Effective field Theories

Second Training School of COST Action CA18108
(Belgrade, Serbia)

Oleg Antipin

Contents

1. Introduction

The journey we are going to take in these very short lectures is a journey through physical phenomena as we change the distance (or, equivalently, energy) scales probed by our experiments. The history of physics itself is such a journey. Starting from classical mechanics describing the movement of object of human size we learned that this theory needs to be generalized to quantum mechanics to deal with movements of particles at subatomic level. In this sub-atomic world, Galelian kinematics of slow sub-relativistic speeds of classical mechanics needed to be generalised to special relativity. Marriage of quantum mechanics and special relativity lead us to quantum field theory (QFT) while marriage of special relativity with Newtonian gravity lead us to general relativity. Moving into even smaller scales we hope to marry QFT with general relativity into something like string theory where we imagine our subatomic particles to be different vibrations modes of a tiny strings.

On this journey, to make a stop at some fixed scale and describe the physical system at that scale we need to:

1. Determine relevant d.o.f. (fields). As we change our "microscope" relevant d.o.f. change. For example, new collective excitations appear or composite particles

2. Symmetries (types of interactions between fields)

3. Expansion parameters (power counting)

Since d.o.f. will change we need to learn how to identify the relevant ones and remove irrelevant. This can be done by procedure of "integrating out" d.o.f. as we change the energy scale. Since this logic will be the central part of the EFT examples below let me illustrate the procedure in a very simple example.

2. Invitation

2a. 1D Ising model of spins on a circle

To learn how to "integrate out" d.o.f. let us consider a simple 1d model of spins on a circle with Hamiltonian:

$$H = -J\sum_{i=1}^{N} S_i S_{i+1} = -J(S_1 S_2 + S_2 S_3 + \ldots + S_N S_1) \tag{1}$$

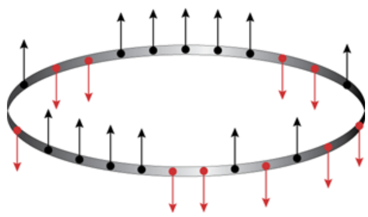

Figure 1: Spins on a circle.

In general, this system of spins is placed in an environment at a given fixed temperature T. Laws of thermodynamics teach us that system prefers to arrange spins to be in a state with minimum free energy $F = H - TS$. Consider $T = 0$ case: We need to find a state where F=H has minimum. Clearly this is the state where all spins point up or all point down because total interaction energy is minimized $H = -NJ$.

Consider now $T > 0$ case. We have to account for all possible configurations of spins and weight them according to their energies. For a canonical ensemble that is classical and discrete this defines the canonical partition function

$$Z(K,N) = \sum_{states} e^{-H/T} = \sum_{S_1=-1}^{+1} \sum_{S_2=-1}^{+1} \dots \sum_{S_N=-1}^{+1} e^{K(S_1S_2+S_2S_3+\dots+S_NS_1)} \tag{2}$$

where $K \equiv J/T$. $F = -T \log Z(K,N)$.

Let us now sum over the two possibilities $S_2 = \pm 1$ for spin S_2:

$$Z(K,N) = \sum_{S_1=-1}^{+1} \sum_{S_3=-1}^{+1} \dots \sum_{S_N=-1}^{+1} \left[e^{K(S_1+S_3)} + e^{-K(S_1+S_3)} \right] e^{K(S_3S_4+S_4S_5+\dots+S_NS_1)} . \tag{3}$$

In the same fashion let us sum over the two possibilities $S_4 = \pm 1$ for spin S_4:

$$Z(K,N) = \sum_{S_1=-1}^{+1} \sum_{S_3=-1}^{+1} \dots \sum_{S_N=-1}^{+1} \left[e^{K(S_1+S_3)} + e^{-K(S_1+S_3)} \right] \left[e^{K(S_3+S_5)} + e^{-K(S_3+S_5)} \right] e^{K(S_5S_6+S_6S_7+\dots+S_NS_1)} . \tag{4}$$

and we can repeat the exersize to sum over all even numbered spins:

$$Z(K,N) = \sum_{S_1=-1}^{+1} \sum_{S_3=-1}^{+1} \dots \sum_{S_{N-1}=-1}^{+1} \left[e^{K(S_1+S_3)} + e^{-K(S_1+S_3)} \right] \left[e^{K(S_3+S_5)} + e^{-K(S_3+S_5)} \right] \left[e^{K(S_5+S_7)} + e^{-K(S_5+S_7)} \right] \dots . \tag{5}$$

Rewrite the remaining sums defining:

$$\sum_{S=-1}^{+1} \sum_{S'=-1}^{+1} e^{K(S+S')} + e^{-K(S+S')} \equiv f(K) e^{-K'SS'} \tag{6}$$

where both $f(K)$ and K' are functions of K. Now we have:

$$\begin{aligned} Z(K,N) &= f(K)^{N/2} \sum_{S_1=-1}^{+1} \sum_{S_3=-1}^{+1} \dots \sum_{S_{N-1}=-1}^{+1} e^{-K'S_1S_3} e^{-K'S_3S_5} e^{-K'S_5S_7} \dots \\ &= f(K)^{N/2} \sum_{S_1=-1}^{+1} \sum_{S_3=-1}^{+1} \dots \sum_{S_{N-1}=-1}^{+1} e^{-K'(S_1S_3+S_3S_5+S_5S_7)} \dots = f(K)^{N/2} Z(K', N/2) . \end{aligned} \tag{7}$$

Note that we rewrote the $Z(K,N)$ we started with in terms of new function $Z(K',N/2)$ i.e. a function with parameters that describe the model with half the number of spins and a different coupling parameter $K' = J'/T$. Let us find the functions $f(K)$ and K' implied by the transformation Eq.6. It is easy to show that they are given by:

$$K' = \frac{1}{2}\log(cosh(2K)) \quad f(K) = 2cosh^{1/2}(2K) \tag{8}$$

Exercise: Derive the result Eq.8.

