Noncommutative $D = 5$ Chern-Simons gravity: Kaluza-Klein Reduction and Chiral Gravitation Anomaly

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Gravity

It is known that the most general action in general relativity in four dimensions is given by an Einstein-Hilbert action with a cosmological constant *(Lovelock's theorem)*.

$$L = c_1 \varepsilon_{abcd} R^{ab} e^c e^d + c_2 \varepsilon_{abcde} e^a e^b e^c e^d e^e.$$

We will work in the first order formalism (vielbein and spin connection are treated independently). There exists many motivations to study gravity in more than four dimensions. In our work, we will start from five dimension. Generalisation of Lovelock theorem allows us to consider the most general action (without explicit torsion) of the form

$$L = c_1 \varepsilon_{abcde} R^{ab} R^{cd} e^e + c_2 \varepsilon_{abcde} R^{ab} e^c e^d e^e + c_3 \varepsilon_{abcde} e^a e^b e^c e^d e^e.$$

Connection with CS

For suitable choice of c_i coefficients, local symmetry group is enlarged to AdS/dS/Poincare. This can be seen if we introduce a gauge connection A, a (locally) one form valued in $\mathfrak{so}(4,2)$ Lie algebra [Zanelli, Chamseddine,...]

$$A = \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{l} e^a P_a.$$

Parameter l has to be introduced on dimensional grounds. Later, we will change our notation slightly. This connection can be used to form a gauge invariant, background independent ("Topological") action: Chern Simons action.

Introduction

Let A be a (local) connection, defined as a pullback of a Ehresmann connection by a local section $(A = s^*\omega)$. We will assume that our principal bundle is trivial, and therefore A can be considered as a (globally) Lie algebra valued one form. Let F be a curvature two form $F = \mathrm{d}A + A^2$ (most of the time we suppress wedge product). Chern-Weill theorem asserts that (Tr can be a symmetric invariant tensor of the algera, but we will use a trace in an explicit matrix representation)

$$\mathrm{Tr}(F^3) = \mathrm{d}Q^5_{CS}, \tag{1}$$

where we have

$$Q^5_{CS} = 3\int_0^1 t^2 \mathrm{Tr}\left(A(F + (t-1)A^2)^2\right) \mathrm{d}t. \tag{2}$$

Introduction

We define an action

$$S^5_{CS} = \alpha \int\limits_{\mathcal{M}_5} Q^5_{CS} = \int\limits_{\mathcal{M}_5} L^5_{CS}.$$

A simple computation gives

$$S^{(5)}_{CS} = \alpha \int\limits_{\mathcal{M}_5} F^2 A - \frac{1}{2} F A^3 + \frac{1}{5} A^5.$$

Note that there is no notion of metric in the action.

AdS

To obtain AdS gravity, we work with Lie algebra $\mathfrak{so}(4,2)$. Commutation relations in this algebra are:

$$[J_{\mathbb{AB}}, J_{\mathbb{CD}}] = G_{\mathbb{AD}} J_{\mathbb{BC}} + G_{\mathbb{BC}} J_{\mathbb{AD}} - (\mathbb{C} \leftrightarrow \mathbb{D}),$$

where $G_{\mathbb{AB}} = (-++++-)$. By splitting indices, we have

$$\begin{aligned}
[J_{AB}, J_{CD}] &= G_{AD} J_{BC} + G_{BC} J_{AD} - (C \leftrightarrow D), \\
[J_{AB}, J_{C5}] &= G_{BC} J_{A5} - G_{AC} J_{B5}, \\
[J_{A5}, J_{C5}] &= J_{AC}.
\end{aligned}$$

Explicit representation of this algebra can be found using gamma matrices. In what follows, trace will stand for the trace in this representation.

Gravitation

Using mentioned representation, we compute $\mathrm{Tr}\left(F^3\right) = \frac{3i}{8}\varepsilon_{ABCDE}F^{AB}F^{CD}F^{E5}$. Taking $A = \frac{1}{2}\Omega^{AB}J_{AB} + \frac{1}{l}E^A J_{A5}$, we get

$$S_{CS}^{(5)} = \frac{k}{8}\int \varepsilon_{ABCDE}\left(\frac{1}{l}R^{AB}R^{CD}E^E + \frac{2}{3l^3}R^{AB}E^C E^D E^E + \frac{1}{5l^5}E^A E^B E^C E^D E^E\right). \quad (3)$$

