Dispersion Relations in κ-Noncommutative Cosmology

Andrzej Borowiec & Anna Pachoł

University of Wrocław

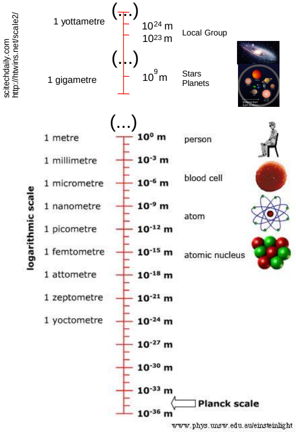
based on results developed in collaboration with
P. Aschieri and A. Pachoł
in J. High Energ. Phys. 2017, 152 (2017) [arXiv:1703.08726]
and JCAP 04 (2021) 025 [arXiv:2009.01051].

Plan:

1. Motivation and general framework
2. Part I: Flat spacetime
   Poincaré Casimir and twisted observables
3. Part II: Noncommutative cosmology
   - Twisted differential calculus
   - Dispersion relations

Quantum Gravity

gravitational interactions of matter and energy described by quantum theory



General Relativity

Quantum Mechanics

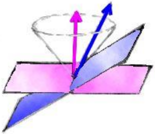
Quantum Gravity
= *quantum effects and gravitational interactions are equally strong*

It happens at the **Planck scale**
(unless extra dimensional theories are correct).

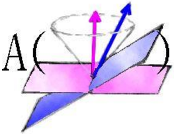
Noncommutative Geometry: origin of quantum space-times

points on manifold M ⟶ algebra of functions on M

http://visualrelativity.com



Minkowski spacetime (M, η) with the position of an event $p = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$



"Minkowski algebra" (abelian, associative,unital) with coordinate functions $x^\mu \in C^\infty(\mathbb{R}^4)$
$x^\mu : \mathbb{R}^4 \to \mathbb{C}$
$[x^\mu, x^\nu] = 0, \quad \mu, \nu = 0, 1, 2, 3.$

At the Planck scale $\quad x^\mu \longrightarrow \hat{x}^\mu$

Example: **κ - Minkowski space-time**

$$[\hat{x}^0, \hat{x}^k] = \frac{i}{\kappa}\hat{x}^k \quad , \quad [\hat{x}^i, \hat{x}^k] = 0$$

The phase space of Quantum Mechanics

$$[x^\mu, P_\nu] = i\delta^\mu_\nu \hbar, \qquad [x^\mu, x^\nu] = 0, \qquad [P_\mu, P_\nu] = 0$$

Generated by the position $\boldsymbol{x^\mu}$ and momentum $\boldsymbol{P_\mu}$ generators (the Heisenberg algebra)

admitting Hilbert space operator representation (CCR)

- Archetype of a **noncommutative space.**

  Replacing 'space' by a **noncommutative algebra**.

- The Heisenberg uncertainty principle: $\Delta x^\mu \Delta P_\nu \geq \frac{\hbar}{2}$.

- Noncommutative geometry - generalised notion of geometry taking into account noncommuatative algebraic structure
- The deformation nature allows for obtaining quantum gravitational corrections to the classical (commutative) solutions.
- Can be helpful in providing the phenomenological models quantifying the effects of quantum gravity.
- One of the mostly studied possible phenomenological effects of quantum gravity is the **modification in wave dispersion**. Such investigations were inspired by the observations of gamma ray bursts (GRBs).

Quantum symmetries

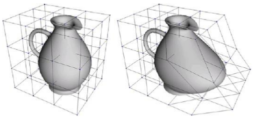
Deformed relativistic symmetries = **Hopf algebras**
quantum spacetimes = **Hopf module algebras**

- Hopf algebra $H(\mu, \eta, \Delta, \epsilon, S)$ is a structure composed by
    1. a (unital associative) algebra (H, μ, η)
    2. a (counital coassociative) coalgebra (H, Δ, ϵ)

  with $S : H \to H$ the antipode.

From any Lie algebra g one can make a Hopf algebra

$$H = (Ug, \Delta_0, S_0, \epsilon)$$



Quantum deformations

Classical —— deformation = quantization ——→ Quantum

$\mathcal{F} \in (Ug \otimes Ug)[[h]]$

Space-time

$\mathcal{A} = (C^\infty(M), \mu)$

$[x^\mu, x^\nu] = 0$

$x^\mu \to \hat{x}^\mu$ ——→

$\mathcal{A}^\mathcal{F} = (C^\infty(M)[[h]], \star)$

$x^\lambda \star x^\nu \doteq \mu \circ \mathcal{F}^{-1}(x^\lambda \otimes x^\nu)$

$[\hat{x}^\lambda, \hat{x}^\nu] = x^\lambda \star x^\nu - x^\nu \star x^\lambda$

Symmetry

Lie Algebra	Hopf Algebra (Quantum Group)
$\mathfrak{g}$ ——→	$\mathcal{H} = (\mathcal{U}(\mathfrak{g}), \Delta_0, \epsilon, S_0)$