Notice that $K' < K$. Eq.8 represent the "RG group functions". Now, repeat the procedure to integrate out another half of spins to arrive at $Z(K'',N/4)$ then $Z(K''',N/8)$ and so on. Since $K > K' > K'' > K'''$ after many iterations the coupling parameter becomes negligibly small. Also notice that with each iteration the distance between the neighboring spins doubles in size.

We found that $K = 0$ ($J = 0$) is an attractive *fixed point* of the RG transformation in 1d model of spins. At this fixed point interaction between spins vanishes. Therefore the temperature effects will determine the emergent behavior at large distances. These thermal fluctuations will tend to align spins randomly and at long distances system is disordered.

The Ising model undergoes a phase transition between an ordered and a disordered phase in two dimensions or more. There is a nontrivial fixed point between two phases at critical $K = K_c$. At this point, changing the scale does not change physics because the system is in a critical fractal state.

2b. Relevant, marginal and irrelevant operators

We saw in the 1d Ising model that coupling strength K decreases as we perform the RG transformations towards larger distances between spins. We call such interactions irrelevant. Interactions whose strength increases as we "integrate out" d.o.f. are called relevant and marginal operators are those whose strength does not change. In d dimensions and at the classical level, operators with dimension$<$ ($>$)d are relevant (irrelevant) while operators with dimension$= d$ are marginal. Quantum corrections will change classical (engineering) dimensions of operators so that, for example, classically marginal operators can become relevant or irrelevant at the quantum level.

3. Constructing SM

The main example of the EFT in these lectures will be Standard Model of particle interactions. Let us try to understand how far we can go in building this model just using the notions of relevant, marginal and irrelevant operators above.

As we discussed above to describe the physical system at some energy scale we need to:

1. Determine relevant d.o.f. (fields). For SM, the fermionic fields will be quarks and leptons, vector ones are gauge bosons $W^{\pm}, Z, \gamma$ and gluons and finally the only scalar particle will be famous Higgs.

2. Symmetries. This is given by the semi-simple product of three gauge groups $SU(3)_{color} \times SU(2)_w \times U(1)_Y$. Gauge bosons mediate these interactions: $W_{1,2,3}$ mediate weak $SU(2)_w$ interactions, gluons mediate strong (QCD) interactions $SU(3)_{color}$ and hypercharge boson mediate the $U(1)_Y$ interactions. Due to spontaneous symmetry breaking of $SU(2)_w \times U(1)_Y \to U(1)_{EM}$ photon is a massless linear combination of the gauge boson of $U(1)_Y$ and W_3.

3. Expansion parameters are given by some mass of the SM particle (say, mass of ones of the quarks) divided by the cutoff of the SM theory. This cutoff may be Planck scale or grand unified scale or whatever is the scale up to which the SM EFT is valid.

Of course, here I just stated the results for the d.o.f. and symmetries of the SM. It took incredible amount of research to arrive at this construction through deep theoretical ideas and experimental efforts.

Now, let us try to stay agnostic and build all possible operators out of fields above consistent with the gauge symmetry of the SM classifying them according to their dimension. Remember that we are working in four dimensions (three space and one time) so that dimension$= 4$ operators will be marginal.

- Dimension-0 operator: this is just identity operator $\mathbf{1}$. We will compute coefficient of this operator in the SM later.

- Dimension-2: These are "mass-terms" and in SM this will be the Higgs mass.

- Dimension-4: marginal operators and these we will present now. They are quartic, Yukawa and gauge interactions between the SM fields.

Summarizing, our SM Lagrangian so far schematically looks like:

$$\mathcal{L} = \rho \cdot \mathbf{1} + \mathbf{m}^2 \phi^\dagger \phi + \mathcal{L}_4 + \ldots \tag{9}$$

where $\phi(x)$ is the Higgs field.

I purposefully omitted operators with dimension=1 and dimension=3. What is their role?

3a. SM Lagrangian

So now we present the SM Lagrangian writing only the dim-2 and dim-4 operators:

