LECTURE 1

The Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2 \tag{1}$$

becomes singular at $r = 2M$. Since a photon of frequency ω at radius r is redshifted to frequency $\sqrt{-g_{tt}(r)}\,\omega$ as it travels to infinity (where $-g_{tt} = 1$), then $r = 2M$ is a surface of infinite redshift. We have nevertheless seen that this surface can be reached in finite affine parameter by null (light-ray) trajectories. Indeed, using Eddington-Finkelstein coordinates adapted to ingoing light rays, we found that the geometry is smooth there[1].

In order to further illuminate the geometry near $r = 2M$ we will explore it in other ways.

1.a Rindler spacetime near the horizon and surface gravity

Go close to $r = 2M$ by taking

$$r - 2M \simeq \frac{\xi^2}{8M} \tag{2}$$

with $\xi \ll \sqrt{M}$.

- Prove that, then,

$$ds^2 \simeq -\frac{\xi^2}{16M^2} dt^2 + d\xi^2 + 4M^2 d\Omega_2 \,. \tag{3}$$

The term for the 2-sphere with constant radius $2M$ is not important in what follows[2]. It is the (t, ξ) part of the metric that matters to us here: it is reminiscent of the plane in polar coordinates

$$ds^2 = \rho^2 d\phi^2 + d\rho^2 \,, \tag{4}$$

and in fact it becomes of this form if we make $\phi \to it/(4M)$, $\rho \to \xi$. We know that in this case $\rho = 0$ is just a coordinate singularity, and we can remove it by changing to cartesian coordinates $x = \rho \cos\phi$, $y = \rho \sin\phi$. Following this lead, change coordinates as

$$X = \xi \cosh(t/(4M)) \,, \qquad T = \xi \sinh(t/(4M)) \,. \tag{5}$$

- Show that the metric becomes

$$-\frac{1}{16M^2}\xi^2 dt^2 + d\xi^2 = -dT^2 + dX^2 \,. \tag{6}$$

Thus we see that the metric on the left, which is called Rindler spacetime, is locally equivalent to 2D Minkowski space. So even if the Rindler metric is singular at the horizon ($\xi = 0$), a simple change of coordinates allows to extend it across the horizon as smoothly as in Minkowski space.

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[1]It is not difficult to see that $r = 2M$ is also reached in finite proper time by timelike (particle) trajectories. The so-called Painlevé-Gullstrand coordinates are adapted to radially infalling particles and yield a metric that is manifestly regular at $r = 2M$.

[2]Strictly, since we are focusing on length scales much smaller than the sphere radius $2M$, this sphere should be approximated by a plane.

Notice that $X^2 - T^2 = \xi^2$. Thus the horizon $\xi = 0$ is manifestly a null surface in Minkowski (actually two surfaces: $T \pm X = 0$), and trajectories of constant $\xi \neq 0$ are hyperbolas, which we know are trajectories of uniform acceleration equal to $1/\xi$. So Rindler spacetime is the geometry of observers following trajectories of uniform acceleration. Of course we know that in order to hover at fixed r above a black hole and not fall in it, you must accelerate away from it. We have found that, close to the horizon, you are approximately a Rindler observer.

More generally, one can prove (see Optional exercise for the static case) that the geometry near a black hole horizon takes the form

$$ds^2 \approx -\kappa^2 x^2 d\tau^2 + dx^2 + r_0^2 d\Omega\,, \tag{7}$$

where κ is a constant called the *surface gravity* of the horizon. For the Schwarzschild black hole we have

$$\kappa = \frac{1}{4M}\,. \tag{8}$$

κ has dimensions of inverse time, which is like acceleration (in natural units $c = 1$). Its operational meaning is as follows: Imagine an observer at a large distance $r \gg 2M$, who is slowly lowering towards the black hole a unit mass that is attached to the endpoint of a rope. As the mass is lowered, the tension of the rope increases, and the observer must exert a stronger force to keep it in place. When the horizon is approached, the tension of the rope near the mass diverges, but this tension is redshifted (as can be seen from stress-energy conservation) upwards along the rope. As a result, to hold the unit mass hovering right above the horizon, the observer at infinity must pull the rope with a finite force equal to κ.

