Noncommutative Field and Gauge Theory

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Outline

- Motivations
- Quantum Mechanics as the prototypical NC geometry
- The "classical picture" of NC theory of gauge and matter fields
  - The differential calculus
  - Scalar field theory
  - Gauge theory
  - NCQED on Moyal space
  - NCQED on $\mathbb{R}^3_\lambda$
  - Symmetries: Covariance vs Twist
- Other approaches
  - Twist
  - Noncommutative field theory from angular twist
- Open problems
- Perspectives

Motivations for NCG

- Space-time noncommutativity as a signature of Quantum gravity
- Gedanken experiments which challenge the Riemannian structure of space-time at scales where both quantum mechanics and general relativity are relevant [Bronstein '36, Doplicher-Fredenhagen-Roberts '94]
- Regularization of QFT in the UV regime [Heisenberg '30, Snyder '47]
- Space-time discreteness emerging from different models of quantum gravity [e.g. LQG where the spectrum of area and volume operators is discrete [Ashtekhar '01]; Group Field Theory [Oriti '06]]
- Low energy regimes of strings in the presence of a background field B [Seiberg-Witten '99] already in [Witten '86] in the context of string field theory

DFR argument

Attempts to localize with extreme precision cause gravitational collapse so that spacetime below the Planck scale $\lambda_P = (\frac{G\hbar}{c^3})^1/2 \simeq 1.6 \times 10^{-33}$ has no operational meaning

- Heisenberg uncertainty principle: measuring the spacetime coordinate of a particle with great accuracy, a, causes an uncertainty in momentum of order $\frac{1}{a}$ (in natural units)
  $\Longrightarrow$ an energy of order $1/a$ is transmitted to the system and concentrated at some time in the localization region;
- General Relativity: the associated energy momentum tensor $T_{\mu\nu}$ generates a gravitational field solution of Einstein's equation for Minkowski metric

$$R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu} = 8\pi T_{\mu\nu}$$

- the smaller the uncertainty Δx_μ the stronger will be the gravitational field generated
- as $\Delta x \to 0$ the field becomes so strong as to prevent light or other signals from leaving the region
  $\Longrightarrow$ operational meaning can no longer be attached to the localization

By requiring that no blackhole is produced DFT infer that the Δx_μ cannot be made simultaneously arbitrary small
$\Longrightarrow$ Uncertainty relations among coordinates emerge

$$\Delta x_\mu \Delta x_\nu \geq \lambda_P^2$$

Learning from quantum mechanics: uncertainty relations can be explained by admitting that coordinates be noncommuting

$$[x_\mu, x_\nu] \neq 0$$

$\Longrightarrow$ Noncommutative, or Quantum Spacetime

- Spacetime observables (what where smooth functions on classical spacetime) become operators
- States (what where points of classical spacetime, namely "evaluation maps" on the space of classical observables $\omega : f \to f(\omega)$ become "quantum evaluation maps"

The simplest instance of NCG is Quantum Mechanics

- Classical Phase-Space as a differentiable manifold is lost
- Classical observables $\longrightarrow$ Operators
- Phase space coordinate functions $q, p \longrightarrow$ noncommuting operators
- The uncertainty principle $\Delta q \Delta p \geq \frac{\hbar}{2}$ implies the existence of a minimal area in phase space
- classical states (points on phase space) $\longrightarrow$ vectors in Hilbert space

The Wigner-Weyl-Moyal approach

QM can be described in a classical-like setting

- Operators $\longrightarrow$ Symbols (functions on $T\mathbb{R}^n$)

$$\hat{A} \longrightarrow f_{\hat{A}}(q,p) = \mathrm{Tr}\, \hat{A}\hat{\Omega}(q,p)$$

with

$$\hat{\Omega}(q,p) = \int \mathrm{d}\eta \mathrm{d}\xi\, e^{i(\eta\cdot\hat{P}+\xi\cdot\hat{Q})} e^{-i(\eta\cdot p+\xi\cdot q)} \qquad (\hbar = 1)$$

the Weyl-Stratonovich operator or simply quantizer (dequantizer) operator

- state $\rho \longrightarrow W_{\hat{\rho}}(q,p) = \mathrm{Tr}\, \hat{\rho}\hat{\Omega}(q,p)$ the Wigner function
- operator product $\longrightarrow$ star product $\star$

$$\hat{A}\cdot\hat{B} \longrightarrow f_{\hat{A}} \star f_{\hat{B}}(q,p) = \mathrm{Tr}\, (\hat{A}\hat{B}\hat{\Omega}(q,p))$$

- this yields in particular $q \star p - p \star q = i$
- $(\mathcal{F}(T^*R^n), \star_\hbar)$ prototype NC algebra

Standard picture of gauge and matter fields-Review

- $M = \mathbb{R}^4$ space–time
- matter fields describing particles are vector fields, namely maps from space–time to vectors: such maps are formalised as sections of vector bundles
- what kind of vectors: they carry a representation of the gauge group determined by the interaction they feel; physics says that the representation is the fundamental one (the group characterises the kind of vector bundle)

  electrically charged matter fields are 1-dim complex vector fields (wrt the group $U(1)$

  fields carrying a weak charge are two-dim complex vector fields (wrt the group $SU(2)$

  fields carrying strong charge are three-dim complex vector fields (wrt the group $SU(3)$

  Namely matter fields are organised in multiplets, of dimension depending on the interaction. They can carry more that one representation (e.g. the electron is a 1-dim complex vector field under $U(1)$ but part of a doublet, with its neutrino under $SU(2)$