$$\begin{aligned}
\mathcal{L}_{SM} =& -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} tr\left(\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu}\right) - \frac{1}{2} tr\left(\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}\right) && \text{(U(1), SU(2), and SU(3) gauge terms)} \\
& + (\bar{\nu}_L, \bar{e}_L)\, \tilde{\sigma}^\mu i D_\mu \binom{\nu_L}{e_L} + \bar{e}_R \sigma^\mu i D_\mu e_R + \bar{\nu}_R \sigma^\mu i D_\mu \nu_R + \text{(h.c.)} && \text{(lepton dynamical term)} \\
& - \frac{\sqrt{2}}{v}\left[(\bar{\nu}_L, \bar{e}_L)\, \phi M^e e_R + \bar{e}_R \bar{M}^e \bar{\phi} \binom{\nu_L}{e_L}\right] && \text{(electron, muon, tauon mass term)} \\
& - \frac{\sqrt{2}}{v}\left[(-\bar{e}_L, \bar{\nu}_L)\, \phi^* M^\nu \nu_R + \bar{\nu}_R \bar{M}^\nu \phi^T \binom{-e_L}{\nu_L}\right] && \text{(neutrino mass term)} \\
& + \left(\bar{u}_L, \bar{d}_L\right) \tilde{\sigma}^\mu i D_\mu \binom{u_L}{d_L} + \bar{u}_R \sigma^\mu i D_\mu u_R + \bar{d}_R \sigma^\mu i D_\mu d_R + \text{(h.c.)} && \text{(quark dynamical term)} \\
& - \frac{\sqrt{2}}{v}\left[\left(\bar{u}_L, \bar{d}_L\right) \phi M^d d_R + \bar{d}_R \bar{M}^d \bar{\phi} \binom{u_L}{d_L}\right] && \text{(down, strange, bottom mass term)} \\
& - \frac{\sqrt{2}}{v}\left[\left(-\bar{d}_L, \bar{u}_L\right) \phi^* M^u u_R + \bar{u}_R \bar{M}^u \phi^T \binom{-d_L}{u_L}\right] && \text{(up, charm, top mass term)} \\
& + \overline{\left(D_\mu \phi\right)} D^\mu \phi - \frac{m_h^2 \left[\bar{\phi}\phi - \frac{v^2}{2}\right]^2}{2v^2} && \text{(Higgs dynamical and mass term)}
\end{aligned}$$

where (h.c.) means Hermitian conjugate of preceding terms, $\bar{\phi} = \phi^\dagger = \phi^{*T}$, and the covariant derivative operators are:

$$D_\mu \binom{\nu_L}{e_L} = \left[\partial_\mu - \frac{ig_1}{2} B_\mu + \frac{ig_2}{2} \mathbf{W}_\mu\right] \binom{\nu_L}{e_L}$$

$$D_\mu \binom{u_L}{d_L} = \left[\partial_\mu - \frac{ig_1}{6} B_\mu + \frac{ig_2}{2} \mathbf{W}_\mu + ig\mathbf{G}_\mu\right] \binom{u_L}{d_L}$$

$$D_\mu \nu_R = \partial_\mu \nu_R \qquad D_\mu e_R = \left[\partial_\mu - ig_1 B_\mu\right] e_R$$

$$D_\mu u_R = \left[\partial_\mu + \frac{i2g_1}{3} B_\mu + ig\mathbf{G}_\mu\right] u_r \qquad D_\mu d_R = \left[\partial_\mu - \frac{ig_1}{3} B_\mu + ig\mathbf{G}_\mu\right] d_R$$

$$D_\mu\phi = \left[\partial_\mu + \frac{ig_1}{2}B_\mu + \frac{ig_2}{2}\mathbf{W}_\mu\right]\phi$$

where ϕ is a 2-component complex Higgs field. Since $\mathcal{L}$ is SU(2) gauge invariant, a gauge can be chosen so ϕ has the form:

$$\phi^T = \frac{(0, \nu + h)}{\sqrt{2}} \qquad\qquad <\phi>_0^T = (\text{expectation value of } \phi) = \frac{(0,\nu)}{\sqrt{2}}$$

where ν is a real constant such that the Higgs potential $\mathcal{V}_\phi = \frac{m_h^2\left[\bar{\phi}\phi - \frac{\nu^2}{2}\right]^2}{2\nu^2}$ is minimized, and h is a residual Higgs field. B_μ, $\mathbf{W}_\mu$, and $\mathbf{G}_\mu$ are the gauge boson vector potentials, and $\mathbf{W}_\mu$ and $\mathbf{G}_\mu$ are composed of 2×2 and 3×3 traceless Hermitian matrices respectively. Their associated field tensors are:

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad \mathbf{W}_{\mu\nu} = \partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + ig_2\frac{\left(\mathbf{W}_\mu\mathbf{W}_\nu - \mathbf{W}_\nu\mathbf{W}_\mu\right)}{2}$$

$$\mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{G}_\nu - \partial_\nu \mathbf{G}_\mu + ig\left(\mathbf{G}_\mu\mathbf{G}_\nu - \mathbf{G}_\nu\mathbf{G}_\mu\right).$$

The fermions include the leptons e_R, e_L, ν_R, ν_L and quarks u_R, u_L, d_R, d_L. They all have implicit 3-component generation indices, $e_i = (e,\mu,\tau)$, $\nu_i = \left(\nu_e,\nu_\mu,\nu_\tau\right)$, $u_i = (u,c,t)$, $d_i = (d,s,b)$, which contract into the fermion mass matrices $M^e_{ij}, M^\nu_{ij}, M^u_{ij}, M^d_{ij}$, and implicit 2-component Pauli indices which contract into the Pauli matrices:

$$\sigma^\mu = \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]$$

$$\tilde{\sigma}^\mu = \left[\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3\right] \qquad tr\left(\sigma^i\right) = 0 \qquad \sigma^{\mu\dagger} = \sigma^\mu \qquad tr\left(\sigma^\mu\sigma^\nu\right) = 2\delta^{\mu\nu}$$

We have included right-handed neutrino ν_R and thus wrote the Dirac mass term for neutrino. We will discuss the issue of generating neutrino mass later.