$\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$

- Poincare – Weyl '2009: $\mathcal{U}^{\mathcal{F}}_{\mathfrak{pw}} \quad (M_{\mu\nu}, P_\mu, D)$
- Conformal '2015: $\mathcal{U}^{\mathcal{F}}_{\mathfrak{so}(2,4)} \quad (M_{\mu\nu}, P_\mu, D, K_\mu)$
- Inhom. Gen. Lin. '2009: $\mathcal{U}^{\mathcal{F}}_{\mathfrak{igl}(n)} \quad (L_{\mu\nu}, P_\mu)$

Lie algebra of vector fields as Hopf algebra

- **Deformations of spacetime symmetries** - Lie algebra g of vector fields ξ
- In the coordinate basis: $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \partial_\mu$.
- This algebra generates the diffeomorphism symmetry; one can also consider subalgebras of g like Poincaré algebra or conformal algebra as symmetry.
- Universal enveloping algebra $U\Xi$ of vector fields includes linear differential operators.

Lie algebra of vector fields as Hopf algebra

1. Ug as Hopf algebra $(Ug, \Delta_0, \epsilon, S_0)$, for $\xi \in g$ (in the coordinate basis : $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \partial_\mu$):

$$\begin{aligned} [\xi, \eta] &= (\xi^\mu \partial_\mu \eta^\rho - \eta^\mu \partial_\mu \xi^\rho)\partial_\rho, \\ \Delta_0(\xi) &= \xi \otimes 1 + 1 \otimes \xi, \\ \varepsilon(\xi) &= 0, \quad S(\xi) = -\xi. \end{aligned}$$

Lie algebra of vector fields as Hopf algebra

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2. The **module algebra** $\mathcal{A} \ni x^\mu, x^\nu$ is an underlying spacetime of given symmetry:

$$\xi \triangleright (x^\mu \cdot x^\nu) = (\xi_1 \triangleright x^\mu) \cdot (\xi_2 \triangleright x^\nu)$$

where $\Delta(\xi) = \xi_1 \otimes \xi_2$ (Sweedler notation).

Drinfeld twisting techniques

provides quantized universal enveloping algebras

$$(Ug, \mathcal{A}) \rightleftarrows (Ug^{\mathcal{F}}, \mathcal{A}^{\mathcal{F}})$$

The twist $\mathcal{F}$ is an invertible element of $(Ug \otimes Ug)[[h]]$.

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h),$$

which provides an undeformed case at the zero-th order in the **deformation parameter** h.

Notation:

$$\mathcal{F} = \mathrm{f}^{\alpha} \otimes \mathrm{f}_{\alpha}, \quad \mathcal{F}^{-1} = \bar{\mathrm{f}}^{\alpha} \otimes \bar{\mathrm{f}}_{\alpha},$$

(sum over $\alpha = 1, 2, ...$ assumed, in fact infinite formal power series in h) $\bar{\mathrm{f}}^{\alpha} \in Ug$ and $\mathrm{f}^{\alpha} \in Ug$ and

$$\mathcal{F} = \sum_{\alpha=0}^{\infty} (\mathrm{f}^{\alpha} \otimes \mathrm{f}_{\alpha}) h^{\alpha}$$

The twist changes the symmetry to twisted symmetry (as deformed Hopf algebra) $Ug^{\mathcal{F}}$

$$
\begin{aligned}
[\xi, \eta] &= (\xi^\mu \partial_\mu \eta^\rho - \eta^\mu \partial_\mu \xi^\rho)\partial_\rho, \\
\Delta^{\mathcal{F}}(\xi) &= \mathcal{F}\Delta_0(\xi)\mathcal{F}^{-1} \\
\varepsilon(\xi) &= 0, \quad S^{\mathcal{F}}(\xi) = \mathrm{f}^\alpha S_0(\mathrm{f}_\alpha) S_0(\xi) S_0(\bar{\mathrm{f}}^\beta)\bar{\mathrm{f}}_\beta
\end{aligned}
$$

- the algebra $([\cdot, \cdot])$ remains undeformed;
- the deformation depends on formal parameter h;

Coassociativity of the deformed coproduct and associativity of the star-multiplication is ensured by the two-cocycle condition:

$$
(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}
$$

Star-product

$$A = (C^{\infty}(M), \cdot) \qquad \Longrightarrow \qquad A^{\mathcal{F}} = (C^{\infty}(M), \star)$$

the algebra of smooth functions becomes a **noncommutative spacetime** with the twisted $\star$-product

$$x^{\mu} \star x^{\nu} \quad = \quad \cdot \quad \mathcal{F}^{-1}(x^{\mu} \otimes x^{\nu}) = \bar{\mathrm{f}}^{\alpha}(x^{\mu})\bar{\mathrm{f}}_{\alpha}(x^{\nu})$$

$x^{\mu}, x^{\nu} \in C^{\infty}(M)$.

