Lecture 1: The black hole as a tale of light and darkness (1/2)

miércoles, 31 de agosto de 2022 15:56

- **General Relativity**: a geometric theory of relativistic gravitation

Basic element:

Metric $ds^2 = g_{ij}(x)\, dx^i dx^j$

Coord Trafos: $x^i \to x'^i(x^j)$ change the metric.
(diffeos)

Infinitesimal: $x^i \to x^i + \varepsilon v^i(x)$ $\varepsilon \ll 1$

$$g_{ij} \to g_{ij} + \varepsilon(\partial_i v_j + \partial_j v_i) \qquad \partial_i \equiv \frac{\partial}{\partial x^i}$$

but physical effects must not change under diff.

Tensors:

$T = T^{ij\ldots}{}_{kl}\, \partial_i \otimes \partial_j \otimes \ldots \otimes dx^k \otimes dx^l \ldots$: diff invariant

$T^{ij\ldots}{}_{kl\ldots}$ are covariant objects

Curvature Tensor:

$$\left.\begin{array}{ll} \text{Riemann} & R^i{}_{jkl} \\ \text{Ricci} & R_{ij} = R^k{}_{ikj} \\ \text{Scalar curvature} & R = R^i{}_i \end{array}\right\} \sim g^{-1}\partial^2 g$$

Einstein equations:

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi \frac{G}{c^2} T_{ij}$$

Geometry Matter

Relativistic Gravity } G and c allow to connect (transform) matter and geometry

$$\text{Curvature} = \frac{G}{c^2} \text{ Mass density}$$

(R here is curvature radius ~ system size)

$$\frac{1}{R^2} \sim \frac{G}{c^2} \frac{M}{R^3} \Rightarrow R \sim \frac{GM}{c^2}$$

Mass can be measured in meters

$$1\,\text{kg} \sim 10^{-29}\,\text{m}$$

Your weight $\sim 10^{-27}$ m << your physical size

$\Rightarrow$ You're not very attractive

Vacuum equations : $R_{ij} - \frac{1}{2} g_{ij} R = 0$

$$\Leftrightarrow R_{ij} = 0$$

• The Schwarzschild solution (Dec 1915)

Simplest non-trivial solution:

$$ds^2 = -\left(1 - \frac{r_0}{}\right) dt^2 + \frac{dr^2}{} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$(c = 1)$

- Indep of t: static
- Spherically symmetric
- Length parameter r_0:

Newtonian limit: $-g_{tt} \simeq 1 + 2\Phi_N$, $|\Phi_N| \ll 1$

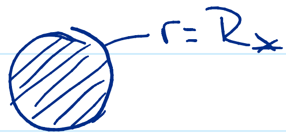
Newtonian potential of localized mass source M:

$$\Phi_N = -\frac{GM}{r} \quad \Rightarrow \quad r_0 = 2GM$$

Restoring c: $r_0 = 2\frac{G}{c^2} M$

as anticipated above

- Star (spherical, static)

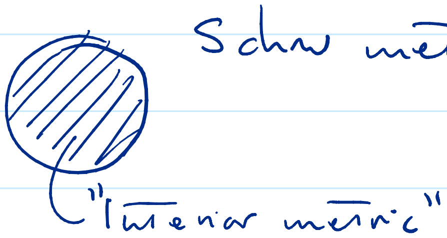


Exterior: $R_{ij} = 0 \Rightarrow r > R_*$: Schwarzschild metric

Interior: $R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij} \rightarrow$ star matter

Model-dependent metric

Schw metric



• The Horizon

- Consider no matter anywhere:

$(G = c = 1)$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2$$

Metric is singular, as a matrix, at $r = 2M$

But metric changes w/ diffeos.