The quarks also have implicit 3-component color indices which contract into $\mathbf{G}_\mu$. So $\mathcal{L}$ really has implicit sums over 3-component generation indices, 2-component Pauli indices, 3-component color indices in the quark terms, and 2-component $SU(2)_w$ indices.

- Dimension-5: There is unique dimension=5 Weinberg operator in the SM and we will discuss it later when we discuss the neutrino mass.

- Dimension-6: There are many dim-6 operators one can build using SM fields. These operators are generated at a new physics scale Λ, which is not known. There are eight different classes of operators: X^3, H^6, H^4D^2, X^2H^2, ψ^2H^3, ψ^2XH, ψ^2H^2D and ψ^4 in terms of their field content, where X, H, D and ψ stand for gauge field strength, Higgs field, covariant derivative and fermion field respectively. The SM EFT classifying these operators called SMEFT in the literature.

Let us stop here even though we could continue the list but clearly higher dimensional operators will be suppressed by more powers of the cutoff scale and so their effects will be smaller.

4. BSM: Addressing problems of the SM from EFT point of view

In the previous section we built all possible operators out of SM fields classifying them according to their dimension:

$$\mathcal{L} = \rho \cdot \mathbf{1} + \mathbf{m}^2 \phi^\dagger \phi + \mathcal{L}_\mathbf{4} + \mathcal{L}_\mathbf{5} + \mathcal{L}_\mathbf{6} + \cdots \tag{10}$$

Let us now discuss some of the problems related to these operators, going in the direction of increasing operator dimensions.

- Dark energy: Let us calculate the contribution to the cosmological constant (C.C) ρ, a coefficient of the unit operator, from the Higgs condensate. For example, if ϕ_{vac} is the value of the Higgs field $\phi(x)$ which minimizes the potential $V(\phi)$, then the lowest state has $T_{\mu\nu} = g_{\mu\nu} V(\phi_{vac})$, which is the classical scalar field contribution to the vacuum energy. Concretely, minimizing Higgs potential

  $$V(\phi) = -m^2 \phi^\dagger \phi + \frac{\lambda}{2} (\phi^\dagger \phi)^2, \tag{11}$$

  Higgs condensate contribution (at the classical level) to the cosmological constant is

  $$\rho_{Higgs} = -\frac{m^4}{2\lambda}. \tag{12}$$

  Exercise: Derive Eq.12.

Besides Higgs condensate, there are other contributions to the C.C., for example from the QCD vacuum, possible GUT scale physics, etc. The experimentally measured physical value of the C.C. ρ_{phys} is given by

$$\rho_{phys} \approx 10^{-47}\ \mathrm{GeV}^4\,. \tag{13}$$

The problem now is that if we use $M_H \sim m = 125\,\mathrm{GeV}$ then the corresponding value $\left|\rho_{Higgs}\right| \simeq 10^8\,\mathrm{GeV}^4$. In order to keep the QFT consistent with the observations, one has to demand that the parts contributing to the ρ_{phys} should cancel with the great accuracy. For example, adding the vacuum contribution to the C.C. ρ_{vac}, which we can always add to the Lagrangian, the ρ_{vac} and ρ_{Higgs} should cancel with the precision of 55 decimal orders. This is the C.C. fine-tuning problem.

- Hierarchy problem: Thinking of SM as an EFT with the cutoff scale Λ, the Higgs mass term (dim-2 operator) is naturally expected to have a form $\Lambda^2\phi^\dagger\phi$. The expected quadratic Λ^2 dependence of the coefficient leads to the so-called "hierarchy problem": the Higgs mass gets a correction of order $\Lambda \gg$ electroweak scale. Indeed, from the sample diagram in Fig.2 coming from some hypothetical

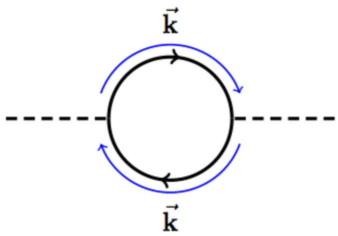

Figure 2: Higgs mass correction

Yukawa interaction of some fermion with the Higgs $y\bar{f}f\phi$, dimensional analysis suggests that

$$m_\phi^2 \sim y^2 \int_0^\Lambda d^4k \times \frac{1}{k} \times \frac{1}{k} \sim \Lambda^2. \tag{14}$$

Notice however that the sensitivity of m_ϕ on Λ follows because we are computing low-energy observable (Higgs mass) in terms of remnant of high-energy parameters represented by Λ which is not in the spirit of the EFT where we are suppose to integrate out high energy physics. Nevertheless, many extensions of the SM were motivated by this problem among which are SUSY, extra-dimensional models, technicolor and composite Higgs. In SUSY there is a cancellation between fermions and bosons protecting corrections to Higgs mass. In technicolor and composite Higgs models, Higgs is a composite particle (similar to mesons in QCD) and so as we reach compositeness scale we have to change description. Extra-dimensional models are conceptually similar to technicolor and composite Higgs models via holographic AdS/CFT correspondence.