1.b Outgoing Eddington-Finkelstein coordinates

In the Schwarzschild spacetime, argue that radially outgoing light rays are given by

$$t = r_* + \text{const} \tag{9}$$

where the tortoise coordinate r_* is defined by

$$dr_* = \frac{dr}{\left|1 - \frac{2M}{r}\right|}\,. \tag{10}$$

• Show that if we now introduce a new coordinate $u = t - r_*$, such that outgoing light rays are $u =$ const, and change coordinates $(t, r) \to (u, r)$, then the metric in these coordinates is regular at $r = 2M$.

• Find that the equation for radial light rays is also solved by: $r = 2M$ (which are the light rays that generate the horizon); and $u = -2r_* +$ const. Verify that the latter agrees with the solution that we found using the coordinate $v = t + r_*$, i.e, $v =$ const.

• Draw the trajectories of these light rays in a diagram where the vertical axis is $u+r$ (which at large r and t approaches t) and the horizontal axis is r. Observe that, now, the outgoing light rays $u =$ const cross the horizon *outwards*, while the ingoing light rays $u = -2r_* +$ const *never* cross the horizon, but only approach it asymptotically to the future. That is, light (and henceforth particles) can escape from inside the horizon, but never enter it — precisely the opposite of what we found earlier! What is going on here? (You are encouraged to try exercise 1.d for a fuller understanding).

Optional exercises for Lecture 1

1.c Surface gravity for static spherical black holes

Consider a static, spherically symmetric metric of the generic form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega \,. \tag{11}$$

Assume that $f(r)$ and $g(r)$ vanish linearly at $r = r_0$, *i.e.*, $f(r) = (r - r_0)f'_0 + \mathcal{O}(r - r_0)^2$ and similarly for $g(r)$.

- Show that near $r = r_0$ they take the form of Rindler spacetime[3]

$$ds^2 \approx -\kappa^2 x^2 d\tau^2 + dx^2 + r_0^2 d\Omega \,, \tag{12}$$

with

$$\kappa = \frac{1}{2}\sqrt{f'(r_0)g'(r_0)} \,. \tag{13}$$

NB: horizons with $\kappa \neq 0$ are called non-extremal, non-degenerate, or bifurcate horizons. Horizons with $\kappa = 0$ (such as when f and g have double zeroes) are called extremal or degenerate, and they require separate treatment.

1.d Kruskal coordinates

Change to null ingoing and outgoing coordinates $(t, r) \to (u = t - r_*, v = t + r_*)$ and examine whether in (u, v) coordinates the metric at $r = 2M$ is regular or not (Answer: it is not).

Try instead a related set of null coordinates (U, V), defined by

$$U = -2Me^{-u/4M} \,, \qquad V = 2Me^{v/4M} \,, \tag{14}$$

and, from them, introduce new time and space coordinates

$$T = \frac{1}{2}(U + V) \,, \qquad X = \frac{1}{2}(U - V) \,. \tag{15}$$

- Verify that, near $r = 2M$, these coordinates become, up to constant factors, the same as the (T, X) coordinates we introduced above in the Rindler limit of the solution. Therefore, in these coordinates, the horizon will be manifestly smooth.
- Write the Schwarzschild solution in terms of them, to find

$$ds^2 = 8M\frac{e^{-r/2M}}{r}\left(-dT^2 + dX^2\right) + r^2 d\Omega_2 \tag{16}$$

where $r(T, X)$ is given implicitly by the relation

$$2M(r - 2M)e^{r/2M} = -T^2 + X^2 \,. \tag{17}$$

[3] See footnote 2 again. The transverse space does not play any role in this analysis.

The only singularity in these coordinates is at $r = 0$. The geometry can then be smoothly (analytically) extended along all the horizons (future and past), resulting in the Kruskal maximal analytic extension of the Schwarzschild solution. You can try to piece together all of the information we have obtained above to try to draw a picture of the Kruskal geometry in the (T, X) plane. (This takes some work, but the result is rewarding).