- gauge fields, A_μ, $F_{\mu\nu}$ represent the radiation fields, namely the bosons which mediate the interactions (electromagnetic, weak, strong, *gravitational in some sense*); they are Lie algebra valued components of forms

$$A = A^a_\mu dx^\mu \tau_a \quad \tau_a \in \mathfrak{g} \qquad \mathfrak{g} = \mathfrak{u}(1), \mathfrak{su}(2), \mathfrak{su}(3)$$
$$F = F^a_{\mu\nu} dx^\mu \wedge dx^\nu \tau_a \quad F^a_{\mu\nu} = \partial_\mu A_\nu - \partial A_\mu - i A^b_\mu A^c_\nu f^a_{bc}$$

  More formally: A is a Lie algebra valued connection one-form; F is the curvature two-form of A: $F = DA = dA + A \wedge A$

- gauge group: smooth maps from space–time to some unitary Lie group

$$\widehat{G} = \{g : x \in \mathbb{R}^4 \to g(x) \in G\}$$

- radiation fields are responsible for the modification of derivatives of vector fields: a connection is needed (think in analogy with gravitational field, which curves space–time)

$$\partial_\mu \boldsymbol{\psi} \to \nabla_\mu \boldsymbol{\psi}$$

$\boldsymbol{\psi} = \mathbf{e}_i \psi^i$, $\mathbf{e}_i$ are basis sections; $i = 1, ..n$ runs over the dimension of the representation

$$\nabla_\mu \boldsymbol{\psi} = \mathbf{e}_i \partial_\mu \psi^i + \nabla_\mu(\mathbf{e}_i)\psi^i$$

∇ : *vectors* $\to$ *vectors* is the (Koszul) connection, namely how derivatives should be performed when acting no longer on scalars, but on vectors

$$\nabla(\mathbf{e}_i) = -i(A)_i^j \mathbf{e}_j \longrightarrow \nabla_\mu(\mathbf{e}_i) = -i(A(\partial_\mu))_i^j \mathbf{e}_j$$

$A(\partial_\mu) = A_\mu$ is the connection one form component in space-time. It is also a $n \times n$ matrix, n the dimension of the representation
$F(\partial_\mu, \partial_\nu)\boldsymbol{\psi} = [\nabla_\mu, \nabla_\nu]\boldsymbol{\psi}$ is the field strength; It is a two-form component in space–time. It is also a $n \times n$ matrix, n the dimension of the representation

NC theory of gauge and matter fields

[Connes, Dubois-Violette, Grosse, Madore, Wess, Chaichian, Gracia-Bondia, Jurco, Schupp, Schraml, Szabo, Sheikh-Jabbari, Wallet, Wulkenhaar, Steinacker, Lizzi, Buric, Radovanovic, Presnajder, Chepelev, Roiban, Seiberg, Witten, van Raamsdonk, Alvarez- Gaumé, Rivasseau, Aschieri, Zoupanos, Dimitrijevic, Jonke ....]

The"classical picture" of noncommutative gauge and matter fields is described in terms of

- a noncommutative algebra $(\mathcal{A}, \star)$ representing space-time (it replaces $\mathcal{F}(M)$)
- a right $\mathcal{A}$-module, $\mathbb{M}$, representing matter fields (it replaces vector bundles)
- a group of unitary automorphisms of $\mathbb{M}$ acting on fields from the left, representing gauge transformations.

The dynamics of fields is described by means of a natural differential calculus based on derivations of the NC algebra;

The gauge connection is the standard noncommutative analogue of the Koszul connection.

Therefore, the first problem to address is to have a well defined differential calculus, namely, an algebra of $\star$-derivations of $\mathcal{A}$ such that

$$D_a(f \star g) = D_a f \star g + f \star D_a g$$

Differential calculus

Given the star product of fields in the form

$$f \star g = f \cdot g + \frac{i}{2}\,\Theta^{ab}(x)\,\partial_a f \partial_b g + \dots$$

ordinary derivations violate the Leibniz rule,

$$\partial_c(f \star g) = (\partial_c f) \star g + f \star (\partial_c g) + \frac{i}{2}\,\partial_c \Theta^{ab}(x)\,\partial_a f \partial_b g + \dots$$

unless Θ is constant $\Longrightarrow$ star derivations are realised by star commutators

$$D_a f = (\Theta^{-1})_{ab}[x^b, f]_\star \stackrel{\Theta \to 0}{\longrightarrow} \partial_a f$$

Lie algebra type star products, $[x^j, x^k]_\star = c_l^{jk} x^l$ do admit a generalisation according to

$$D_j f = k[x^j, f]_\star$$

with k a suitable dimensionful constant, but the limit, $\Theta \to 0$, does not yield the standard commutative result.

Alternatively, one can use twisted differential calculus for those NC algebras whose star product is defined in terms of a twist.

Summarising: ordinary derivations in general violate the Leibniz rule, whereas twisted or star derivations might not reproduce the correct commutative limit.

The problem is not new

Derivations based differential calculus

[Dubois-Violette, Michor, Madore, Masson, Wallet...] It generalises the algebraic description of standard differential calculus to the NC case. In the commutative case vector fields are identified with derivations of $\mathcal{F}(M)$, one-forms and the exterior derivative d are defined by duality

$df(X) = X(f); \alpha = g \cdot df; d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$

$d^2 f(X, Y) = X(df(Y)) - Y(df(X)) - df([X, Y]) =$

$X(Y(f)) - Y(X(f)) - [X, Y](f) = 0$

Higher forms are constructed analogously.

Thus, to define a differential calculus on a noncommutative algebra, $\mathcal{A}$ we need a Lie algebra $\mathcal{L}$ and a representation of $\mathcal{L}$ in terms of derivations of $\mathcal{A}$. Derivations, have to be independent and sufficient (A set of derivations is said to be sufficient when the only elements which are annihilated by all of them are in the centre of the algebra). That is, we need $\mathcal{L}$, ρ such that

$$\rho(X)\,(f \star g) = (\rho(X) f) \star g + f \star (\rho(X) g)\,, \quad X \in \mathcal{L}, \quad f, g \in \mathcal{A}$$

Assuming such structures are given, the first step for the construction of a differential calculus is the identification of zero forms with the algebra itself $\Omega^0 = \mathcal{A}$.

