Thermal CFT correlators

Borut Bajc

J. Stefan Institute, Ljubljana, Slovenia

BB, Lugo - work in progress

Introduction

In d-dimensional Euclidean CFT the 2-point correlation function (propagator) of the operator $\mathcal{O}(x)$ of anomalous dimension Δ is constrained by conformality to be

$$G_2(x) \propto \frac{1}{|x|^{2\Delta}} \quad , \quad |x|^2 = \vec{x}^2 + t^2$$

with the Fourier transform

$$G_2(k) \propto |k|^{2\Delta - d} \quad , \quad |k|^2 = \vec{k}^2 + \omega^2$$

The splitting $x \to \vec{x}, t(= x^0)$ or $k \to \vec{k}, \omega(= k^0)$ not necessary here, but is necessary for temperature $T \neq 0$

A generic CFT is usually nonperturbative so not obvious how to introduce the temperature

But if we assume that such a CFT has a gravitational dual, then we could use the AdS/CFT correspondence

The temperature in the CFT corresponds to the Hawking temperature of the horizon in the bulk, which comes as the black hole solution or, in our case, the black brane solution (large mass black hole limit) in the AdS

We will be interested in the small temperature corrections, $T \ll k$ and $T \ll \omega$

It turns out that the opposite limit, large T corrections, is much simpler

The equation

CFT thermal propagator from AdS/CFT correspondence:

Solution of the e.o.m. : black brane in Euclidean AdS_{d+1} $(L=1)$

$$ds^2 = \frac{1}{z^2}\left(\frac{dz^2}{f(z)} + f(z)dt^2 + d\vec{x}^2\right)$$

$$f(z) = 1 - \left(\frac{z}{z_h}\right)^d$$

$z \to \infty$ (boundary)

$z \to z_h$ (horizon)

Hawking temperature: $T = \frac{d}{4\pi z_h}$

Put a scalar field into this non-dynamical background

$$S_{bulk} = \frac{1}{2} \int d^{d+1}x \sqrt{det\ g_{ab}} \left(\partial_a \phi \, g^{ab} \, \partial_b \phi + m^2 \phi^2 \right)$$

According to the AdS/CFT dictionary, the operator in the boundary CFT dual to ϕ has anomalous dimension

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$$

assume Δ non-integer

The linearised excitation of $\xi(z) = \xi(z,k)$ defined as

$$\phi(x,z) = \int \frac{d^d k}{(2\pi)^d} e^{i\omega t + i\vec{k}\vec{x}}\, \xi(z)$$

becomes $(k^2 = \omega^2 + \vec{k}^2)$

$$f(z) z^2\, \xi''(z) - (d - f(z)) z\, \xi'(z)$$
$$- \quad \left(z^2 \left(k^2 + \omega^2 \frac{1 - f(z)}{f(z)} \right) + \Delta(\Delta - d) \right) \xi(z) = 0$$

$$f(z) = 1 - (z/z_h)^d$$

According to AdS/CFT the solution close to the boundary gives the propagator

$$z \to 0 \; : \; \xi(z) \propto z^{d-\Delta} + G_2(\omega, k) z^{\Delta}$$

However to pick the right solution the following boundary condition on the horizon must be satisfied

$$z \to z_h \; : \; \xi(z) < \infty$$

This amounts essentially to solve the usual connection problem:

given the independent solution $h_a^{1,2}(z)$ known expanded around point a

$$h_a^{1,2}(z) = (z - z_a)^{\alpha_{1,2}} \sum_{n=0}^{\infty} c_n^{1,2} (z - z_a)^n$$

how are they related to two other independent solutions $h_b^{1,2}(z)$ expanded around point b ?

$$h_b^{1,2}(z) = (z - z_b)^{\beta 1,2} \sum_{n=0}^{\infty} d_n^{1,2} (z - z_b)^n$$

In our case: $a = z_h$ and $b = 0$

i.e. the connection problem is to find q_{ij}, $i, j, = 1, 2$:

$$\begin{aligned} h_a^1(z) &= q_{11} h_b^1(z) + q_{12} h_b^2(z) \\ h_a^2(z) &= q_{21} h_b^1(z) + q_{22} h_b^2(z) \end{aligned}$$