- such $\star$-product is noncommutative and associative.
- $A^{\mathcal{F}}$ can be represented by deformed, $\star$-commutators of noncommutative coordinates:

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = [x^{\mu}, x^{\nu}]_{\star} = x^{\mu} \star x^{\nu} - x^{\mu} \star x^{\mu}$$

Quantum (noncommutative) spacetimes

1. Canonical (Moyal-Weyl) spacetime A_θ:
   $$[\hat{x}^\mu, \hat{x}^\nu] = ih\theta^{\mu\nu}$$
   with deformation parameter h of length^2 (L_P) dim.

   S. Doplicher, K. Fredenhagen, J. E. Roberts, Commun. Math. Phys. 172 (1995), [arXiv:hep-th/0303037].

2. Lie-algebraic type spacetime:
   $$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa}\theta^{\mu\nu}_\rho \hat{x}^\rho$$
   with deformation parameter $\kappa = L_p^{-1}$ of mass (M_P) dim.

   Special case: A_κ

   $$[\hat{x}^0, \hat{x}^k] = \frac{i}{\kappa}\hat{x}^k \quad , \quad [\hat{x}^i, \hat{x}^k] = 0$$

   - the so-called: κ-Minkowski spacetime.

   S. Majid, H. Ruegg Phys.Lett. B334 (1994) [hep-th/9405107] ; S. Zakrzewski J. Phys. A 127 (1994).

Twisted generators

P. Aschieri, A. Schenkel, Adv. Theor. Math. Phys. 18 3 (2014), arXiv:1210.0241.

Within the Hopf algebra $\mathcal{H}^{\mathcal{F}} = \left(Ug^{\mathcal{F}}, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}\right)$ we can introduce a notion of **quantum Lie algebra** g^F.

- g and of $g^{\mathcal{F}}$ are in 1-1 correspondence, for all $\chi \in g$ we have

$$\chi^{\mathcal{F}} = \bar{\mathrm{f}}^{\alpha}(\chi)\,\bar{\mathrm{f}}_{\alpha} \in g^{\mathcal{F}}$$

  where $\xi(\chi) = \xi_1 \chi S(\xi_2)$ is the Ug adjoint action.

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  where $\xi(\chi) = \xi_1 \chi S(\xi_2)$ is the Ug adjoint action.
- the subspace $g^{\mathcal{F}}$ generates $Ug^{\mathcal{F}}$.
- has a deformed Lie bracket $[\ ,\ ]_{\mathcal{F}} : g^{\mathcal{F}} \otimes g^{\mathcal{F}} \to g^{\mathcal{F}}$ - given by the **adjoint action** of $Ug^{\mathcal{F}}$:

$$[\chi, \xi]_{\mathcal{F}} = \chi_{1^{\mathcal{F}}} \xi S^{\mathcal{F}}(\chi_{2^{\mathcal{F}}}) \in g^{\mathcal{F}}$$

  where $\Delta^{\mathcal{F}}(\chi) = \chi_{1^{\mathcal{F}}} \otimes \chi_{2^{\mathcal{F}}}$.

Twisted differential calculus - general framework

[S. Majid, R. Oeckl, Commun.Math.Phys. 205 (1999)
arXiv:math/9811054
P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess , Class.Quant.Grav. 23 (2006)
arXiv:hep-th/0510059]

The star-product between functions $g \in C^\infty(M)$ and 1-forms $\omega \in \Omega^r(M)$:

$$g \star \omega = \bar{\mathrm{f}}^\alpha(g)\bar{\mathrm{f}}_\alpha(\omega)$$

- the action of $\bar{\mathrm{f}}_\alpha$ - via the Lie derivative;
- Cartan's formula for the Lie derivative along the vector field ξ of an arbitrary form ω

$$\mathcal{L}_\xi \omega = \mathrm{d} i_\xi \omega + i_\xi \mathrm{d}\omega.$$

  where d is the exterior derivative and i_ξ is the contraction along the vector field ξ.
- The $\star$-wedge product on two arbitrary forms ω and ω' is

$$\omega \wedge_\star \omega' = \bar{\mathrm{f}}^\alpha(\omega) \wedge \bar{\mathrm{f}}_\alpha(\omega')$$

- The exterior derivative $\mathrm{d} : A \to \Omega$ satisfies:

$$\begin{aligned} \mathrm{d}(f \star g) &= \mathrm{d}f \star g + f \star \mathrm{d}g, \\ \mathrm{d}^2 &= 0, \\ \mathrm{d}f &= (\partial_\mu f)\mathrm{d}x^\mu \end{aligned}$$

- The usual exterior derivative d commutes with the Lie derivative which enters in the definition of the $\star$-product.

