Lecture 2: The black hole that vibrates (1/2)

miércoles, 31 de agosto de 2022 20:33

$$ds^2 = g^{(0)}_{ij}(x)\,dx^i dx^j = -f\,dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$f = 1 - \frac{2M}{r}$$

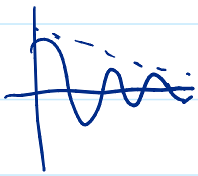
Is the black hole stable?

i.e. if we wiggle it a little, will it return back to the initial state?

When it relaxes to equilibrium, what signal does it emit, and what information does it carry?

This is the ring-down problem and the Quasinormal mode spectrum of the black hole.

A black hole relaxes down to equilibrium like a bell does: with damped oscillations



$$e^{-i\omega t - t/\tau}$$: quasinormal vibrations

Not "normal", but "quasi-normal", since the spectrum of vibrational frequencies has an imaginary part, owing to absorption by the black hole (dissipation).

We can first study the problem in linear perturbation theory, and furthermore, only for linear-mode stability

Full linear stability (not just modes) and non-linear

Full linear stability (not just modes) and non-linear stability are harder problems.

We can also put the metric in a Num Rel computer code and evolve it numerically

Linear perturbations:

$$g_{ij}(x) = g^{(0)}_{ij} + \varepsilon h_{ij}(x) \qquad \varepsilon \ll 1$$

Insert this into Einstein's equations, with

$$R_{ij}(g^{(0)}) = 0$$

and expand them to first order in ε.

Then we get the Lichnerowicz eqn, a complicated set of coupled, 2^{nd} order PDEs for the $h_{ij}(x)$

We can use the symmetries: Time-independence
spherical symmetry

To decompose the $h_{ij}(x)$

$$h_{ij}(x) = e^{-i\omega t} Y_{\ell m}(\theta, \phi)\, h^{(\omega,\ell,m)}_{ij}(r)$$

$Y_{\ell m}(\theta,\phi)$: spherical harmonics (scalar, vector, tensors of $SO(3)$)

subtlety: even/odd parity = polar/axial = scalar/vector

scalars do not deform shape of S^2

vectors do not change size of S^2

In $D=4$ they're Regge-Wheeler (vectors)

Zerilli (scalars)

[Zerilli (scalars)
but they're isospectral!]

Then we get a set of coupled 2^{nd} order ODEs.
Moreover, some of these perturbations are unphysical, or pure gauge, since they just correspond to coord changes $h_{ij} = \partial_i v_j(x) + \partial_j v_i(x)$
(eg a translation of origin, a boost, or a silly change)

With some work (and luck) these can be combined and reduced to a single <u>master equation</u> for a <u>master variable</u> $\Psi_{\omega \ell m}(r)$ that is invariant under gauge transformations

$r_* =$ tortoise coordinate

$$\frac{d^2 \Psi_{\omega \ell m}}{dr_*^2} + \left(\omega^2 - V_{\ell m}(r)\right) \Psi_{\omega \ell m} = 0 \qquad dr_* = \frac{dr}{1 - 2M/r}$$

with $V_{\ell m}(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)$ here $r = r(r_*)$

Then all the $h_{ij}(r)$ can be obtained (up to gauge) from $\Psi_{\omega \ell m}(r)$

We can consider this equation as describing the propagation of weak gravitational (spin-2) fields propagation of weak gravitational (spin-2) fields around a black hole.

We could also consider spin-0 (scalar) fields $\Phi(x)$ or spin-1 (gauge vector) fields $A_\mu(x)$, which satisfy

$$\Box \Phi(x) = 0 \,, \qquad \nabla_\mu F^{\mu\nu} = 0 \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

ie $$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi \right) = 0 \qquad \text{[EXERCISE]}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, F^{\mu\nu} \right) = 0$$

These also reduce to master equations of the type above

$$\frac{d^2 \Psi^{(s)}_{\omega l m}}{dr_*^2} + \left(\omega^2 - V^{(s)}_{lm}(r) \right) \Psi^{(s)}_{\omega l m} = 0$$

for master variables $\Psi^{(s)}_{\omega l m}$ $\quad s = \begin{cases} 0 & \text{scalar field} \\ 1 & \text{gauge field} \\ 2 & \text{grav field} \end{cases}$

$$V^{(s)}_{lm} = \left(1 - \frac{2M}{r} \right) \left(\frac{l(l+1)}{r^2} + (1 - s^2) \frac{2M}{r^3} \right)$$

(scalar field is easiest to derive).