(non-singular metric: g_{ij} finite, $\det(g_{ij})$ finite (invertible))

Let's shine light:

ingoing radial light rays:

θ, ϕ constant

$ds^2 = 0$ $\Rightarrow$ $dt^2 = \dfrac{dr^2}{\left(1 - \frac{2M}{r}\right)^2}$

$$dt = -\frac{dr}{\left|1 - \frac{2M}{r}\right|}$$

$\Rightarrow$ $t = -r_* + \text{constant}$

where $dr_* = \dfrac{dr}{|1 - 2M/r|}$

$r_* = r + 2M \log|r - 2M|$ ("Tortoise coord")

Note: $r_* \to r \to \infty$ as $r \to \infty$

$r_* \to -\infty$ as $r \to 2M$

Ingoing light rays are

Ingoing light rays are

$$v \equiv t + r_* = \text{constant}$$

As $r \to 2M$ along a light ray, $t \to +\infty$

But t is a coordinate whose meaning is clear near ∞, less so at smaller r.

We may choose v as coordinate, adapted to ingoing light rays, instead of t.

i.e. change $(t, r) \longrightarrow (v, r)$

$$dt = dv - dr_* = dv - \frac{dr}{1 - \frac{2M}{r}}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega_2$$

$$(g_{ij}) : \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & r^2\Omega_2 \end{pmatrix}$$

: non-singular at $r = 2M$

$g_{vv} = 1 - \frac{2M}{r}$: finite

$\det g_{ij} = -2 r^2 \sin^2\theta$ finite

Ingoing light rays encounter $r = 2M$ as a smooth place.

Analyze all radial light rays:

$$ds^2 = 0 \Rightarrow \left(1 - \frac{2M}{r}\right) dv^2 = 2\,dv\,dr$$

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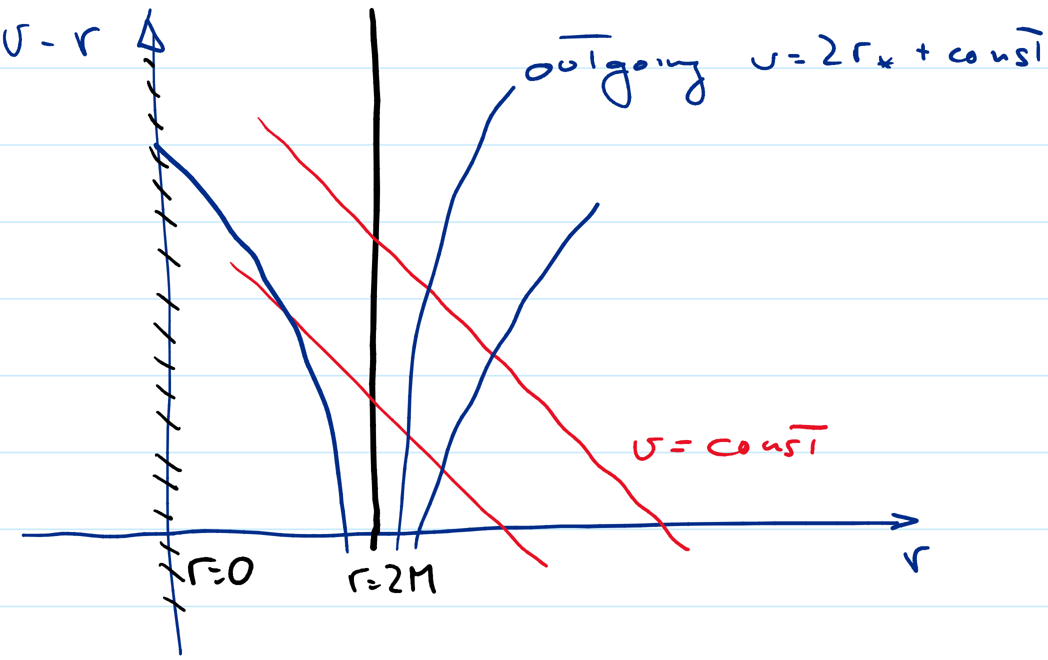
$\Rightarrow$ $v =$ const : ingoing

$r = 2M$: constant radius

$\left(1-\frac{2M}{r}\right)dv = 2\,dr \Rightarrow v = 2r_* + \text{const}$: outgoing