Then the exterior derivative is implicitly defined by $df(X) = \rho(X)f$ It automatically verifies the Leibniz rule because $\rho(X)$ are $\star$-derivations

$$d(f \star g)(X) = (\rho(X)f) \star g + f \star (\rho(X)g)$$

moreover $\quad d^2 = 0$
because the $\star$-derivations close a Lie algebra. The second step consists in defining Ω^1 as a left $\mathcal{A}$ module that is

$$gdf(X) = g \star (\rho(X)f)$$

Because of noncommutativity, the wedge product

$$df \wedge_\star dg(X, Y) = df(X) \star dg(Y) - df(Y) \star dg(X)$$

is not anticommutative $df \wedge_\star dg \neq -dg \wedge_\star df$.
In a similar way to Ω^1, Ω^2 is defined as a left $\mathcal{A}$ module, $\omega = f \star dg \wedge_\star dh$
Higher Ω^p are built analogously.
Derivations have to be independent: namely no functions belonging to the center of the algebra exist s.t. $f_\mu X_\mu = 0$ and sufficient, namely if $\alpha(X_\mu) = 0 \ \forall \mu \to \alpha$ is central

Moyal space:

It is the simplest noncommutative space, modelled on the phase space of quantum mechanics:

First, go to dual description in terms of algebra of functions on classical phase space

Quantize (make it "noncommutative phase space")

Phase space is not a smooth manifold anymore

Noncommutativity can be described in terms of a star product: quantum mechanics in the Moyal approach

- Do the same for space-time $\rightarrow [\hat{x}_i, \hat{x}_j] = i\theta_{ij}$

  - θ constant

  - replace with an algebra of functions on space-time (assume it even-dim.), with noncommutative star product. For coordinate functions

$$x_i \star x_j - x_j \star x_i = i\theta_{ij}$$

The Moyal algebra $\mathcal{A} = \mathbb{R}^{2n}_\theta$

$(\mathcal{F}(R^{2n}), \star_\theta) =: \mathbb{R}^{2n}_\theta$ is the Moyal algebra

- Technically the star product is defined for Schwartz functions $\mathcal{S}(\mathbb{R}^{2n})$

$$f \star g(x) = \frac{1}{(2\pi)^{2n}} \int f(x + \frac{1}{2}\Theta u) g(x+v) e^{iu\cdot v}$$

  Θ is block diagonal, antisymmetric with θ real.

$$\Theta = \theta \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & .. \end{pmatrix}$$

- Extended $\Longrightarrow \mathbb{R}^{2n}_\theta$ is unital and involutive under complex conjugation. It contains $\mathcal{S}$, polynomials, constants [Varilly, Gracia-Bondia IJMP '89, Soloviev arxiv-1012.0669]

$$f \star_\theta g(x) = \exp\left(\frac{i}{2}\Theta^{\mu\nu} \frac{\partial}{\partial u^\mu} \frac{\partial}{\partial v^\mu}\right) f(u) g(v)|_{u=v=x}$$

$$[x^\mu, x^\nu]_{\star_\theta} = i\theta^{\mu\nu}$$

which describes space-time noncommutativity and implies the presence of a minimal area $\simeq \theta$

The differential calculus over the Moyal algebra

Minimal derivation based differential calculus [DuboisViolette-Masson-Wallet, Marmo-V.-Zampini]

As a minimal Lie algebra we can choose translations $\{P_\mu\}$ (but we could choose a bigger algebra: the largest algebra of derivations being $\mathfrak{isp}(4,\mathbb{R})$)

$$\rho(P_\mu) := \partial_\mu = -i\theta^{-1}_{\mu\nu}[x^\nu, \cdot]_\star$$

generate the minimal Lie algebra of derivations of $\mathbb{R}^{2n}_\theta$

These are

- inner
- not a left module over $\mathbb{R}^{2n}_\theta$, but only over the center of the algebra because

$$f \star \partial_\mu(g \star h) \neq f \star \partial_\mu g \star h + g \star f \star \partial_\mu h$$

- d, i_{P_μ} defined algebraically,

$$df(P_\mu) = P_\mu(f) = -i\theta^{-1}_{\mu\nu}[x^\nu, f]_\star,$$

$$i_{P_\mu}\omega(P_\nu) = \omega(P_\mu, P_\nu) = f \star (dg(P_\mu) \star dh(P_\nu) - dg(P_\nu) \star dh(P_\mu));$$

Integration

$$\int f \star g = \int g \star f = \int f \cdot g$$

$\Longrightarrow$ the integral is a trace

The scalar action functional

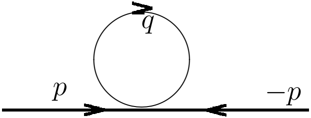
Once we have a differential calculus and an integral we can make sense of the Euclidean action functional

$$S[\varphi] = \int_{\mathbb{R}^4} D_\mu \varphi \star D^\mu \varphi + m^2 \varphi^{\star 2} + \frac{\lambda}{4!} \varphi^{\star 4}$$

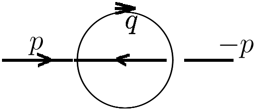
where $D_\mu \to \partial_\mu$ are the star-derivations above.
Since the product is closed, the free action is the same as the undeformed theory, as well as the tree level propagator. But the 4-vertex is deformed. In momentum space

$$\Delta^{(0)} = \frac{1}{p^2 + m^2}, \qquad V_\star = -i \frac{\lambda}{4!} \delta^3 \left(\sum_{a=1}^{4} k_a \right) \prod_{a<b} \exp(-\frac{i}{2} \theta^{ij} k_{ai} k_{bj})$$