Constant k is introduced as $\alpha = -ik/3$. We can rewrite this as

$$S_{CS}^{(5)} = \frac{1}{16\pi G^{(5)}}\int \mathrm{d}^5x\sqrt{-g}\Big[R - 2\Lambda + \frac{l^2}{4}\left(R^2 - 4R^{\mu\nu}R_{\nu\mu} + R^{\mu\nu\rho\sigma}R_{\rho\sigma\mu\nu}\right)\Big] \quad (4)$$

NC field theory

First, we start from canonical noncommutativity $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu} = const$. Noncommutativity is imposed by deforming the algebra of functions.

$$f \star g = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}} f(x)g(y)|_{x\to y}$$

It is more natural to deal with CS Lagrangians using differential forms; we introduce an Abelian Drinfeld Twist

$$\tau_p \wedge_\star \tau'_q \equiv \sum_{n=0}^{+\infty} \left(\frac{i}{2}\right)^n \theta^{I_1 J_1} \ldots \theta^{I_n J_n} (\ell_{I_1} \ldots \ell_{I_n} \tau_p) \wedge (\ell_{J_1} \ldots \ell_{J_n} \tau_q)$$

Lie derivatives are taken with respect to a set of mutually commuting vector fields X_I.

NC Gauge Theories, SW mapping

In a commutative gauge theory, we have

$$\delta_\epsilon A = -\mathrm{d}\epsilon - [A, \epsilon] = -\mathrm{d}\epsilon - A \wedge \epsilon + \epsilon \wedge A,$$
$$\delta_\epsilon F = [\epsilon, F] = \epsilon \wedge F - F \wedge \epsilon,$$

where $\epsilon = \epsilon^K T_K$ is a gauge transformation parameter. For NC fields, we define

$$\widehat{\delta}_{\widehat{\epsilon}} \widehat{A} = -\mathrm{d}\widehat{\epsilon} - \widehat{A} \wedge_\star \widehat{\epsilon} + \widehat{\epsilon} \wedge_\star \widehat{A},$$
$$\widehat{\delta}_{\widehat{\epsilon}} \widehat{F} = \widehat{\epsilon} \wedge_\star \widehat{F} - \widehat{F} \wedge_\star \widehat{\epsilon}.$$

We note that

$$[\widehat{\delta}_{\widehat{\epsilon}_1}, \widehat{\delta}_{\widehat{\epsilon}_2}]\widehat{F} = \widehat{\delta}_{-[\widehat{\epsilon}_1, \widehat{\epsilon}_2]_\star} \widehat{F} = -[[\widehat{\epsilon}_1, \widehat{\epsilon}_2]_\star, \widehat{F}]_\star, \qquad (5)$$

where

$$[\widehat{\epsilon}_1, \widehat{\epsilon}_2]_\star = \frac{1}{2}\left([\widehat{\epsilon}_1^K, \widehat{\epsilon}_2^L]_\star \{T_K, T_L\} + \{\widehat{\epsilon}_1^K, \widehat{\epsilon}_2^L\}_\star [T_K, T_L]\right). \qquad (6)$$

NC Gauge Theories, SW mapping

Anticomutator in the last expression implies that the commutator does not close in the algebra. In order to cure this problem, one introduces an Enveloping algebra. Unfortunately, this introduces (infinity) new degrees of freedom. Seiberg and Witten showed that it is possible to express all the degrees of freedom in terms of the commutative ones, in a perturbative expansion in θ. Thus, NC gauge field theory is connected with a commutative gauge field theory. We demand:

$$\widehat{\delta}_{\widehat{\epsilon}}\widehat{A}(A) = \widehat{A}(A + \delta_{\epsilon}A) - \widehat{A}(A),$$

Then one can obtain (keeping only linear terms in θ):

$$\hat{A} = A - \frac{i}{4}\theta^{IJ}\{A_I, \ell_J A + F_J\}, \tag{7}$$

$$\hat{\epsilon} = \epsilon - \frac{i}{4}\theta^{IJ}\{A_I, \ell_J \epsilon\}. \tag{8}$$

NC Correction

In the last formula, A_I stands for contracted connection along the direction of X_I. NC CS action in $5D$ reads

$$S^{(5)}_{CS,NC} = -\frac{ik}{3}\int \mathrm{Tr}\Big(\widehat{F}\wedge_\star \widehat{F}\wedge_\star \widehat{A} -\frac{1}{2}\widehat{F}\wedge_\star \widehat{A}\wedge_\star \widehat{A}\wedge_\star \widehat{A} +\frac{1}{10}\widehat{A}\wedge_\star \widehat{A}\wedge_\star \widehat{A}\wedge_\star \widehat{A}\wedge_\star \widehat{A}\Big). \qquad (9)$$

θ independent part is precisely the action we already considered. We again use $\mathfrak{so}(4,2)$ algebra and decompose (classical) connection as before, to obtain the first order correction of this action.