- vacuum instability (dimension-4 operator): The analysis of the vacuum stability requires the knowledge of the effective potential of the model at hand. The standard model effective potential is known up to two loops. For large field values $\phi \gg v = 246$ GeV, we may neglect the Higgs mass term because at high energy it becomes irrelevant and the potential is very well approximated by its RG-improved tree-level expression,

$$V(\phi) = -m^2\phi^\dagger\phi + \lambda(\mu)(\phi^\dagger\phi)^2 \approx \lambda(\mu)(\phi^\dagger\phi)^2 \,, \tag{15}$$

with RG scale $\mu = \mathcal{O}(\phi)$ itself. Therefore if one is simply interested in the condition of absolute stability of the potential, it is possible to study the RG evolution of λ and determine the largest scale $\Lambda < M_{pl}$, with M_{pl} the Planck scale, above which the coupling becomes negative. The RG evolution of the Higgs quartic coupling in the SM is shown on the right in Fig.3 and we observe that the coupling becomes negative around 10^{10}GeV.

To illustrate how BSM physics can solve this problem we postulate an additional complex singlet scalar S and to study its effect on the stability of the Higgs potential, we consider a combined tree-level scalar potential for both scalars of the form

$$V_0 = \lambda\left(\phi^\dagger\phi - v^2/2\right)^2 + \lambda_S\left(S^\dagger S - w^2/2\right)^2 + 2\lambda_{\phi S}\left(\phi^\dagger\phi - v^2/2\right)\left(S^\dagger S - w^2/2\right) \,. \tag{16}$$

This model leads to the tree-level effect through which the new singlet can affect the stability bound. Let us consider the limit in which the mass of S, $M_S \sim w$ is much larger than the Higgs mass ($w^2 \gg v^2$). At the scale M_S we can "integrate out" the field S using its equation of motion :

$$S^\dagger S \approx \frac{w^2}{2} - \frac{\lambda_{\phi S}}{\lambda_S}\left(\phi^\dagger\phi - \frac{v^2}{2}\right) \tag{17}$$

where we neglected the kinetic term for S since we are building an EFT for momenta $p \ll M_S$. Plugging eq. (17) in V_0, we obtain the effective potential

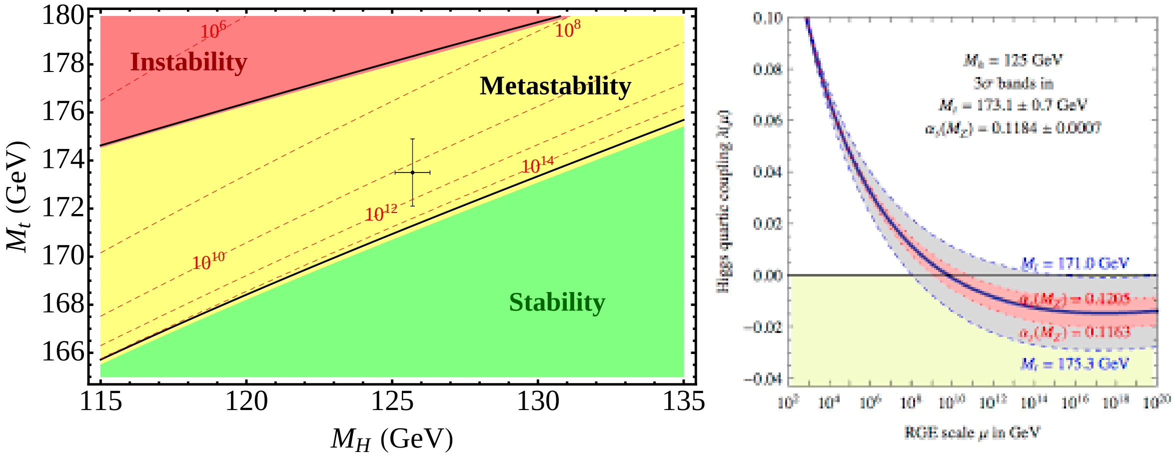

Figure 3: (Left) Standard model stability analysis based on the effective standard model Higgs quartic coupling. The red region indicates instability, the yellow metastability and the green absolute stability. The point with error bars shows the experimental values of the top and Higgs masses. The red dashed lines show the value in GeV at which λ crosses zero (Right) Running of the Higgs quartic coupling in the SM.

below the scale M_S:

$$V_{\text{eff}} = \lambda' \left(H^\dagger H - \frac{v^2}{2} \right)^2, \qquad \lambda' = \lambda - \frac{\lambda_{HS}^2}{\lambda_S} \,. \tag{18}$$

This shows that the matching condition at the scale $p = M_S$ of the Higgs quartic coupling gives a tree-level shift, $\delta\lambda \equiv \lambda_{HS}^2/\lambda_S$, as we go from λ just above M_S to λ' just below M_S. Since $\lambda_S > 0$ is needed for the stability of the potential in the S direction for large S values, we see that $\lambda' < \lambda > 0$ which means that starting from the electroweak scale from which we have the running of λ', before we approach the instability scale of around 10^{10}GeV we should have a threshold effect increasing the effective quartic coupling to λ' and thus avoiding instability.

Finally, let me comment that RG running of the Higgs quartic coupling in the SM is very sensitive to the value of the top quark mass. This is illustrated on the left plot in Fig.3 where we see that changing the value of the top mass by $\mathcal{O}(1GeV)$ may bring us back to the (green) stability region.