Exercise: Compute the one-loop corrections to the propagator



(a)



(b)

$$\Delta_{pl}^{(1)} = \frac{1}{3} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m^2} \qquad \Delta_{np}^{(1)} = \frac{1}{6} \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq \wedge p}}{q^2+m^2}$$

UV/IR mixing

[$Minwalla-VanRaamsdonk-Seiberg$, $Chepelev-Roiban(2000)$]

in D=4 $\Pi^{(1)}_{np} = \frac{C_1}{(\theta p)^2} + m^2 C_2 \log(\theta p)^2 + F(p)$ UV finite by IR divergent when inserted in higher loops. The model is non-renormalizable

Linear noncommutativity: the case $\mathbb{R}^3_\lambda$

In order to appreciate the importance of differential calculus consider the case $\Theta = \Theta(x)$. The simplest case is $\Theta^{ij}(x) = c^{ij}_k x^k$, with c^{ij}_k structure constants.

An example is the noncommutative space $\mathbb{R}^3_\lambda$ first introduced in [Hammou, Lagraa, Sheikh-Jabbari' 01] as quadratic subalgebra of $(\mathbb{R}^4_\theta, \star_V)$.

$$\varphi \star \psi\,(z_a, \bar{z}_a) = \varphi(z, \bar{z})\exp(\theta \overleftarrow{\partial}_{z_a} \overrightarrow{\partial}_{\bar{z}_a})\psi(z, \bar{z}), \quad a = 1, 2$$

by means of $x_\mu = \frac{1}{2}\bar{z}^a \sigma^{ab}_\mu z^b, \quad \mu = 0, .., 3$. The subalgebra generated by x_μ is closed wrt the star product implying

$$[x_i, x_j]_\star = i\lambda\epsilon^k_{ij}x_k \quad \textit{check!}$$

and

$$\sum_i x_i^2 = x_0^2$$

and x_0 star-commutes with x_i. Thus we can alternatively define $\mathbb{R}^3_\lambda$ as the star-commutant of x_0.

The algebra $\mathbb{R}^3_\lambda$

The induced $\star$-product for $\mathbb{R}^3_\lambda$ reads

$$\varphi \star \psi\,(x) = \exp\left[\frac{\lambda}{2}\left(\delta_{ij}x_0 + i\epsilon_{ij}^k x_k\right)\frac{\partial}{\partial u_i}\frac{\partial}{\partial v_j}\right]\varphi(u)\psi(v)|_{u=v=x}$$

$\Longrightarrow$ for coordinate functions

$$\begin{aligned} x_i \star x_j &= x_i x_j + \frac{\lambda}{2}\left(x_0\delta_{ij} + i\epsilon_{ij}^k x_k\right) \\ x_0 \star x_i &= x_i \star x_0 = x_0 x_i + \frac{\lambda}{2}x_i \\ x_0 \star x_0 &= = x_0(x_0 + \frac{\lambda}{2}) = \sum_i x_i \star x_i - \lambda x_0 \end{aligned}$$

One can introduce a matrix basis [V., Wallet '13]:

$$v^j_{m\tilde{m}}(x) = \frac{e^{-2\frac{x_0}{\lambda}}}{\lambda^{2j}}\frac{(x_0+x_3)^{j+m}(x_0-x_3)^{j-\tilde{m}}\,(x_1-ix_2)^{\tilde{m}-m}}{\sqrt{(j+m)!(j-m)!(j+\tilde{m})!(j-\tilde{m})!}} \quad j \in \frac{\mathbb{N}}{2},\; m,\tilde{m} \in (-j,j)$$

$\Longrightarrow$

$$v^j_{m\tilde{m}} \star v^{\tilde{j}}_{n\tilde{n}}(x) = \delta^{j\tilde{j}}\delta_{\tilde{m}}$$

Then, the star product in $\mathbb{R}^3_\lambda$ becomes a block-diagonal infinite-matrix product and the integral becomes a trace.

In the matrix basis

$$
\begin{aligned}
x_+ &= \lambda \sum_{j,m} \sqrt{(j+m)(j-m+1)}\, v^j_{mm-1} \\
x_- &= \lambda \sum_{j,m} \sqrt{(j-m)(j+m+1)}\, v^j_{mm+1} \\
x_3 &= \lambda \sum_{j,m} m v^j_{mm} \\
x_0 &= \lambda \sum_{j,m} j v^j_{mm}
\end{aligned}
$$

$$
\begin{aligned}
x_+ \star v^j_{m\tilde{m}} &= \lambda \sqrt{(j+m+1)(j-m)}\, v^j_{m+1\tilde{m}} \\
x_- \star v^j_{m\tilde{m}} &= \lambda \sqrt{(j-m+1)(j+m)}\, v^j_{m-1\tilde{m}} \\
x_3 \star v^j_{m\tilde{m}} &= \lambda\, m\, v^j_{m\tilde{m}} \\
x_0 \star v^j_{m\tilde{m}} &= \lambda\, j\, v^j_{m\tilde{m}}
\end{aligned}
$$

and analogous expressions when star multiplying from the right

Derivations of the algebra $\mathbb{R}^3_\lambda$

In order to introduce a dynamics described by an action functional we need derivations. In the commutative case one uses the Kustaanheimo-Stiefel (KS) map:

- $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^4 - \{0\}$ are given the structure of trivial bundles over spheres, $\mathbb{R}^3 - \{0\} \simeq S^2 \times \mathbb{R}^+$, $\mathbb{R}^4 - \{0\} \simeq S^3 \times \mathbb{R}^+$;
- then use the Hopf fibration $\pi_H : S^3 \to S^2$, with the identification of S^3 with $SU(2)$,

$$\pi_H : s \in SU(2) \to \vec{x} \in S^2, \ \ : s\sigma_3 s^{-1} = x^i \sigma_i$$

  where $s = y_0\sigma_0 + iy_i\sigma_i$, y_μ are real coordinates on $\mathbb{R}^4$ s.t. $y_\mu y^\mu = 1$;
- extend the Hopf map to $\mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$, relaxing the radius constraint $\Rightarrow y_\mu y^\mu = R^2$;
- finally introduce $g = Rs$ and define

$$\pi_{KS} : g \in \mathbb{R}^4 - \{0\} \to \vec{x} \in \mathbb{R}^3 - \{0\}, \quad x^k \sigma_k = g\sigma_3 g^\dagger = R^2 s\sigma_3 s^{-1};$$

  which gives quadratic expressions for the x_μ and $x_0 = R^2/4$. (Exercise)

Derivations of the algebra $\mathbb{R}^3_\lambda$

Projectable vector fields are defined by the condition $[D_i, Y_0] = 0$, with $Y_0 = y^0\partial_{y^3} - y^3\partial_{y^0} + y^1\partial_{y^2} - y^2\partial_{y^1}$ generator of the fibre $U(1)$.
They correspond to the three rotation generators and the dilation $Y_i = y_0\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial y_0} - \epsilon_{ijk}y_j\frac{\partial}{\partial y_k}, \qquad D = y_\mu\frac{\partial}{\partial y_\mu} \implies$

$$\pi_{KS*}(Y_i) = X_i = \epsilon_{ijk}x_j\frac{\partial}{\partial x_k}\,, \quad \pi_{KS*}(D) = x_i\frac{\partial}{\partial x_i}$$

When passing to the noncommutative case the three rotations are still derivations of the algebra $\mathbb{R}^3_\lambda$ and may be given the form of inner derivations

$$X_i(\varphi) = -\frac{i}{\lambda}[x_i, \varphi]_\star\,, \quad i = 1, .., 3$$

- they satisfy the Leibniz rule
- they are independent (even though $x_i \star X_i(\varphi) + X_i(\varphi) \star x_i = 0$, derivations are not a module over the algebra in the NC case)
- and sufficient ("constant" functions are in the center of the algebra)

The dilation is not a derivation as it does not satisfy the Leibniz rule (check by applying it to the star product of coordinates).

The scalar field theory $g\,\varphi^{\star 4}$

- ✓ star product
- ✓ derivations
- ✓ integration

Well defined scalar action:

$$S[\varphi] = S[\varphi] = \int \varphi \star (\Delta + \mu^2)\varphi + \frac{g}{4!}\varphi \star \varphi \star \varphi \star \varphi$$

with the Laplacian: $\Delta\varphi = \alpha \sum_i D_i^2 \varphi + \beta\, x_0 \star x_0 \star \varphi$
The second term is introduced to reproduce radial dynamics,
$x_0 \star \varphi = x_0 \varphi + \frac{\lambda}{2} x_i \partial_i \varphi$.
Other proposals exist. There remain two main problems:

- the commutative limit;
- the radial dynamics (clear in the matrix basis: j does not change)

The model has been studied at one-loop in the matrix basis [Vitale, Wallet '13]

Noncommutative gauge theory on R^{2n}_θ

To make sense of noncommutative gauge and matter fields we need

- ✓ a noncommutative algebra $(\mathcal{A}, \star)$ representing space-time (it replaces $\mathcal{F}(M)$)
- ✓ A differential calculus based on derivations of the NC algebra which allows to introduce the dynamics;
- a NC analogue of matter fields, compatible with $\star$ multiplication by functions, which replaces the notion of vector bundles
- a group of unitary automorphisms acting on fields from the left, representing gauge transformations;
- a NC analogue of gauge connection

For QED the gauge group is $\widehat{U(1)}$, implying that charged matter fields are 1-dim complex vector fields (sections of 1-d complex vector bundle), namely a right module over $\mathcal{F}(\mathbb{R}^4)$

$\Longrightarrow$ The NC generalization is

- a 1-dim complex right module (one generator) over $\mathbb{R}^{2n}_\theta$

$$\mathcal{H} = \mathbb{C} \otimes \mathbb{R}^{2n}_\theta$$

with Hermitian structure h : $h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) = \psi_1^\dagger \star \psi_2$

General setting

For non-Abelian gauge theories (gauge group $\widehat{SU(N)}$) charged matter fields are typically complex vector fields in the fundamental representation of the group (–> sections of N-dim complex vector bundles)

$\Longrightarrow$ The NC generalization is

- a N-dim complex right module (N generators) over $\mathbb{R}^{2n}_\theta$

$$\mathcal{H} = \mathbb{C}^N \otimes \mathbb{R}^{2n}_\theta$$

- Gauge transformations are defined as automorphisms of $\mathcal{H}$ compatible both with the structure of right $\mathbb{R}^{2n}_\theta$-module

$$g(\psi f) = g(\psi) f$$

and with the Hermitian structure $h : \mathcal{H} \times \mathcal{H} \to \mathbb{R}^{2n}_\theta$

$$h(g\psi_1), g(\psi_2)) = h(\psi_1, \psi_2) \quad \forall \psi_1, \psi_2 \in \mathcal{H}$$

- A connection (*discuss classical definition on the bb*) is a linear map $\nabla : \mathrm{Der}(\mathbb{R}^{2n}_\theta) \times \mathcal{H} \to \mathcal{H}$ satisfying