NC Correction

The first order correction of our action is given by [Aschieri, Castellani, '14]

$$S^{(5)}_{CS,\theta} = \frac{k\theta^{IJ}}{12} \times \int \Big(F^{AB}(F_I)_{BC}(\mathrm{D}_\Omega F_J)^C{}_A + \frac{1}{l^2} F^{AB}(F_I)_{BC}(T_J)^C E_A$$
$$+\frac{1}{l^2} F^{AB}(T_I)_B(\mathrm{D}_\Omega T_J)_A + \frac{2}{l^2} F^{AB}(T_I)_B(F_J)_{AC}E^C$$
$$+\frac{1}{l^2} T^A(T_I)^B(\mathrm{D}_\Omega F_J)_{BA} + \frac{1}{l^2} T^A(\mathrm{D}_\Omega T_I)^B(F_J)_{BA}$$
$$+\frac{1}{l^2} T_A(F_I)^{AB}(F_J)_{BC}E^C + \frac{2}{l^4} T_A(T_I)_B(T_J)^{[B}E^{A]} \Big).$$

KK Reduction

Chamseddine showed that CS gravity can be connected to a topological $\sim \mathrm{Tr}\phi F^n$ gravity by a dimensional reduction (with truncation).

We truncate components Ω_4^{ab}, E_4^a, Ω_μ^{a4}, и E_μ^4 (consistent with the residual $SO(3,2)$ symmetry). We furthermore label $\Omega_\mu^{ab} \equiv \omega_\mu^{ab}$, $E_\mu^a \equiv e_\mu^a$ and $\phi^a \equiv -l^2\Omega_4^{a4}$, $\varphi \equiv lE_4^4$. We can arrange remaining fields as

$$\mathcal{A} = \frac{1}{2}\omega^{ab}J_{ab} + l^{-1}e^a J_{a5},$$
$$\mathcal{F} = \frac{1}{2}F^{ab}J_{ab} + F^{a5}J_{a5},$$
$$\Phi = \Phi^a J_{4a} + \Phi^5 J_{45} = \phi^a J_{4a} + \varphi J_{45}, \tag{10}$$

in order to write

$$S_{red} = \frac{ik(2\pi R)}{l^2}\int \mathrm{Tr}\left(\mathcal{F}\mathcal{F}\Phi\right) \tag{11}$$

Symmetry Breaking

Obtained theory of gravity can be considered, but due to it's enlarged local symmetry group, one cannot immediately connect it with observable physics. We therefore insist on a symmetry breaking of $SO(3,2)$ down to $SO(3,1)$, that is done (motivated by MacDowell-Mansouri-Chamseddine-Stelle-West approach) by $\phi^a = 0$ и $\varphi = l$. We are left with

$$S_{red} = \frac{k(2\pi R)}{8l^3} \int \varepsilon_{abcd} \times \left(l^2 R^{ab} R^{cd} + 2R^{ab} e^c e^d + \frac{1}{l^2} e^a e^b e^c e^d \right). \qquad (12)$$

This is Einstein-Hilbert action with cosmological constant (together with the topological Gauss-Bonnet term), written in the first-order formalism.

Noncommutative version

We apply the mentioned procedure to the action obtained as a first order correction to a classical CS gravity action. After calculation, the final result is

$$S_{red,NC} = S_{red} + \frac{(2\pi R)k}{12}\theta^{I4} \times \int \Bigg[\frac{2}{l^4} R^{ab} T_a (e_I)_b - \frac{4}{l^4} T^a (R_I)_{ab} e^b + \frac{2}{l^4} R^{ab} (T_I)_a e_b + \frac{6}{l^6} T^a e_a (e_I)^b e_b \Bigg] \quad (13)$$

We assumed $\partial_\mu X^4_I = 0$ (consistent from the point of view of a $4D$ theory). We defined $\theta^{I4} \equiv \theta^{IJ} X^4_J$.