- Neutrino masses: Neutrinos are electrically neutral, and so can have either Majorana type or Dirac type mass terms. The existence of a Dirac mass term would necessitate the existence of right-handed neutrinos. In the minimal standard model without right-handed neutrino, there is an effective dimension-5 Weinberg operator which generates Majorana neutrino masses

$$\Lambda^{-1}\phi^0\phi^0\nu_L\nu_L, \tag{19}$$

All models of neutrino mass and mixing (which have the same light particle content as the minimal standard model) can be summarized by this operator. Different models are merely different realizations of this operator. In the following I will show that it has only three *tree-level* realizations. In addition, it also has three 1-loop realizations of radiative neutrino masses.

To obtain the effective operator Eq.19 at the tree level, using only renormalizable interactions, we show now that there are only three ways. To start with recall that in the SM, left-handed neutrino and neutral component of the Higgs are parts of the left-handed doublets $\psi = \binom{\nu_L}{e_L}$ and $\Phi = (\phi^+\phi^0)$. So using the group theory multiplication $2\otimes 2 = 3\oplus 1$ we have the following possibilities:

(I) $\psi\times\Phi \sim (\phi^0\nu_L - \phi^+e_L)$ form a fermion singlet,
(II) $\psi\times\psi$ form a scalar triplet with one of the components $\nu_L\nu_L$,
(III) $\psi\times\Phi$ form a fermion triplet with one of the components $(\phi^0\nu_L + \phi^+e_L)$.

In each case, we always generate the operator Eq.19 among with the other interactions which altogether schematically look like:

$$\text{(I)}\qquad \Lambda^{-1}(\phi^0\nu_L - \phi^+e_L)(\phi^0\nu_L - \phi^+e_L), \tag{20}$$

$$\text{(II)}\qquad \Lambda^{-1}[\phi^0\phi^0\nu_L\nu_L - \phi^+\phi^0(\nu_Le_L + e_L\nu_L) + \phi^+\phi^+e_Le_L], \tag{21}$$

$$\text{(III)}\qquad \Lambda^{-1}[(\phi^0\nu_L + \phi^+e_L)(\phi^0\nu_L + \phi^+e_L) - 2\phi^+\nu_L\phi^0e_L - 2\phi^0e_L\phi^+\nu_L]. \tag{22}$$

The intermediate heavy particle in the first case is clearly a fermion singlet (right-handed neutrino) and this is well-known type-I seesaw mechanism. In the second case intermediate heavy particle is a heavy scalar triplet $\xi = (\xi^{++}, \xi^+, \xi^0)$ realizing type-II seesaw. Finally, we have a heavy Majorana fermion triplet $(\Sigma^+, \Sigma^0, \Sigma^-)$ and obtain type-III seesaw mechanism. Clearly each seesaw mechanism has its own unique implications about physics beyond the standard model.

Note that the singlet combination of ψ_i and ψ_j is $\nu_il_j - l_i\nu_j$ and does not generate Eq.19.

- Dark matter: Particle physics proposes a plausible and effective solution to this problem in terms of an electrically neutral and weakly interacting massive particle that is stable at cosmological scales. DM particles are predicted by many extensions of the SM, including the well motivated ones that address other important theoretical or experimental issues above. Because of the large number of possibilities for DM candidates, it has become customary and quite useful to consider EFT approaches, which allow to study in a model-independent manner the phenomenology of these particles. It is typically assumed that the new state is either a scalar, a vector or a fermion. In order to work with a manageable theory some restrictions on the DM sector need to be imposed. Possible assumptions are:

  1. In order to stabilize the DM particle, we impose a discrete Z_2 symmetry.
  2. The field content of the theory is given by the SM one, including the Higgs doublet, and a single extra multiplet X that belongs to some irreducible representation of the SM gauge group $G_{SM} = SU(3)_{color} \times SU(2)_w \times U(1)_Y$. Under the Lorentz group, X transforms either as a scalar, a spinor or a vector. All SM fields are even under Z_2, while X is odd.

  The DM-EFT Lagrangian can be schematically written as:

  $$L = L_{SM} + \sum_{dim \leq 6} c_i O_i \tag{23}$$

  where operators O_i include SM and DM field X.

- Flavour problem (dim-6 operators). SM does not explain the fermion masses and their mixing angles. These parameters are very different. Also, why there are 3 generations of quarks and leptons? From the Lagrangian point of view this problem connected to Yukawa coupling of the fermions to the Higgs and Weinberg operator in neutrino sector:

  $$Y_{ij} \psi_L^i \psi_R^j \phi + \Lambda^{-1} \phi^0 \phi^0 \nu_L^i \nu_L^j \tag{24}$$

  where i, j are indices of the SM three generations. The main goal of the flavor physics model building is the identify the symmetries and symmetry-breaking patterns beyond those present in the SM which would explain fermion masses and mixing angles. The dim-6 operators build from the SM fields are very important as they lead, for example to $B - \bar{B}$ mixing: $(\bar{d}_L \gamma_\mu b_L)(\bar{d}_L \gamma^\mu b_L)$.