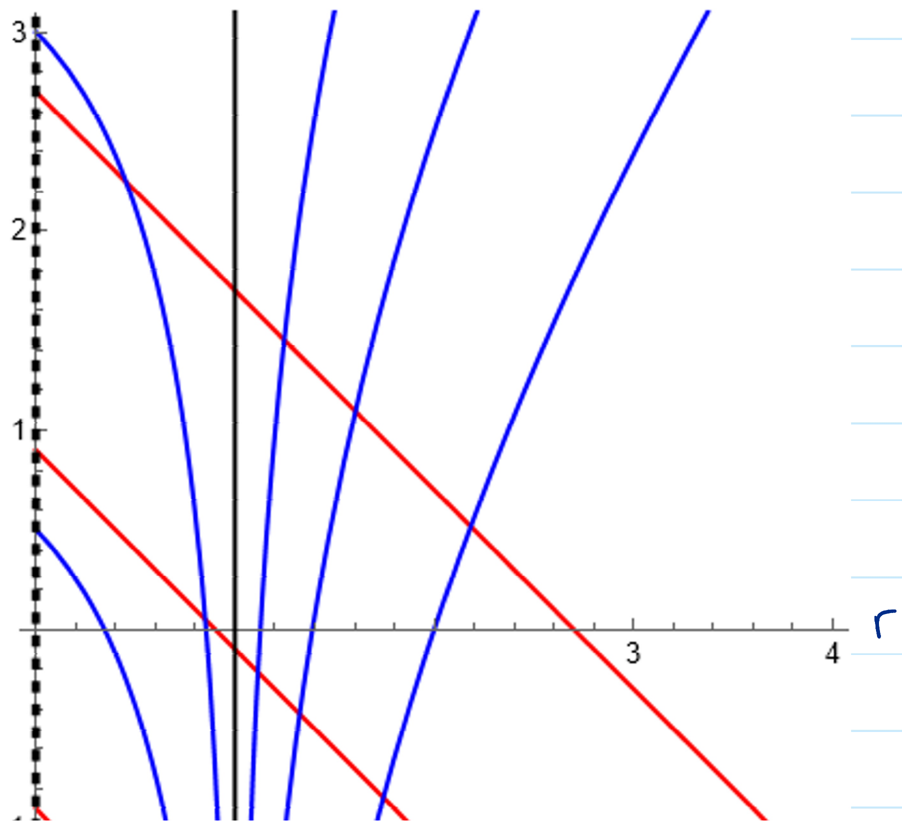
Plot $v - r$

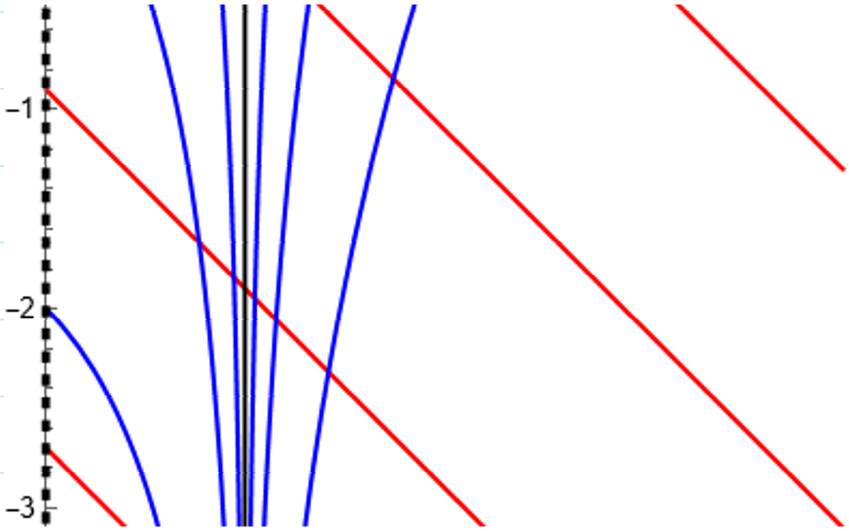


$v - r = t + (r_* - r)$

$\sim t$ for $r \gg 2M$

$v - r$





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