Part I
Flat spacetime

Symmetry

- Poincaré-Weyl-Lie algebra

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\lambda}] &= i(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda}), \\
[M_{\mu\nu}, P_{\rho}] &= i(\eta_{\nu\rho}P_{\mu} - \eta_{\mu\rho}P_{\nu}) \quad , \qquad [P_{\mu}, P_{\lambda}] = 0, \\
[D, P_{\mu}] &= iP_{\mu} \quad , \quad [D, M_{\mu\nu}] = 0.
\end{aligned}
$$

The differential representation of the generators of Poincaré-Weyl algebra is

$$
P_{\mu} = -i\partial_{\mu} \quad ; \quad M_{\mu\nu} = -i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \quad ; \quad D = -ix^{\mu}\partial_{\mu}
$$

Symmetry

- Poincaré-Weyl-Lie algebra

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\lambda}] &= i(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda}), \\
[M_{\mu\nu}, P_{\rho}] &= i(\eta_{\nu\rho}P_{\mu} - \eta_{\mu\rho}P_{\nu}) \quad , \qquad [P_{\mu}, P_{\lambda}] = 0, \\
[D, P_{\mu}] &= iP_{\mu} \quad , \quad [D, M_{\mu\nu}] = 0.
\end{aligned}
$$

The differential representation of the generators of Poincaré-Weyl algebra is

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P_\mu = -i\partial_\mu \quad ; \quad M_{\mu\nu} = -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \quad ; \quad D = -ix^\mu\partial_\mu
$$

Universal enveloping algebra of Poincaré-Weyl algebra - as Hopf algebra :

$$
\Delta_0(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}
$$

$$
\Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu \quad \text{and} \quad \Delta_0(D) = D \otimes 1 + 1 \otimes D
$$

with antipodes

$$
S(M_{\mu\nu}) = -M_{\mu\nu};\ S(P_\mu) = -P_\mu;\ S(D) = -D
$$

and counits

$$
\epsilon(M_{\mu\nu}) = \epsilon(P_\mu) = \epsilon(D) = 0
$$

Jordanian twist

A. Borowiec, A.P., Phys.Rev.D79:045012 (2009) [arXiv:0812.0576].

For the deformation we can use Jordanian twist (with support in Poincaré-Weyl Hopf algebra)

- $$\mathcal{F} = \exp\left(-iD \otimes \sigma\right) \quad ; \quad \sigma = \ln\left(1 + \frac{1}{\kappa}P_0\right)$$

- κ - deformation parameter (classical limit when $\kappa \to \infty$)

- it provides

$$[x^0, x^k]_\star = x^0 \star x^k - x^k \star x^0 = \frac{i}{\kappa}x^k \quad , \quad [x^i, x^k]_\star = 0$$

κ-Minkowski spacetime

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- $$\mathcal{F} = \exp\left(-iD \otimes \sigma\right) \quad ; \quad \sigma = \ln\left(1 + \frac{1}{\kappa}P_0\right)$$
- $$\mathcal{F}^{-1} = 1 \otimes 1 + iD \otimes \frac{1}{\kappa}P_0 + \frac{1}{2}iD(iD-1) \otimes \frac{1}{\kappa^2}P_0^2 + \ldots$$
- κ - deformation parameter (classical limit when $\kappa \to \infty$)
- it provides
$$[x^0, x^k]_\star = x^0 \star x^k - x^k \star x^0 = \frac{i}{\kappa}x^k \quad , \quad [x^i, x^k]_\star = 0$$

κ-Minkowski spacetime

Twisted generators

P. Aschieri, A. Borowiec, A.P.,JHEP 152 (2017) [arXiv:1703.08726].

- Twisted generators of Poincaré-Weyl algebra:

$$P_\mu^{\mathcal{F}} = \bar{\mathrm{f}}^\alpha(P_\mu)\bar{\mathrm{f}}_\alpha = P_\mu \frac{1}{1+\frac{1}{\kappa}P_0}$$

$$M_{\mu\nu}^{\mathcal{F}} = M_{\mu\nu} \quad ; \quad D^{\mathcal{F}} = D$$

- Twisted Poincaré Casimir from $P_\mu^{\mathcal{F}}$

$$\Box^{\mathcal{F}} = P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}} = P_\mu P^\mu \frac{1}{\left(1+\frac{1}{\kappa}P_0\right)^2}$$

- Twisted commutation relations

$$\begin{aligned}\left[\Box^{\mathcal{F}}, P_\mu^{\mathcal{F}}\right]_{\mathcal{F}} &= 0 = \left[\Box^{\mathcal{F}}, M_{\mu\nu}^{\mathcal{F}}\right]_{\mathcal{F}} \\ \left[\Box^{\mathcal{F}}, D^{\mathcal{F}}\right]_{\mathcal{F}} &= -2i\Box^{\mathcal{F}}\end{aligned}$$

Twisted Poincaré Casimir

- Poincaré Casimir $\Box = P_\mu P^\mu$ can be deformed through twist into:

$$\Box^{\mathcal{F}} = \frac{P_\mu P^\mu}{\left(1 + \frac{1}{\kappa} P_0\right)^2}$$

- This type of invariant on momentum space leading to deformed dispersion relation was already considered in DSR framework.