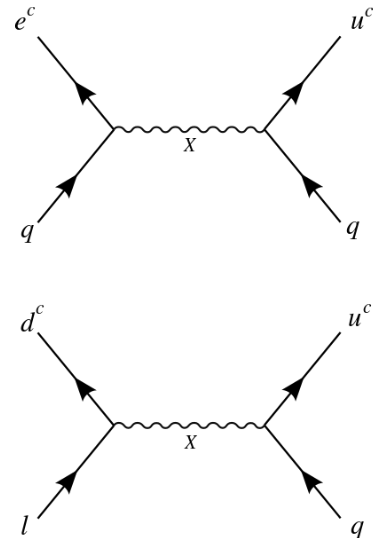

Figure 4: Proton decay mediated by the X boson $(3,2)_{-5/6}$ in $SU(5)$ GUT.

- Proton decay (dim-6 operators) In the SM, the proton is stable because it is the lightest baryon and baryon number (quark number) is conserved. Many BSM models explicitly break the baryon number symmetry, allowing protons to decay. For example, in grand unified theories (GUTs) it can decay via the new X vector bosons (see fig.4) . Integrating out heavy X bosons (we will illustrate this process of integrating out a heavy vector particle again later, when we will integrate out the W and Z bosons to obtain the Fermi theory) we are left with dimension-6 operators $\frac{\bar{e^c}\bar{u^c}qq}{\Lambda^2_{GUT}}$ and $\frac{\bar{u^c}\bar{d^c}ql}{\Lambda^2_{GUT}}$. All of these operators violate both baryon number (B) and lepton number (L) conservation but not the combination $B-L$. This operators mediate the decay of the proton to positron and neutral pion: $p \to e^+ + \pi^0$. Breaking of the baryon number symmetry is also important to explain the matter-antimatter asymmetry as we observe in our Universe.

In summary, notice that puzzles related to relevant dim<4 operators lead to "hierarchy" problems due to the fact that related observables expected to pick up contribution proportional to the cutoff scale Λ whereas experimentally they are at the low scale. Problems related to irrelevant dim>4 operators have opposite, "decoupling" nature. If we take a cutoff scale infinitely large, the effects of these operators will be unobservable.

5. EFTs in general

Having discussed SM EFT and it problems we now discuss how to construct EFT in general. There are two ways to do it:

1. Top-down: in this approach we integrate out heavy particles and match onto a low energy theory. We find new operators and new low energy constants.

2. Bottom-up: here you write down the most general possible operators/interactions consistent with symmetries. Couplings of your EFT will be unknown but can be fitted to experiment.

5a. Examples of EFTs

Let me give some examples of EFTs keeping in mind that the list is not exhaustive. First three examples will be examples of top-down approach while the last three will be bottom-up.

- Heavy quark effective theory (HQET): describes the low-energy dynamics of hadrons (composite particles built from quarks and thus interacting via QCD interactions) containing a heavy quark. The theory is usually applied to hadrons containing b and c quarks. The expansion parameter is Λ_{QCD}/m_Q, where $m_Q = m_b, m_c$ is the mass of the heavy quark and Λ_{QCD} is dynamical scale generated in QCD theory. Since hadrons are build from quarks which interact with gluons, coefficients of this EFT also have an expansion in powers of $\alpha_s(m_Q)/(4\pi)$ where $\alpha_s(m_Q) \sim g^2(m_Q)$ is evaluated at the heavy quark mass and g is the coupling constant of QCD Lagrangian (see SM Lagrangian in Sec.3a). The matching from QCD to HQET can be done in perturbation theory, since $\alpha_s(m_Q)/(4\pi)$ is small, for example $\alpha_s(m_b) \sim 0.22$, $\alpha_s(m_b)/(4\pi) \sim 0.02$.

- Fermi theory of weak interactions: this is EFT for weak interactions at energies below the W and Z masses. Expansion parameter is p/M_W where p is the momenta of a particle in the weak decay (which is related to b-quark mass in a b-decay, for example). We start with the amplitude for the $b \to c$ decay as our simple example:

$$A = \left(\frac{-ig}{2\sqrt{2}}\right)^2 V_{cb}\bar{c}\gamma_\mu(1-\gamma_5)b \quad \bar{\ell}\gamma_\nu(1-\gamma_5)\nu_\ell\left(\frac{-ig^{\mu\nu}}{p^2 - M_W^2}\right) \tag{25}$$

For low momentum transfers, $p \ll M_W$, we can expand the W propagator:

$$\frac{1}{p^2 - M_W^2} = -\frac{1}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \dots\right). \tag{26}$$

Keeping only the first term we obtain the local Lagrangian:

$$A = \left(\frac{-ig}{2\sqrt{2}}\right)^2 \frac{i}{M_W^2} V_{cb}\bar{c}\gamma_\mu(1-\gamma_5)b \quad \bar{\ell}\gamma^\mu(1-\gamma_5)\nu_\ell + \mathcal{O}\left(\frac{1}{M_W^4}\right) \tag{27}$$

This EFT no longer has dynamical W bosons, and the effect of W exchange in the SM has been included via this dimension-six four-fermion operator.

