} P_1^2$$

Casimir Operator

$$C = \frac{4}{\ell^2} \sinh^2\left(\frac{\ell}{2} P_0\right) - (P_1)^2 e^{\ell P_0}$$

Non-commutative space-time

$$[x_1, x_0] = i\ell x_1$$

Total momentum

The standard interpretation is to regard the generators P_0, P_1, N as conserved charges of free particle motion.

If more than one particle is considered the coproducts usually inspire the composition law of said charges from which we obtain the total momentum:

$$(p \oplus k)_0 = p_0 + k_0$$

$$(p \oplus k)_1 = p_1 + k_1 + e^{-\ell p_0} k_1$$

$$\left(N_p \oplus N_k\right) = N_p + N_k + e^{-\ell p_0} N_k$$

Ambiguities for the total momentum

However, this framework suffers from ambiguities in defining such total momentum.

A first ambiguity comes from the fact that in the free particle case, even just the single momentum of each particle is a conserved charge.

And even if we construct the total momentum using the $\oplus$ inspired by the copruducts there are ambiguities due to the non commutativity of the composition law:

$$(p \oplus k)_1 \neq (k \oplus p)_1$$

What can dynamics tell us about total momentum?

Spatial "(2+0)D" κ-Minkowski (Vanilla) (1)

Let us consider the following map:

$$\ell \to i\ell \qquad x_0 \to ix_2 \qquad P_0 \to -iP_2 \qquad N \to -iR$$

The Hopf Algebra becomes:

$$[P_1, P_2] = 0 \qquad [R, P_1] = \frac{i}{2\ell}\left(1 - e^{-2\ell P_2}\right) + \frac{i\ell}{2} P_1^2 \qquad [R, P_2\,] = -iP_1$$

$$\Delta P_1 = P_1 \otimes 1 + e^{-\ell P_2} \otimes P_1 \qquad \Delta P_2 = P_2 \otimes 1 + 1 \otimes P_2$$

$$\Delta R = R \otimes 1 + e^{-lP_2} \otimes R$$

G.Amelino-Camelia, ``Planck-scale soccer-ball problem: a case of mistaken identity,'' Entropy {19} (2017) no.8, 400

Spatial "(2+0)D" κ-Minkowski (2)

The Casimir operator now reads

$$C = \frac{4}{\ell^2}\sinh^2\left(\frac{\ell}{2}P_2\right) + (P_1)^2 e^{\ell P_2} \xrightarrow{\ell \to 0} P_1^2 + P_2^2$$

and the coordinate non-commutativity becomes

$$[x_1, x_2] = i\ell x_2$$

Now x_0 is a commutative variable and we can set up standard Hamiltonian analysis in the deformed Galilean Relativistic regime.

Furthermore, to ensure Jacobi identities are satisfied, the symplectic structure is also deformed

$$[P_1, x_1] = i \qquad [P_2, x_2] = i \qquad [P_1, x_2] = 0 \qquad [P_1, x_2] = -i\ell P_1$$

The elastic potential

In standard QM, a system of two particles interacting via an elastic potential is described by the following Hamiltonian:

$$H = \frac{\vec{p}_A^{\,2}}{2m} + \frac{\vec{p}_B^{\,2}}{2m} + \frac{1}{2}\rho(\vec{q}_A - \vec{q}_B)^2$$

The conserved charges read

$$\vec{P} = \vec{p}_A + \vec{p}_B \qquad \vec{L} = \vec{L}_A + \vec{L}_B \qquad \vec{L}_I = \vec{x}_I \wedge \vec{p}_I$$

Indeed

$$[H, \vec{L}] = [H, \vec{P}] = 0$$

Our objective will be to construct a deformed Hamiltonian that commutes with total momentum and total angular momentum, as inspired from non-commutativity.

The deformed Hamiltonian

Keeping our analysis up to second order in the deformation parameter ℓ^2, the deformed kinetic term is given by

$$H_K^I = \frac{(p_1^I)^2}{2m} + \frac{(p_2^I)^2}{2m} + \ell \frac{(p_1^I)^2 p_2^I}{2m} + \frac{\ell^2 (p_1^I)^2 (p_2^I)^2}{4m} + \frac{\ell^2 (p_2^I)^4}{24m}$$

while the most general ansatz for the quadratic potential reads

$$V(\vec{x}^A, \vec{x}^B) = \frac{1}{2}\rho(\vec{x}^A - \vec{x}^B)^2 + \ell\rho \sum \alpha_{ijk}^{IJK} p_i^I x_j^J x_k^K + \ell^2 \rho \sum \beta_{ijkh}^{IJKH}\, p_i^I p_j^J x_k^K x_h^H$$

