# Lecture 3: The black hole that spins (1/2)

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Collapse typically involves angular momentum, so the geometry in the exterior of a collapsing object won't be Schwarzschild.

If the collapse doesn't produce a black hole, there are many possible exterior (vacuum) solutions. But if a bh forms, then the only possible final state, when all transients have died away, is the Kerr solution.

This is a consequence of the bh uniqueness theorems.

Carter-Robinson-Hawking: The only stationary, AF solution of $R_{\mu\nu}=0$ that is regular on and outside a (non-degenerate) EH is the Kerr black hole w/ mass $M$ and angular momentum $J$.

Moreover, these solutions are dynamically stable:

perturbative (linear) stability: $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

solve for $\delta g_{\mu\nu}$ $\Rightarrow$ perturbation decays in time

Proving non-linear stability is a much more difficult problem.

But in numerical simulations, the Kerr solution always appears as the unique, stable final state of collapse to a black hole, so even if the uniqueness and stability of the Kerr black holes may not have a rigorous mathematical proof, they are strongly believed to be true.

They have a striking consequence:

> The Kerr solution gives the exact description of all astrophysical black holes in the universe

In linearized gravity $g_{ij} = \eta_{ij} + h_{ij}$

$\Box h_{ij} \propto T_{ij}$

If there's linear momentum along $x$: $P_x \sim T_{tx}$

angular momentum along $\phi$: $J_\phi \sim T_{t\phi}$

hence if $P_x \neq 0 \Rightarrow h_{tx} \neq 0$

$J_\phi \neq 0 \Rightarrow h_{t\phi} \neq 0$ : spacetime is dragged by rotating matter, and it will drag matter around it

<u>Kerr solution</u> $\quad G = 1$

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - 2a \sin^2\theta \, \frac{2Mr}{\Sigma} \, dt \, d\phi + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta \, d\phi^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right)$$

$\Delta = r^2 - 2Mr + a^2$ $\quad$ $M, a$ : parameters

$\Sigma = r^2 + a^2 \cos^2\theta$ $\quad$ $a = 0$ : Schwarzschild $\quad$ $g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta$

$$\frac{\Delta}{r^2} = 1 - \frac{2M}{r} + \frac{a^2}{r^2}$$

$\hookrightarrow$ centrifugal potential barrier

Independent of $t$ : stationary but not static because $g_{t\phi} \neq 0$

Independent of $\phi$ : axisymmetric

Mass : as $r \to \infty$ $\quad -g_{tt} \simeq 1 - \frac{2M}{r} + O(1/r^2)$ $\quad$ Mass $= M$

Angular momentum: $\quad g_{t\phi} \simeq -\frac{2J}{r}\sin^2\theta + O(1/r^2)$ $\quad$ Angular momentum $= J$

Kerr $\quad g_{t\phi} \simeq -\frac{2Ma}{r}\sin^2\theta + O(1/r^2)$ $\quad$ $J = Ma$

$a = J/M$

Will take $a \geq 0$ w/o l.o.g

Metric has singularities at $\Sigma = 0$ and $\Delta = 0$ $\quad$ ($\theta = 0, \pi$ : symmetry axis)

$\det g_{\mu\nu} = -\Sigma^2 \sin^2\theta$

$\Sigma = 0 = r^2 + a^2\cos^2\theta$ : curvature singularity $(\text{Riemann})^2 \to \infty$

$(r = 0, \; \theta = \pi/2)$ $\quad$ Won't deal with this in these lectures.

$\Delta = 0$ : $r = r_\pm = M \pm \sqrt{M^2 - a^2}$ $\quad$ if $r_\pm \in \mathbb{R}$ then these are horizons

$M \geq a$

Observe that the metric where $g_{tt} = 0$ is regular, and different

Observe that the metric where $g_{tt}=0$ is regular, and different than $\Delta = 0$. We'll return to this

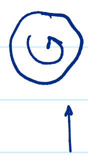
One can verify that $\exists$ Eddington-Finkelstein coordinates that make $r = r_\pm$ smooth places.

$\exists$ horizons only if $a \leq M$, ie $\boxed{J \leq M^2 \quad \text{ie} \quad J \leq \frac{G}{c} M^2}$ $\left( = Mc \frac{GM}{c^2} \right)$

$\Rightarrow$ Angular momentum for a given mass is bounded above.

Upper limit is <u>extremal Kerr</u> : $a = M \quad J = M^2$

Physical mechanisms to spin up a black hole fail to get beyond this limit : eg



: Throw particle towards bh.

If impact parameter is large, it misses the black hole

Surface gravity:

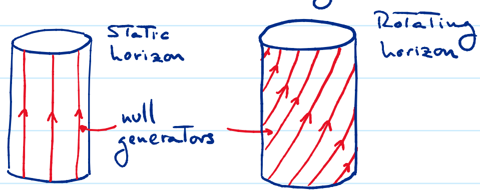
$$\kappa_\pm = \frac{\pm\sqrt{M^2 - a^2}}{r_\pm^2 + a^2} = \frac{\pm\sqrt{M^2 - a^2}}{2Mr_\pm}$$

$\kappa_+$ is $< \kappa_+(a=0)$ : rotation reduces attraction

$\kappa_- < 0$ (!)

Notice that the generator of the horizon is not $\partial/\partial t$.

This allows us to specify in what sense the black hole is rotating. The null geodesic generators are rotating relative to static asymptotic observers who follow trajectories with four-velocity $u = \frac{\partial}{\partial t}$




The velocity of rotation of the horizon is $\Omega_H = \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+}$

This is the velocity relative to static observers at infinity