- $\nabla_X(\psi f) = \psi X(f) + \nabla_X(\psi) f, \nabla_{cX}(\psi) = c\nabla_X(\psi)$ c in the center
- $\nabla_{X+Y}(\psi) = \nabla_X(\psi) + \nabla_Y(\psi)$
- Hermiticity:
  $X(h(\psi_1, \psi_2)) = h(\nabla_X(\psi_1), \psi_2) + h(\psi_1, \nabla_X(\psi_2)), \forall \psi_1, \psi_2 \in \mathcal{H}$

- Curvature is the linear map $\mathbf{F}(X, Y) : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathbf{F}(X, Y)\psi = i\,([\nabla_X, \nabla_Y]\psi - \nabla_{[X,Y]})\psi$$

Noncommutative QED on R^{2n}_θ

In this case $\mathcal{H}$ has only one generator, $\mathbf{e}$ —> $\boldsymbol{\psi} = \mathbf{e}\psi, \psi \in R^{2n}_\theta$
• The connection is completely determined by its action on the module generator: $\nabla_X(\boldsymbol{\psi}) = \nabla_X(\mathbf{e})\psi + \mathbf{e}X(\psi)$, with $\nabla_X(\mathbf{e})^\dagger = -\nabla_X(\mathbf{e})$.
$\Longrightarrow$ The 1-form connection $\mathbf{A}$:

- $\mathbf{A} : X \to \mathbf{A}(X) := i\nabla_X(\mathbf{e}), \ \ \forall X \in \mathrm{Der}(\mathbb{R}^{2n}_\theta)$
- $\nabla_\mu(\mathbf{e}) =: -i\mathbf{A}(\partial_\mu) = -i\mathbf{e}A_\mu$
- so that

$$\nabla_\mu \boldsymbol{\psi} := \nabla_\mu(\mathbf{e}\psi) = \mathbf{e}(\partial_\mu\psi - iA_\mu \star \psi)$$

• Gauge transformations can be identified with the unitaries $\mathcal{U}(\mathbb{R}^{2n}_\theta)$

Indeed

$$g(\boldsymbol{\psi}) = g(\mathbf{e}\psi) = g(\mathbf{e}) \star \psi = \mathbf{e}\, f_g \star \psi$$
$$h(g(\boldsymbol{\psi}_1), g(\boldsymbol{\psi}_2)) = h(\mathbf{e},\mathbf{e})\overline{(f_g \star \psi_1)} \star f_g \star \psi_2 = h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) \longrightarrow$$
$$\overline{f_g} \star f_g = 1$$
$$\Longrightarrow f_g \in \mathcal{U}(\mathbb{R}^{2n}_\theta)$$

Properties of the gauge connection

- gauge covariance: $(\nabla^A_\mu)^g(\psi) := g(\nabla^A_\mu(g^{-1}\psi)) = \nabla^{A^g}_\mu(\psi)$
  with
  $A^g_\mu = f_g \star A_\mu \star f_{g^{-1}} + i f_g \star \partial_\mu f_{g^{-1}}$
- Curvature:
  $\mathbf{F}_{\mu\nu} = ([\nabla^A_\mu, \nabla^A_\nu] - \nabla^A_{[\partial_\mu,\partial_\nu]}) = \mathbf{e}(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star)$
  $\mathbf{F}^g_{\mu\nu} = ([\nabla^A_\mu, \nabla^A_\nu] - \nabla^A_{[\partial_\mu,\partial_\nu]}) \overset{\text{check}}{=} \mathbf{e}(f_g \star F_{\mu\nu} \star f_{g^{-1}})$

Implying
$F^g_{\mu\nu} \star F^g{}_{\mu\nu} = f_g \star F_{\mu\nu} \star F_{\mu\nu} \star f_{g^{-1}}$

The QED action on R_θ^{2n}

A natural candidate is

$$S = \int d^{2n}x \;\; F_{\mu\nu} \star F^{\mu\nu}$$

Symmetries

- because of cyclicity of the product it is gauge invariant
- it is invariant under standard observer Poincaré transformations
- but yields new pathologies w.r.t. the commutative case: UV/IR mixing, Gribov ambiguity

Space-time symmetries
Moyal product has been shown to be covariant under observer (passive) transformations belonging to the Weyl group (*undeformed* Poincaré + dilations; -more generally under linear affine transformations-) [GraciaBondia- R.Ruiz-Lizzi-Vitale '06]

$$[\Omega \cdot f] \star_{\Omega\cdot\Theta} [\Omega \cdot g] = \Omega \cdot (f \star_\Theta g), \quad \Omega = (L, a)$$

$$[\Omega \cdot f](x) = f(L^{-1}(x - a), \quad \Omega \cdot \Theta = L\Theta L^t$$

Infinitesimal generators:

- They are the standard ones $G = \epsilon^\alpha_\beta x^\beta \partial_\alpha + a^\beta \partial_\beta$
- not derivations of the star product (precisely because the Lie derivative of Θ has to be taken into account)
- However: since the product depends on Θ even if starting functions don't, it is convenient to consider a (x, Θ)-space on which

$$\Omega \cdot (x, \Theta) = (Lx + a, L\Theta L^t) \Longrightarrow$$

the infinitesimal generators in (x, Θ)-space are

$$P^\Theta_\mu = -\partial_\mu, \quad D^\Theta = -x \cdot \partial - \theta^{\mu\nu} \frac{\partial}{\partial \theta^{\mu\nu}}$$

$$M^\Theta_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu + \theta^\rho_\mu \frac{\partial}{\partial \theta^{\rho\nu}} - \theta^\rho_\nu \frac{\partial}{\partial \theta^{\rho\mu}}$$