The full Hamiltonian for two particles is simply $H = H_K^A + H_K^B + V(\vec{x}^A, \vec{x}^B)$

Vanilla κ-Minkowski : two particle dynamics

Adopting the usual physical interpretation, for the total charges to be symmetries of the dynamics we require

$$[(p^A \oplus p^B)_1, H] = 0 \qquad [(p^A \oplus p^B)_2, H] = 0 \qquad [(R^A \oplus R^B), H] = 0$$

This is equivalent to a system of equations involving coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. At first order in ℓ , we find the following solution

$$H = H_K^A + H_K^B + \frac{1}{2}\rho(\vec{x}_A - \vec{x}_B)^2 + \frac{1}{2}\ell\rho\left[p_2^A x_1^A\left(x_1^A - x_1^B\right) - \left(p_1^A x_1^A - 2p_1^B x_1^B\right)\left(x_2^A - x_2^B\right)\right] + O(\ell^2)$$

Vanilla κ-Minkowski : three particle dynamics

A possible separable three particle Hamiltonian for this model simply reads

$$H^{ABC} = H_K^A + H_K^B + H_K^C + V(\vec{x}^A, \vec{x}^B) + V(\vec{x}^A, \vec{x}^C) + V(\vec{x}^B, \vec{x}^C) + O(\ell^2)$$

And it can be explicitly checked that

$$[(p^I \oplus p^J \oplus p^K)_i, H^{ABC}] \neq 0 \quad [(R^I \oplus R^J \oplus R^K)_i, H^{ABC}] \neq 0 \; I,J,K = A,B,C \quad i = 1,2$$

Even the most general ansatz for a non-separable Hamiltonian fails ⇒ No three-particle dynamics in spatial κ-Minkowski space with coproduct inspired composition laws!

Proper-dS framework

Momentum space model with κ-Minkowski algebra structure but composition laws given by

$$(p^A \oplus p^B)_1 = p_1^A + p_1^B - \ell(p_2^A p_1^B + p_1^A p_2^B) + \frac{\ell^2}{2}[(p_2^A p_1^B + p_1^A p_2^B)(p_2^A + p_2^B) - p_1^A (p_1^B)^2 - (p_1^A)^2 p_1^B]$$

$$(p^A \oplus p^B)_2 = p_2^A + p_2^B + \ell p_1^A p_1^B - \frac{\ell^2}{2}[(p_1^A)^2 p_2^B + (p_1^B)^2 p_2^A - p_1^A p_1^B (p_2^A + p_2^B)]$$

$$(R^A \oplus R^B) = R^A + R^B$$

The composition law for momentum is commutative but non-associative.

G.Amelino-Camelia, G.Gubitosi and G.Palmisano, ``Pathways to relativistic curved momentum spaces: de Sitter case study,'' Int. J. Mod. Phys. D {25} (2016) no.02, 1650027

Proper-dS: dynamics

Requiring that the total charges are symmetries of the dynamics, we obtain infinitely many solutions up to second order in $\ell^2 \Rightarrow$ A potential $V(\vec{x}^A, \vec{x}^B)$ exists.

The three-particle separable Hamiltonian

$$H^{ABC} = H_K^A + H_K^B + H_K^C + V(\vec{x}^A, \vec{x}^B) + V(\vec{x}^A, \vec{x}^C) + V(\vec{x}^B, \vec{x}^C)$$

doesn't work with total charges $\left((p^A \oplus p^B) \oplus p^C\right)_i$ nor $(p^A \oplus (p^B \oplus p^C))_i$. If we add a non-separable term of the form

$$H^{NS} = \frac{1}{2} l^2 \rho[\left(p_2^C p_1^A - p_1^C p_2^A\right)\left(x_2^C x_1^B - x_1^C x_2^B\right) + p_2^B (p_1^C x_1^C x_2^A - p_1^C x_2^C x_1^A + p_1^A\left(p_2^A p_1^C\left(2x_1^C - x_1^A - x_1^B\right) - 2x_1^C x_2^C + x_1^C x_2^B + x_2^C x_1^A\right) + \\ + p_1^B \left(p_2^C \left(x_2^C x_1^A - x_1^C x_2^A\right) + p_1^A x_2^C (2x_2^C - x_2^A - x_2^B\right) + p_2^A x_1^C \left(x_2^A - 2x_2^C\right) + p_2^A x_2^C x_1^B]$$

$\left((p^A \oplus p^B) \oplus p^C\right)_i$ is conserved.

Conclusions

With coproduct inspired composition law, we can not find a compatible Hamiltonian for the three particle system.

If we consider the proper dS scenario we can construct a non separable Hamiltonian compatible with the three particle system.

To conclude we can say that the Hamiltonian is the fundamental element in order to define the total momentum of a multiparticle system.

Thank you!