- SM below EW scale: Below the electroweak scale, one can write a low energy effective theory with quark and lepton fields, and only QCD and QED gauge fields. Since SU(2) gauge invariance is no longer a requirement, there are several new types of operators:
  - There are $\nu_L\nu_L$ operators which give a Majorana neutrino mass for left-handed neutrinos as we discussed in the previous section.
  - There are dimension-five dipole operators, e.g. $\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$
  - There are X^3 and ψ^4 operators as in SMEFT classification, but operators containing Higgs field H are no longer present.
  - There are many four-fermion interactions e.g. $\bar{\psi}\psi\nu_L\nu_L$.
- Chiral perturbation theory (χPT): describes the interactions of pions and nucleons at low momentum transfer p in a bottom-up approach. It is not possible to analytically compute the matching onto the EFT, since the matching is non-perturbative. The two theories, QCD and χPT, are not written in terms of the same fields. The QCD Lagrangian has quark and gluon fields, whereas χPT has meson and baryon fields. The parameters of the chiral Lagrangian are fit to experiment. The expansion parameter of χPT is p/Λ_χ, where $\Lambda_\chi \sim 1$ GeV is referred to as the scale of chiral symmetry breaking
- SMEFT: is the EFT constructed out of SM fields, and is used to analyze deviations from the SM, and search for BSM physics. The higher dimension operators in SMEFT are generated at a new physics scale Λ, which is not known. Unique dim-5 operator is Weinberg operator while at dimension-six level there are eight different operator classes of operators: X^3, H^6, H^4D^2, X^2H^2, ψ^2H^3, ψ^2XH, ψ^2H^2D and ψ^4 as we discussed above.

- General relativity: The field relevant for gravity is the metric, $g_{\mu\nu}$ (whose matrix inverse is denoted $g^{\mu\nu}$). For applications on macroscopic scales we use the most general effective lagrangian consistent with general covariance:

$$L_{grav} = \sqrt{-g}\left(\frac{1}{2}M_p R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \frac{c_4}{m^2}R^3 + \dots\right) \tag{28}$$

  The term linear in R is the usual Einstein-Hilbert action, with M_p denoting the usual Planck mass. The remaining effective couplings c_k are dimensionless and unknown a priori. The scale m stands for the lightest particle (say, the electron) which have been integrated out to obtain this EFT.

5b. Top-down approach: Example of tree-level matching

Before we conclude these lectures let me give one more example of tree-level matching technique from the full theory to the effective one in the top-down approach. We consider $U(1)$ global theory:

$$L = \partial_\mu \phi^* \partial^\mu \phi - \frac{\lambda\left[\bar{\phi}\phi - v^2\right]^2}{4} \tag{29}$$

which has a $U(1)$ global symmetry $\phi \to e^{i\omega}\phi$ for $\partial_\mu \omega = 0$. Redefine $\phi \equiv \chi e^{i\theta}$ to obtain:

$$L = \partial_\mu \chi \partial^\mu \chi + \chi^2 \partial_\mu \theta \partial^\mu \theta - \frac{\lambda\left[\chi^2 - v^2\right]^2}{4} \tag{30}$$

The structure of the theory is now transparent. We see that we have two fields: θ, which is massless, and χ with mass $M = \sqrt{\lambda}v$. As usual, we have to shift the χ field and so we define new fields: $\chi \to v + \frac{\psi}{\sqrt{2}}$ and $\theta = \frac{\xi}{\sqrt{2}v}$. Our Lagrangian becomes:

$$L = \partial_\mu \psi \partial^\mu \psi + \frac{1}{2}(1 + \frac{\psi}{\sqrt{2}v})^2 \partial_\mu \xi \partial^\mu \xi - \frac{\lambda\left[\sqrt{2}v\psi + \psi^2/2\right]^2}{4} \tag{31}$$

To construct our EFT we will need to choose some observable to calculate. Let us use $\xi\xi \to \xi\xi$ scattering, which occurs at tree-level in the full theory through the s, t and u channel processes, all formed from the $\psi \partial_\mu \xi \partial^\mu \xi$ vertex. We will assign momenta to the external lines as follows: p and q to incoming lines, p' and q' to outgoing. Then the amplitude in the full theory is :

$$A_{full} = \frac{2}{v^2}\left(\frac{(p \cdot q)^2}{(p+q)^2 + M^2} + \frac{(p \cdot p')^2}{(p-p')^2 + M^2} + \frac{(p \cdot q')^2}{(p-q')^2 + M^2}\right) \tag{32}$$

Exercise: Derive the result Eq.32.

To order $\mathcal{O}(1/M^2)$ we simply have:

$$A_{LO} = \frac{2}{v^2M^2}\left((p\cdot q)^2 + (p\cdot p')^2 + (p\cdot q')^2\right). \tag{33}$$

We now need to construct the effective Lagrangian for ξ, and calculate the same amplitude using this EFT. We have:

$$L_{eff} = \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - a(\partial_\mu\xi\partial^\mu\xi)^2 + \ldots \tag{34}$$

where a is unknown coefficient. Using this effective Lagrangian we obtain for the amplitude:

$$A_{eff} = 8a\left((p\cdot q)^2 + (p\cdot p')^2 + (p\cdot q')^2\right), \tag{35}$$

so that comparing we obtain $a = \frac{1}{4v^2M^2}$. By matching the coefficient in the effective theory to that produced (approximately) by the full theory we embeded information about the heavy field ψ, which is not itself part of the EFT, into our results.

References

A. V. Manohar, doi:10.1093/oso/9780198855743.003.0002 [arXiv:1804.05863 [hep-ph]].

D. B. Kaplan, [arXiv:nucl-th/9506035 [nucl-th]].

Youtube lectures series by Iain Stewart: https://www.youtube.com/watch?v=WB8r7CU7clk