ExerciseThey close the standard Weyl algebra and are derivations of the star product

$$G^\theta(f \star g) = G^\theta f \star g + f \star G^\theta g$$

Weyl invariance of the QED action

A_α does not depend on Θ –>

- $P^\Theta_\alpha A_\mu = -\partial_\alpha A_\mu$
- $M^\Theta_{\alpha\beta} A_\mu = (x_\alpha \partial_\beta - x_\beta \partial_\alpha) A_\mu + g_{\alpha\mu} A_\beta - g_{\alpha\nu} A_\alpha$
- $D^\Theta A_\mu = -(1 + x \cdot \partial) A_\mu$

For the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_{\star\Theta}$ use the fact that G^Θ are $\star$ derivations –>

- $P^\Theta_\alpha F_{\mu\nu} = \partial_\alpha F_{\mu\nu}$
- $M^\Theta_{\alpha\beta} F_{\mu\nu} = (x_\alpha \partial_\beta - x_\beta \partial_\alpha) F_{\mu\nu} + g_{\mu\alpha} F_{\beta\nu} - g_{\mu\beta} F_{\alpha\nu} + g_{\nu\alpha} F_{\beta\mu} - g_{\nu\beta} F_{\alpha\mu}$
- $D^\Theta F_{\mu\nu} = -(2 + x \cdot \partial) F_{\mu\nu}$

namely the same as for commutative case $\Longrightarrow$ the action is invariant

Remark. There is a difference wrt commutative 4-d QED: Special conformal invariance is lost because quadratic (or higher) in x $\Longrightarrow$

$$[x_\mu x_\nu \partial_\rho]_\Theta (f \star g) \overset{\text{check}}{\neq} [x_\mu x_\nu \partial_\rho]_\Theta f \star g + f \star [x_\mu x_\nu \partial_\rho]_\Theta g$$

with $$[x_\mu x_\nu \partial_\rho]_\Theta = x_\mu x_\nu \partial_\rho + (\theta^\alpha_\mu x_\nu + \theta^\alpha_\nu x_\mu) \frac{\partial}{\partial \theta^{\alpha\rho}}$$

Comparison with the twist approach

Moyal product is not covariant under Poincaré *particle* (active) transformations, where the background field Θ does not change.
But it is covariant under Θ-Poincaré particle transformations: the universal enveloping algebra of the Lie algebra $\mathfrak{p}$, with twisted coproduct (Hopf algebra $U_{\mathcal{F}}(\mathfrak{p})$).

Recap:

Given H Hopf algebra, denote by id the identity map of H onto itself, by Δ the coproduct map, and by η the counit map from the Hopf algebra to the scalars.

Consider an invertible element $\mathcal{F}$ in $H \otimes H$ that satisfies the conditions

$$(1 \otimes \mathcal{F})(\mathrm{id} \otimes \Delta)\mathcal{F} = (\mathcal{F} \otimes 1)(\Delta \otimes \mathrm{id})\mathcal{F} \qquad (\eta \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \eta)\mathcal{F} = 1$$

$\mathcal{F}$ is said to be a counital 2-cocycle for H (*the twist*)

$\Delta_{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}$, with h in H, defines a new coproduct in H

The algebra underlying H endowed with $\Delta_{\mathcal{F}}$ is the Hopf algebra $H_{\mathcal{F}}$ (twisted Hopf algebra)

If H has a representation in an associative algebra $\mathcal{A}$ (here $F(\mathbb{R}^4)$) with product m:

$$m(a \otimes b) = ab$$
$$h \cdot (ab) = h \cdot m\,(a \otimes b) = m\big(\Delta(h) \cdot (a \otimes b)\big)\,, \quad h \in H$$

the twisting of Δ introduces in $\mathcal{A}$ a twisted product $m_{\mathcal{F}}$ defined by

$$m_{\mathcal{F}}(a \otimes b) = m\big(\mathcal{F}^{-1} \cdot (a \otimes b)\big)$$

which is associative.
$H_{\mathcal{F}}$ is represented in $(\mathcal{A}, m_{\mathcal{F}})$ by its action through $\Delta_{\mathcal{F}}(h)$,

$$\begin{aligned} h \cdot m_{\mathcal{F}}(a \otimes b) &= h \cdot m\big(\mathcal{F}^{-1} \cdot (a \otimes b)\big) = m\big(\Delta(h)\mathcal{F}^{-1} \cdot (a \otimes b)\big) \\ &= m\big(\mathcal{F}^{-1}\Delta_{\mathcal{F}}(h) \cdot (a \otimes b)\big) = m_{\mathcal{F}}\big(\Delta_{\mathcal{F}}(h) \cdot (a \otimes b)\big) \quad ** \end{aligned}$$

$\Longrightarrow$ A $\star$-product defined in terms of a twist is *always* twist-covariant, by definition
$\Longrightarrow$ *An action functional invariant under some space-time transformations always yields a twisted action invariant wrt the corresponding twisted transformations; these should be understood as particle (active) transformations*

Back to QED on $\mathbb{R}^4_\Theta$

Consider the Lie algebra of diffeomorphisms, $\mathfrak{D}(\mathbb{R}^4)$, whose generators are vector fields with polynomial coefficients on $\mathbb{R}^4$

- As Hopf algebra H take the enveloping algebra $U(\mathfrak{D})$:
  Δ is first defined for $h \in \mathfrak{D}$ by $\Delta(h) = 1 \otimes h + h \otimes 1$, and then multiplicatively extended to all of $U(\mathfrak{D})$ by $\Delta(hh') = \Delta(h)\Delta(h')$;
- for the algebra $\mathcal{A}$ carrying a representation of $U(\mathfrak{D})$, take the algebra of functions on spacetime with the ordinary multiplication $m(f \otimes g) = fg$;
- for $\mathcal{F}$, take $\mathcal{F}_\Theta = \exp(-\frac{i}{2}\,\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu)$. This is clearly in $U(\mathfrak{D}) \otimes U(\mathfrak{D})$, has an inverse

  $$\mathcal{F}_\Theta^{-1} = \exp(\tfrac{i}{2}\,{}^{\mu\nu}\partial_\mu \otimes \partial_\nu)$$

  and satisfies the cocycle condition

The Moyal product is then recovered as the twisted product

$$m_\Theta(f \otimes g) = m\big(\mathcal{F}_\Theta{}^{-1} \cdot (f \otimes g)\big) = f \star_\Theta g$$

The action of a generator h on the Moyal product is determined by $\Delta_\Theta(h) = \mathcal{F}_\Theta \Delta(h) \mathcal{F}_\Theta{}^{-1}$ and conversely.