Equations of motion

Varying action with respect to e^a and ω^{ab}, we obtain

$$\begin{aligned}
\delta e_d: \quad & \varepsilon_{abc}{}^{d}\left(R^{ab}e^c + \frac{1}{l^2}e^a e^b e^c\right) \\
& -\frac{\theta^{I4}}{3l}\left[\left(R^{db} + \frac{3}{l^2}e^d e^b\right)(\mathrm{D}_\omega e_I)_b - 2(\mathrm{D}_\omega R_I)^{db}e_b\right] = 0, \qquad (14) \\
\delta\omega_{ac}: \quad & \varepsilon^{ac}{}_{bd}T^b e^d + \frac{\theta^{I4}}{3l}\left[\frac{1}{2}R^{ab}e^c(e_I)_b\right. \\
& \left. -\frac{1}{2}R^{cb}e^a(e_I)_b + (R_I)^{ab}e^c e_b - (R_I)^{cb}e^a e_b + \frac{3}{l^2}e^a e^b e^c(e_I)_b\right] = 0. \qquad (15)
\end{aligned}$$

As θ is a small parameter, and we used only first order correction of an action, we solve equations (14) and (15) perturbatively.

AdS Spacetime

Line element in a suitable coordinates is given by

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{l^2}\right)\mathrm{d}t^2 + \frac{1}{\left(1 + \frac{r^2}{l^2}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2.$$

Classically, torsion is zero. We make an ansatz $e^a \to e^a + \tilde{e}^a$ and $\omega^{ab} \to \omega^{ab} + \tilde{\omega}^{ab}$. It is not hard to see that the corrections are actuall zero in this case, and AdS spacetime remains a solution of E.O.M even after introducing the first order correction.

AdS Schwarzschild Black Hole

Line element is given by

$$\mathrm{d}s^2 = -f^2(r)\mathrm{d}t^2 + \frac{1}{f^2(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2,$$

where $f^2(r) \equiv \left(1 - \frac{2m}{r} + \frac{r^2}{l^2}\right)$. It turns out that it is enough to assume that only spin connection receives a correction. In order to simplify our equations, we will assume that only two vector fields X_I are given by ∂_r and ∂_4. First, we obtain torsion components. Next, we compute the components of a spin connection and curvature (see the paper for results). Curvature receives a first order correction.

AdS Schwarzschild Black Hole

Also, even though metric ($g_{\mu\nu} = e^a_\mu e_{a\nu}$) remains the same as in the commutative limit, this solution develops a torsion. We can ask what are the differences between this solution, and the classical one. In order to do this, we compute forms connected with the topological invariants. Euler form ($\varepsilon^{abcd} R_{ab} R_{cd}$) and Nieh-Yan ($T^a T^b - R^{ab} e_a e_b$) form remain the same as in the commutative case. On the other hand, Pontryagin form, that is classically zero, is now given by

$$R^{ab} R_{ab} = \frac{48 m^2 \theta^{14}}{l r^5} \sin\theta \, \mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{16}$$

We work for $r > r_h$, but this expression is finite at the horizon.

Chiral Gravitational Anomaly

Last result is significat, because it implies the presence of a chiral gravitational anomaly (gravity analogy of a $\psi \to e^{i\alpha\gamma^5}\psi$ anomaly in a massless electrodynamics). Upon quantisation of a massless Dirac fermion on a fixed curved background, axial current, that is classically conserved, satisfies

$$\mathrm{d} * j_5 = \frac{1}{96\pi^2} R^{ab} R_{ab}. \tag{17}$$

We have

$$\mathrm{d} * j_5 = \frac{m^2 \theta^{14}}{2\pi^2 l r^5} \sin\theta \, \mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{18}$$

or written in another form:

$$\partial_\mu(\sqrt{-g} j_5^\mu) = \frac{m^2 \theta^{14}}{2\pi^2 l r^5} \sin\theta.$$

Conclusion

- $5D$ CS action has a first order in θ correction.
- This theory can be related with the four dimensional gravity by KK reduction and SB.
- Remaining action is fairly simple, and E.O.M. are solved by AdS spacetime.
- AdS Schwarzschild spacetime developes a torsion, and introduces a chiral gravitational anomaly if one couples a massless Dirac fermion to this fixed NC background.

Thank you!