NB: Although Kruskal coordinates help clarify the global nature of the maximal Schwarzschild geometry, the implicit nature of (17) often makes them impractical. Eddington-Finkelstein coordinates are usually the most efficient way of verifying horizon regularity.

LECTURE 2

2. Wave propagation in Schwarzschild spacetime

Consider the propagation of a massless scalar field in a spacetime of the form[4]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{18}$$

• Write down the form of the wave equation

$$\Box\Phi \equiv \nabla_i\nabla^i\Phi = \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\Phi\right) = 0 \tag{19}$$

(keep it in a compact form).

• Observe that the angular part is the same as in the wave equation in flat space, so the angular dependence can be separated and solved by introducing the spherical harmonics $Y_{lm}(\theta,\varphi)$. Then, write the wave equation for the radial field modes $\phi_{\omega lm}(r)$ in the decomposition

$$\Phi(x^\mu) = e^{-i\omega t}Y_{lm}(\theta,\varphi)\phi_{\omega lm}(r)\,. \tag{20}$$

• We know that in flat space (which is approached as $f \to 1$) it is convenient to introduce a new radial field variable $\psi_{\omega lm}$ defined as

$$\phi_{\omega lm} = \frac{\psi_{\omega lm}}{r}\,. \tag{21}$$

In addition, for the propagation of massless excitations in the black hole background it is convenient to introduce the tortoise coordinate r_* defined as

$$dr_* = \frac{dr}{f(r)}\,. \tag{22}$$

With these changes, you must find an equation of the form

$$-\frac{\partial^2\psi_{\omega lm}(r_*)}{\partial r_*^2} = \left(\omega^2 - V_l(r)\right)\psi_{\omega lm}(r_*) \tag{23}$$

in terms of an effective potential $V_l(r)$ (which you can leave expressed in terms of r, with the understanding that r is a function of r_*).

• For the Schwarzschild spacetime, with

$$f = 1 - \frac{2M}{r}\,, \tag{24}$$

sketch the shape of potential V_l vs. r_*/M for different values of l.

V_l is the effective radial potential that a massless scalar wave feels when propagating in this background. We have mapped this problem to one of waves in a one-dimensional potential that extends in the range $r_* \in (-\infty, +\infty)$. There are many questions that can be answered qualitatively from the shape of this potential. For instance: argue that there are not any bound states of the scalar field in the black hole background. This shows (at the perturbative level) that the black hole does not admit 'scalar hair'.

[4] This is not the most general static spherical metric. Doing this exercise for the general case is only a little more involved.

LECTURE 3

3a. Rotation parameters for stars, planets, and other objects

• Estimate, assuming rigid rotation, the dimensionless rotation parameter $a_* = a/M$, which in conventional units is $a_* = cJ/(GM^2)$, for: (a) the Sun; (b) the Earth; (c) a rapidly rotating neutron star, of mass $\simeq 1.5 M_\odot$, radius $\simeq 10$ km and rotation period $\simeq 1.5$ ms; (d) a ball with radius 1 cm, weight 1 g, spinning at 1 Hz.

• Comment on the results, in particular how/why they differ so much from the maximum value for a Kerr black hole $a_* = 1$.

3b. Extension across the Kerr horizon

In the Kerr solution, change to Eddington-Finkelstein ingoing coordinates $(t, \phi) \to (v, \tilde{\phi})$ as

$$dv = dt + (r^2 + a^2)\frac{dr}{\Delta}, \qquad d\tilde{\phi} = d\phi + a\frac{dr}{\Delta}. \tag{25}$$

• Write the metric in coordinates $(v, r, \theta, \tilde{\phi})$ and show that it is regular at the points $r_\pm$ where $\Delta = 0$.

• Find the change to outgoing E-F coordinates $(t, \phi) \to (u, \hat{\phi})$ that allow to extend the metric across the past horizon.