  [J. Magueijo and L. Smolin in Phys.Rev.Lett.88 (2002), hep-th/0112090; and in Phys.Rev.D67 (2003), gr-qc/0207085.]

Twisted observables

Twisted generators $X^{\mathcal{F}} \in g^{\mathcal{F}}$ as the observables.

- given $\mathcal{F} \longrightarrow$ unique $g^{\mathcal{F}}$;
- $X^{\mathcal{F}}$ act on fields as **quantum infinitesimal transformations**;
- they are the generators of the twisted Lie algebra $g^{\mathcal{F}}$ and are closed under the twisted commutator $[\cdot,\cdot]_{\mathcal{F}}$:

$$
\begin{aligned}
\left[M^{\mathcal{F}}_{\rho\lambda}, M^{\mathcal{F}}_{\mu\nu}\right]_{\mathcal{F}} &= -i(\eta_{\mu\lambda}M^{\mathcal{F}}_{\nu\rho} - \eta_{\nu\lambda}M^{\mathcal{F}}_{\mu\rho} + \eta_{\nu\rho}M^{\mathcal{F}}_{\mu\lambda} - \eta_{\mu\rho}M^{\mathcal{F}}_{\nu\lambda}), \\
\left[M^{\mathcal{F}}_{\mu\nu}, P^{\mathcal{F}}_{\rho}\right]_{\mathcal{F}} &= i(\eta_{\nu\rho}P^{\mathcal{F}}_{\mu} - \eta_{\mu\rho}P^{\mathcal{F}}_{\nu}), \\
[P^{\mathcal{F}}_{\mu}, P^{\mathcal{F}}_{\lambda}]_{\mathcal{F}} &= 0 \quad , \quad \left[D^{\mathcal{F}}, P^{\mathcal{F}}_{\lambda}\right]_{\mathcal{F}} = iP^{\mathcal{F}}_{\lambda}.
\end{aligned}
$$

- $P^{\mathcal{F}}_{\mu}$ are Hermitean.

Twisted observables

- $P_\mu^{\mathcal{F}}$ - have the interpretations as the generators of infinitesimal (deformed) translations;
- we confirm this by recalling their associated differential geometry (Part II);
- they allow us to define the appropriate (Poincaré) Casimir operator in the twisted Lie algebra:

$$\left[\Box^{\mathcal{F}}, P_\mu^{\mathcal{F}}\right] = 0 = \left[\Box^{\mathcal{F}}, M_{\mu\nu}^{\mathcal{F}}\right] \quad , \quad \left[\Box^{\mathcal{F}}, D^{\mathcal{F}}\right]_{\mathcal{F}} = -2i\Box^{\mathcal{F}}$$

Dispersion relation: Flat spacetime

- Deformed wave equation: $\Box^{\mathcal{F}}\phi = P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}}\phi$ for massless particles is equivalent to $\Box\phi = 0$.
- The energy-momentum dispersion relations $P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}} = 0$ are undeformed.

- The group velocity $v_g = \frac{d\omega}{dk} = c$ is as in the classical case due to the fact that the (usual) plane waves are the 'eigenvectors' of the twisted observables.

- $P_\mu^{\mathcal{F}} e^{ik_\mu x^\mu} = k_\mu^{\mathcal{F}} e^{ik_\mu x^\mu}$ evaluation of the energy momentum operator on the monochromatic wave leads to modified Einstein -Planck relatons:

$$E^{\mathcal{F}} = \omega^{\mathcal{F}} = \frac{\omega}{1 - \frac{i}{\kappa}\omega} \quad \text{and} \quad \mathbf{p}^{\mathcal{F}} = \frac{\mathbf{k}}{1 - \frac{i}{\kappa}\omega}$$

$\kappa \to E_p$ (Planck energy).

Differential calculus deformed with Jordanian twist

P. Aschieri, A. Borowiec, A.P.,JHEP 152 (2017) [arXiv:1703.08726].

- For the **twisted differential calculus** we use the coordinate basis where the basis 1-forms are denoted as $\mathrm{d}x^\mu$.