For $s=2$: $\exists$ zero-frequency perturbations with

$l = 0$: add mass

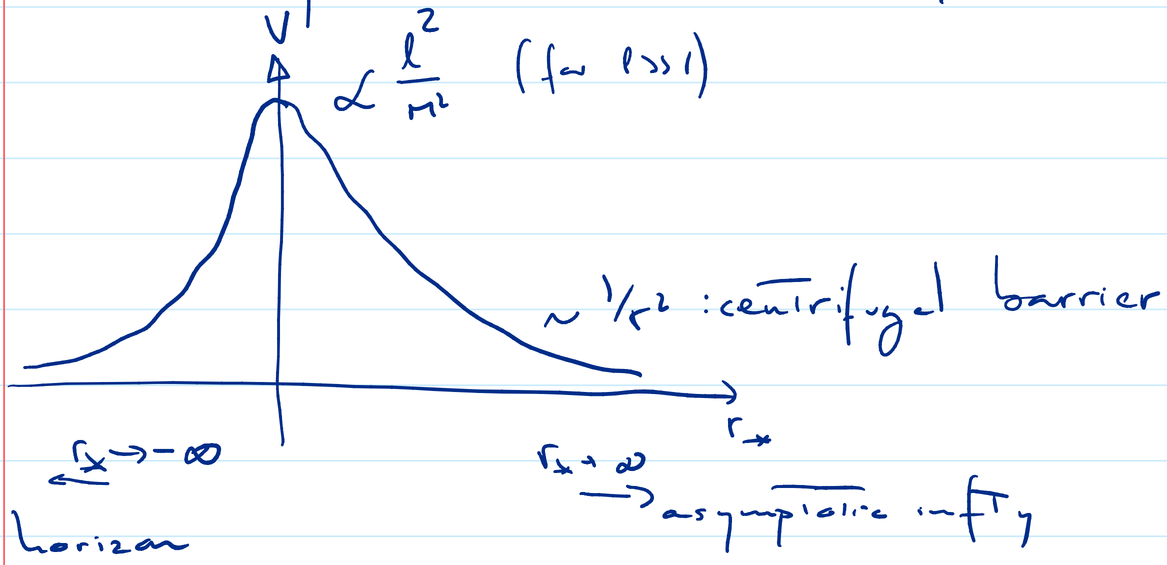
$\ell = 1$: add rotation

For $s=1$: $\exists$ zero-freq with

$\ell = 0$: add charge

For $s=0$: $\nexists$ zero-freq solutions : no scalar hair

We are mostly interested in $s=2$ (grav waves) but all other qualitative features are very similar.



Recall potential for light-rays

$$V_L(r) = \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}$$

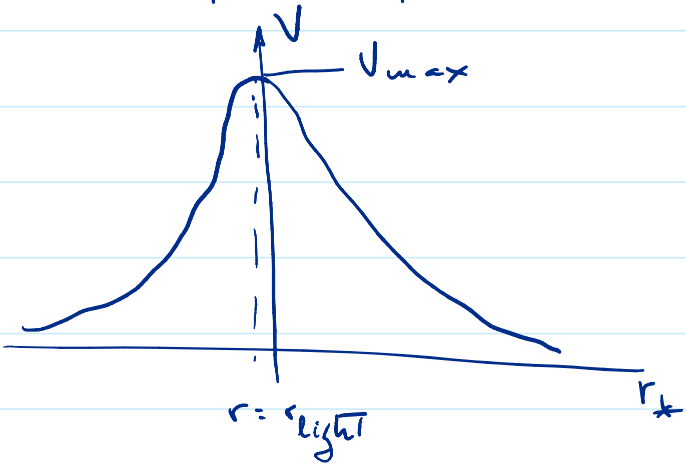
We see that for $\ell \gg 1$ (large "quantum number") eikonal limit

we have

$$V_L(r) \simeq V^{(s)}_{\ell m}(r) \quad \text{with} \quad L \simeq \ell$$

So the peak of $V^{(s)}_{\ell m}(r)$ is $\simeq$ light-ring radius

So the peak of $V_{em}^{(s)}(r)$ is $\simeq$ light-ring radius



This will be extremely important and useful to understand how and why bh's vibrate.

Now, incident waves w/ $\omega^2 > V_{max} \simeq V(r_{light})$ travel balistically into bh (classical bullets) while low-freq waves $\omega^2 \ll V_{max}$ are mostly reflected (bh too small)