3c. Superradiance

Consider a complex massless scalar field Φ, which satisfies the Klein-Gordon equation

$$\nabla_\mu \nabla^\mu \Phi = 0. \tag{26}$$

For this field we can construct a current

$$J_\mu = i(\Phi \nabla_\mu \Phi^* - \Phi^* \nabla_\mu \Phi), \tag{27}$$

which gives the flux of the field (*e.g.,* the flux of particles associated to the field, when it is quantized).

• Show that this current is conserved when the Klein-Gordon equation is satisfied.

Now scatter this field off a rotating black hole. Due to the symmetries of the Kerr metric and the linearity of the Klein-Gordon equation, we can expand the field into modes and then consider them individually, *i.e.,*

$$\Phi = \Phi_{\omega m}(r, \theta) e^{-i\omega t} e^{im\phi}. \tag{28}$$

• Show that the flux F across the horizon

$$F = -J_\mu \xi^\mu, \tag{29}$$

where ξ is the horizon generator

$$\xi = \partial_t + \Omega_H \partial_\phi, \tag{30}$$

is negative for modes satisfying $\omega < m\Omega_H$. In other words, there is a positive flux of these modes out of the horizon: they will reflect off the black hole with larger amplitude than they came in with. This is called superradiance.

LECTURE 4

4. Entropy of astrophysical black holes

The Bekenstein-Hawking entropy

$$S_{BH} = \frac{c^3}{\hbar G}\frac{\mathcal{A}_H}{4} \tag{31}$$

is enormous for astrophysical black holes. Here we have taken Boltzmann's constant $k_B = 1$, so temperature is measured in units of energy and the entropy is dimensionless. Ignoring rotation (which would only introduce corrections by factors of order one) we can write

$$S_{BH} \simeq 10^{77}\left(\frac{M}{M_\odot}\right)^2 , \tag{32}$$

where $M_\odot = 2 \times 10^{33}$ g is the mass of the Sun. In the following, we will try to obtain order-of-magnitude estimates of the entropy of astrophysical black holes and compare it to the entropies of other relevant systems. Our assumptions will be rather crude and may be off by one or even two orders of magnitude, but the comparisons will still be significant.

• **Entropy of a star**. A very crude estimate (but sufficient for our purposes) of the entropy of the Sun is the following. For an ideal gas of n particles, the entropy is $S \sim n$. Regard the Sun as a ball of a gas of particles of mass equal to the proton mass, $m_p \sim 10^{-24}$g. What is then the entropy of the Sun? What is the entropy of a black hole of the same mass?

• **Entropy of a galaxy**. Estimate in the same manner the entropy of the galaxy from the following sources:

1. Luminous matter (stars), if the luminous mass in the galaxy is $M_{\text{galaxy}} \sim 10^{11} M_\odot$.
2. Central black hole. There is significant evidence that our galaxy contains a central black hole with mass $M_\bullet \simeq 4 \times 10^6 M_\odot$.
3. Stellar-mass black holes. The number of black holes of stellar mass $\sim M_\odot$ in the galaxy is estimated to be $\sim 10^8$.

Which of these three contributions to the entropy of the galaxy is the largest?

• **Entropy of the Universe**. Estimate the following contributions to the entropy of the Universe:

1. Luminous matter (stars in galaxies): take the radius of the Universe to be $\sim 10^{10}$ ly (ly = light year), and consider that each galaxy occupies a sphere of radius $\sim 10^5 - 10^6$ ly, with ou galaxy being a typical galaxy.
2. Cosmic microwave background radiation at $T \sim 3\text{K} \sim 10^{-4}$eV. The entropy of a gas of photons at temperature T in a volume V is $S \sim VT^3$ (with $\hbar = 1 = c$). (*Hint:* write T in terms of the wavelength of the radiation).

3. Entropy in black holes at galactic centers. Assume that each galaxy has a central black hole of mass $\sim 10^6 - 10^9 M_\odot$.

You must have found that the total entropy of the Universe is overwhelmingly dominated by black hole entropy. If you have any suggestion for what could be the ultimate meaning (if any) of this striking fact,[5] I would like to know.

[5]Which adds to the better known enigmas that the total energy of the Universe is dominated by dark energy, and the total mass by dark matter.