- The action of a vector fields in the twist is via Lie derivative:

$$\mathcal{L}_{P_\mu}(\mathrm{d}x^\nu) = 0, \quad \mathcal{L}_D(\mathrm{d}x^\mu) = -i\mathrm{d}x^\mu$$

- Using these relations one can show that the basis 1-forms anticommute:

$$\mathrm{d}x^\mu \wedge_\star \mathrm{d}x^\nu = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

Therefore we have:

$$\mathrm{d}x^\mu \wedge_\star \mathrm{d}x^\nu = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = -\mathrm{d}x^\nu \wedge_\star \mathrm{d}x^\mu$$

But the basis 1-forms do not $\star$-commute with functions:

$$f \star \mathrm{d}x^\mu = f\mathrm{d}x^\mu$$
$$\mathrm{d}x^\mu \star f = \mathrm{d}x^\mu(1 + \frac{1}{\kappa}P_0)f$$

Therefore:

$$[f, \mathrm{d}x^\mu]_\star = \frac{i}{\kappa}\mathrm{d}x^\mu \partial_0 f$$

Part II
Noncommutative cosmology

- Noncommutative differential geometry approach is based on Drinfeld twist ($\mathcal{F}$) deformation.
- Can be implemented for any twist ($\mathcal{F}$) and any curved background (g).
- Toy model:
  Jordanian twist - giving κ-Minkowski spacetime in flat space
  $([x^0, x^k]_\star = x^0 \star x^k - x^k \star x^0 = \frac{i}{\kappa} x^k)$
  - in the presence of a Friedman-Lemaitre-Robertson-Walker (FLRW) cosmological background (in 2D).

Wave equation in curved spacetime

- The wave equation in curved spacetime is governed by the Laplace-Beltrami operator (for Lorenzian even dimensional manifolds):

$$\Box_{LB}\varphi = *\mathrm{d} * \mathrm{d}\varphi \,, \quad (+\mathrm{d} * \mathrm{d} * \varphi = 0)$$

- The Laplace-Beltrami operator is a generalization to curved spacetime of the D'Alembert operator and on a scalar field φ we have (using local coordinates)

$$\Box_{LB}\varphi = *\mathrm{d} * \mathrm{d}\varphi = \frac{1}{\sqrt{g}}\partial_\nu \left[\sqrt{g} g^{\nu\mu}\partial_\mu\varphi\right]$$

Hodge star deformed

A linear map $* : \Omega^r(M) \to \Omega^{n-r}(M)$. In local coordinates for an r-form is given by

$$*\omega = \frac{\sqrt{g}}{r!\,(n-r)!}\omega_{\mu_1\ldots\mu_r}\epsilon^{\mu_1\ldots\mu_r}{}_{\nu_{r+1}\ldots\ldots\nu_n}\underset{\cdot}{x}^{\nu_{r+1}}\wedge\ldots\underset{\cdot}{x}^{\nu_n}$$

where $\sqrt{g}$ is the square root of the absolute value of the determinant of the metric, the completely antisymmetric tensor $\epsilon_{\nu_1\ldots\nu_n}$ is normalized to $\epsilon_{1\ldots n} = 1$ and indices are lowered and raised with the metric g and its inverse.

- The deformation of the Hodge $*$ operation is explicitly dependent on the twist form:

$$*^{\mathcal{F}} = \bar{\mathrm{f}}^{\alpha}_{(1)} \triangleright \circ * \circ S\left(\bar{\mathrm{f}}^{\alpha}_{(2)}\right) \triangleright \circ \bar{\mathrm{f}}_{\alpha} \triangleright$$

- For Jordanian twist ($\mathcal{L}_{P_\nu}(\mathrm{d}x^\mu) = 0$) the non vanishing is only the zero-th order:

$$*^{\mathcal{F}}\left(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_s}\right) = *\left(dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_s}\right)$$

Deformed Laplace-Beltrami operator

P. Aschieri, A. B., A.Pachoł,
[arXiv:2009.01051].

- Deformation of the Laplace-Beltrami operator for any twist:

$$\Box^{\mathcal{F}}_{LB}\varphi = *^{\mathcal{F}} d *^{\mathcal{F}} d\varphi$$

- The wave equation for the scalar field in terms of twisted momenta for deformed LB op. with Jordanian twist:

$$\Box^{\mathcal{F}}_{LB}\varphi = \frac{1}{\sqrt{g}} \star \frac{\partial^{\mathcal{F}}_{\rho}}{\left(1+\frac{i}{\kappa}\partial^{\mathcal{F}}_{0}\right)^{1-n}} \left((\sqrt{g} g^{\mu\rho}) \star \frac{\partial^{\mathcal{F}}_{\mu}}{\left(1+\frac{i}{\kappa}\partial^{\mathcal{F}}_{0}\right)^{n-1}} \varphi \right) = 0$$

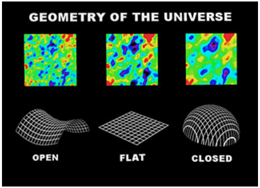
Solutions of deformed wave eq. for FRWL metric

Friedman-Robertson-Walker-Lemaitre (FRWL) metric

(for simplicity in 2 dimensions)

$$g = -dt^2 + a^2(t)\, dx^2$$

where $a(t)$ - scale factor



2-dim twisted wave equation

$$-a \star \partial_0^2 \varphi - (\partial_0 a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \varphi + a^{-1} \star \partial_x^2 \varphi = 0$$