For the generators of translations, Lorentz transformations and dilations the following expressions were obtained [Kulish, Matlock]

$$
\begin{aligned}
\Delta_\Theta(P_\mu) &= P_\mu \otimes 1 + 1 \otimes P_\mu \\
\Delta_\Theta(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\
&\quad + \tfrac{i}{2}\,\theta^{\alpha\beta}\big[(g_{\mu\alpha}P_\nu - g_{\nu\alpha}P_\mu) \otimes P_\beta + P_\alpha \otimes (g_{\mu\beta}P_\nu - g_{\nu\beta}P_\mu)\big] \\
\Delta_\Theta(D) &= D \otimes 1 + 1 \otimes D - i\,\theta^{\mu\nu} P_\mu \otimes P_\nu
\end{aligned}
$$

From these formulas it was concluded that Poincaré invariance can be maintained in noncommutative field theory although twisted.

But this is not specific of Poincaré invariance

Note that Eq. $**$ places *no restriction* on the generator h except that of being an infinitesimal diffeomorphism

This is why the generators K_μ of special conformal transformation could be added to the list of computed $\Delta_\Theta(h)$ [Matlock, Lizzi Vaydia V.].

- Because we are in the enveloping algebra, $**$ applies to differential operators of any order
- the method is thus a recipe to encode the action of arbitrary differential operators with polynomial coefficients on Moyal products

Exercise

Show that

$$\begin{aligned} \partial_\alpha(f \star_\Theta g) &= \partial_\alpha f \star_\Theta g + f \star_\Theta \partial_\alpha g \\ x^\alpha(f \star_\Theta g) &= x^\alpha f \star_\Theta g - \frac{i}{2}\,\theta^{\alpha\beta} f \star_\Theta \partial_\beta g = f \star_\Theta x^\alpha g + \frac{i}{2}\,\theta^{\alpha\beta}\,\partial_\beta f \star_\Theta g \end{aligned}$$

and use it to check the twisted coproduct of infinitesimal spacetime transformations generated by $x^{\mu_1}...x^{\mu_N}\partial_\nu$

$$\Delta_\Theta(x^{\mu_1}...x^{\mu_N}\partial_\nu) = x^{\mu_1}...x^{\mu_N}\partial_\nu \otimes 1 + 1 \otimes x^{\mu_1}...x^{\mu_N}\partial_\nu$$
$$+ \sum_{k=1}^{N}\left(\frac{i}{2}\right)^k \sum_{N\geq c_k>...>c_1\geq 1} \theta^{\mu_{c_1}\alpha_{c_1}}\ldots\theta^{\mu_{c_k}\alpha_{c_k}}\Big[\partial_{\alpha_{c_1}}...\partial_{\alpha_{c_k}} \otimes x^{\mu_1}\ldots \overset{c_1}{\smile}\ldots \overset{c_k}{\smile}\ldots x^{\mu_N}\,\partial_\nu$$
$$+(-1)^k\, x^{\mu_1}\ldots \overset{c_1}{\smile}\ldots \overset{c_k}{\smile}\ldots x^{\mu_N}\,\partial_\nu \otimes \partial_{\alpha_{c_1}}...\partial_{\alpha_{c_k}}\Big]$$

[$\overset{c_1}{\smile}$ indicates that the factor $x^{\mu_{c_1}}$ is removed]
Moreover,

$$m_\Theta\big(\Delta_\Theta(x^{\mu_1}\cdots x^{\mu_N}\partial_\nu)\cdot(x^\alpha \otimes x^\beta - x^\beta \otimes x^\alpha)\big) \overset{check}{=} 0$$

namely, $\theta^{\alpha\beta}$ remains unchanged. The twisted coproduct formulation accounts only for particle transformations

Twist vs covariance

To summarize: for G in the affine group (generators at most linear in coordinates) the relation between the covariant and twist approaches can be accounted by the following equation

$$m_\Theta\big(\Delta_\Theta(G)\cdot(f\otimes g)\big) = G^\Theta m_\Theta(f\otimes g) - \frac{1}{2}\,\delta_G\theta^{\alpha\beta}\,\frac{\partial}{\partial\theta^{\alpha\beta}}\,m_\Theta(f\otimes g)\,,$$

where $\delta_G\theta^{\alpha\beta}$ is the Lie derivative of the tensor $\Theta=\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta$ with respect to G
For instance for dilatations one has

$$m_\Theta\big(\Delta_\Theta(D)\cdot(f\otimes g)\big) = D^\Theta(f\star_\Theta g) + \theta^{\alpha\beta}\,\frac{\partial}{\partial\theta^{\alpha\beta}}\,(f\star_\Theta g)\,.$$

Furthermore, observer and twist covariances boil down to

$$\text{observer}:\ \ G^\Theta m_\Theta = m_\Theta\Delta(G) \qquad \text{twist}:\ \ G\,m_\Theta = m_\Theta\Delta_\Theta(G)\,.$$