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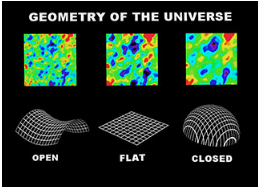

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In the classical limit it reduces to:

$-a\partial_0^2\varphi - \dot{a}\partial_0\varphi + \frac{1}{a}\partial_i^2\varphi = 0$ where $\dot{a} = \partial_0 a(t)$

Classical version of equation

$$-a\partial_0^2\varphi - \dot{a}\partial_0\varphi + \frac{1}{a}\partial_i^2\varphi = 0$$

- separation of variables: $\varphi = \lambda\left(t\right)e^{-ikx}$
- $$a\ddot{\lambda} + \dot{\lambda}\dot{a} + k^2\lambda\frac{1}{a} = 0$$
- it corresponds (in conformal time $d\eta = \frac{dt}{a}$ to harmonic oscillator type equation

$$(\partial_\eta^2 + k^2)\lambda = 0$$

which has the well known solution $\lambda = \exp i\omega\eta$: $\omega^2 = k^2$.

Twisted wave equation

$$a \star \partial_0^2 \varphi + (\partial_0 a) \star \left(1 - \frac{i}{\kappa}\partial_0\right) \partial_0 \varphi - a^{-1} \star \partial_x^2 \varphi = 0$$

- In the noncommutative case in 2 dimensions we consider the solution of the form: $\varphi = \lambda(t) \star e^{-ikx} = \lambda(t)\, e^{-ikx}$

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We simplify the equation as:

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Expand star-product in the first order of $\frac{1}{\kappa}$

$$a\partial_0^2\lambda + \partial_0(a)\left(1 - \frac{i}{\kappa}\partial_0\right)\partial_0\lambda + a^{-1}k^2\lambda - \frac{i}{\kappa}t\left(\partial_0 a\,\partial_0^3\lambda + \partial_0^2 a\,\partial_0^2\lambda + k^2\partial_0 a^{-1}\partial_0\lambda\right) = 0$$

Conformal time - classical case strategy

- As in the classical case - change the coordinates into conformal time η, and $' = \partial_\eta$
- Introduce simplified notation $s = \ln a$; $s' = \frac{a'}{a}$; $\frac{a''}{a} = s'' + (s')^2$;

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- Introduce simplified notation $s = \ln a$; $s' = \frac{a'}{a}$; $\frac{a''}{a} = s'' + (s')^2$;
- Look for the (perturbative) solution of the type:

$$\lambda = \exp\left(i\omega\eta + \frac{i}{\kappa}F\right)$$

- Classical part (at 0-th order) remains:

$$\left(\omega^2 - k^2\right)\lambda = 0$$

- And equation on $F(\eta)$ becomes:

(using the zero-th order solution $\omega = k$),

$$F'' + 2ikF' = \frac{ikt(\eta)}{a^2}\left(2\left(s'\right)^3 - 2s's'' - 2k^2s' + ik\left(s'' - 3\left(s'\right)^2\right)\right) - \frac{ik}{a}s'\left(s' - ik\right) .$$

Group velocity for the wave

Starting from

$$\varphi_k(x,t) = \lambda(t) \star e^{-ikx} = \lambda(t) e^{-ikx} = \exp\left(ik\eta + \frac{i}{\kappa} F\right) e^{-ikx} = e^{i(f_k(t) - kx)}$$

we get:

$$f_k(t) = \left(k\eta + \frac{1}{\kappa} F\right)(t)$$

Group velocity expression

$$v_g = \frac{\partial x}{\partial t} = \frac{\partial}{\partial k} \frac{\partial f_k(t)}{\partial t}$$

$\implies$ we need to compute $\dot{F} = \partial F / \partial t$.

$\Longrightarrow$ we need to compute $\dot{F} = \partial F/\partial t$

- easily obtained from the differential equation for F in the physical regime we are interested in: cosmic time related to large scale structure formation, and high frequency waves.
- There are three frequency parameters in the differential equation on F: $\omega = k$, t^{-1} and the Hubble parameter H;
- we obviously have $\omega >> t^{-1}$ for the present cosmic time as well as the cosmic time of emission of the travelling γ-ray, typically at redshift below $z = 10$.
- Similarly $\omega >> H \sim t^{-1}$

In this regime equation for F simplifies to

$$2ikF' = -\frac{2ik^3 ts'}{a^2}$$

$$\dot{F} = -\frac{k^2 t\dot{a}}{a^3} \ .$$

- The group velocity, at the first order in the $\frac{1}{\kappa}$ deformation, results

$$v_g = \frac{\partial x}{\partial t} = \frac{\partial}{\partial k}\frac{\partial f_k(t)}{\partial t} = \frac{1}{a}+\frac{1}{\kappa}\frac{\partial \dot{F}}{\partial k} = \frac{1}{a}\Big(1-\frac{2}{\kappa}\frac{kt\dot{a}}{a^2}\Big) = \frac{1}{a}\Big(1-\frac{2}{\kappa}\frac{\omega t\dot{a}}{a^2}\Big)$$

- Taking into account the $\frac{1}{a}$ factor due to the comoving coordinates and inserting the flat spacetime speed of light c we see that κ-spacetime noncommutativity in the presence of a FLRW metric leads to a velocity of photons $v_{ph} = v_g a$ given by

$$v_{ph} = c(1 - \frac{2}{\kappa}\frac{\omega t\dot{a}}{a^2}) \ .$$

- If we define (as usual) the energy where classical Lorentz violation (in our case Lorentz deformation) is manifested $E_{LV} := |\kappa|\hbar$.
- The variation of the speed of light v_{ph} with respect to the usual one c (of photons in flat spacetime, or of low energetic photons) is then given by

$$|1 - v_{ph}/c| \sim \frac{E_{ph}}{E_{LV}} \frac{2t\dot{a}}{a^2} \quad .$$

Comments on the results

$$v_{ph} = c(1 - \frac{2}{\kappa}\frac{\omega t \dot{a}}{a^2}) \ .$$

- The **combined effects of noncommutativity and gravity** affect the velocity of light by a term linearly dependent on the frequency ω, the cosmic time t, the Hubble parameter $H = \dot{a}/a$ and inversely proportional to the scale factor.
- We have $v_{ph} < c$ for $\frac{1}{\kappa}$ a positive time (as it is usually considered, and in an expansion phase of the universe $\dot{a} > 0$).
- In flat spacetime ($\dot{a} = 0$) as well as in commutative spacetime ($\kappa \rightarrow \infty$) there are no modified dispersion relations.
- This result offers an explicit cosmological correction to the usually considered models, which assume as the leading power for the correction to the light speed the expression $v_{ph} \ \sim \ c(1 - \frac{E_{ph}}{E_{LV}})$.
- one can actually estimate the order of magnitude of the variation of the speed of light.

Comments on the results

- We can also study the time lag Δt between the arrival of a low energetic and a high energetic photon emitted simultaneously during a gamma ray burst.
- the comoving distance between the gamma ray burst and the observer is the same for both photons;
- for the high energy photon it reads $\int_{t_{em}}^{t_0+\Delta t} v_g \, dt$
- for the low energy one it reduces to $\int_{t_{em}}^{t_0} \frac{c}{a} \, dt$.
- Equating these distances, and considering only first order corrections we obtain that the time delay Δt is given by

$$\Delta t = \frac{2E_{ph}}{E_{LV}} \int_{t_{em}}^{t_0} \frac{t\dot{a}}{a^3} dt = \frac{2E_{ph}}{E_{LV}} \int_0^z t\,(1+z')dz' \ .$$

- For the range of redshifts we are interested into (up to $z \sim 10$) we can use the analytic solution $a(t) = (1+z)^{-1} = (\frac{\Omega_m}{\Omega_\Lambda})^{1/3} \sinh^{2/3}(t/t_\Lambda)$, $t_\Lambda = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}$ and obtain the time lag

$$\Delta t = 2\frac{E_{ph}}{E_{LV}} t_\Lambda \int_0^z \mathrm{arcsinh}\sqrt{\frac{\Omega_\Lambda}{\Omega_m}(1+z')^{-3}}\ (1+z')dz' \ .$$

- Our model gives a time lag that is ~ 3 times the ones considered in the typical 'Lorentz invariance violation' literature.

Comments on the results

- In the present work, as a first approximation, we have considered a commutative gravity background, hence noncommutativity affects only propagation of light.
- In a noncommutative theory of gravity consistently coupled to light, one could consider the backreaction effects of turning on noncommutativity also on the gravitational field.

Summary

- Quantum spacetimes - motivated by the Planck scale effects.
- Twist deformation and Noncommutative Geometry - allow for obtaining quantum corrections to the classical solutions. Twisted generators as observables.
- Framework is valid not only for the flat spacetimes, but allows for more general **curved background** as well.
- The result that the combined effects of noncommutativity and curvature produce modified dispersion relations is expected to be a **general feature** of wave equations in noncommutative curved spacetime.

Summary

- we used a top-down approach that complements the bottom-up one of phenomenological models.
- we applied noncommutative differential geometry to derive the propagation of waves in noncommutative cosmology.
- we studied a **noncommutative deformation of the wave equation in curved background** and we discuss the modification of dispersion relations due to the presence of both noncommutativity and curvature of spacetime.
- as a first approximation we turn on noncommutativity in the usual (classical) homogeneous and isotropic gravity solution given by FLRW spacetime, and derive the wave equation for massless particles in this context.
- This is a first step toward a more comprehensive approach that encompasses both the dynamics of light and of gravity in a noncommutative spacetime. We have considered a classical gravity background.

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Thank you for your attention!