This problem has been recently solved analytically for our case

Dodelson, Grassi, Iossa, Panea Lichtig, Zhiboedov, 22

based on a general solution of the connection problem for Heun equation

Bonelli, Iossa, Lichtig, Tanzini, 22

The propagator is related to the Nekrasov Shatashvili function F

(sums over instantons in $\mathcal{N} = 2$ SQCD, gauge SU(2) with massive $N_f = 4$)

and the Matone relation (between the parameters of the model)

- very nice for expansion in large T/k for the black hole case
- but very bad for expansion in small T/k for the black brane case

In fact all instantons contribute, coupling large

Essentially it boils down to infinite sums very difficult to evaluate

So this solution is not useful for our program, perturbative expansion in positive powers of T/k

Which are the parameters of the problem?

To count them it is useful to rescale

$z \to z_h z$:

$$f(z) z^2 \xi''(z) - (d - f(z)) z \xi'(z)$$
$$- \left(z^2 \left((k z_h)^2 + (\omega z_h)^2 \frac{1 - f(z)}{f(z)} \right) + \Delta(\Delta - d) \right) \xi(z) = 0$$

$$f(z) = 1 - z^d \quad , \quad k z_h = \frac{k}{\pi T} \quad , \quad \omega z_h = \frac{\omega}{\pi T}$$

$z = 0 \ldots$ boundary

$z = 1 \ldots$ horizon

The equation has 4 parameters:

1. d, we will take $d = 4$ from now on

2. T/ω (first dimensionless ratio)

3. T/k (second dimensionless ratio)

   We will be interested in expansion in positive powers of T/ω and T/k (small temperature)

4. Δ, for large $\Delta \gg 1$ one can approximate the propagator as the exponent of the geodesic length between the two points in spacetime. Solutions have been found to order T^8

$$G_2(x) = \frac{1}{|x|^{2\Delta}} \Bigg(1 + \frac{\Delta \pi^4 T^4}{120} C_2^{(1)}(\eta)|x|^4 + \frac{\Delta^2 \pi^8 T^8}{28800} \left(C_4^{(1)}(\eta) + C_2^{(1)}(\eta) + C_0^{(1)}(\eta) \right) |x|^8 + \mathcal{O}(T^{12}) \Bigg)$$

$|x|^2 = t^2 + \vec{x}^2$

$C_n^{(1)}$ ... Gegenbauer polynomials

$\eta = t/|x|$

Rodriguez-Gómez, Russo, 21

Our goal is to calculate these coefficients but for general values of Δ

By

$$\xi(z)=\frac{z^{3/2}}{f^{1/2}(z)}h(z)$$

the equation becomes

$$h''(z)=a(z)h(z)$$

$$a(z)=a_0(z)+a_T(z)$$

with the $T = 0$ potential

$$a_0(z) = k^2 + \frac{\nu^2 - 1/4}{z^2} \quad , \quad \Delta = d/2 + \nu \to 2 + \nu$$

and the $T \neq 0$ part

$$\begin{aligned} a_T(z) &= (1 - f(z)) \left(\frac{k^2}{f(z)} + \frac{\omega^2}{f^2(z)} + \frac{\nu^2}{z^2 f(z)} - \frac{d^2}{4z^2 f^2(z)} \right) \\ &= \sum_{n=1}^{\infty} \left(k^2 + n\omega^2 + \frac{\nu^2 - nd^2/4}{z^2} \right) \left(\frac{z}{z_h} \right)^{nd} \\ &= \sum_{n=1}^{\infty} \frac{a_n(z)}{(kz_h)^{nd}} \end{aligned}$$

We will transform now this second order differential equation into a system of two first order ones:

$$\frac{d}{dz}\begin{pmatrix} h(z) \\ h'(z) \end{pmatrix} = A(z)\begin{pmatrix} h(z) \\ h'(z) \end{pmatrix}$$

with

$$\begin{aligned} A(z) = \begin{pmatrix} 0 & 1 \\ a(z) & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ a_0(z) & 0 \end{pmatrix} + \sum_{n=1}^{\infty}\begin{pmatrix} 0 & 0 \\ a_n(z) & 0 \end{pmatrix}\frac{1}{(kz_h)^{nd}} \\ &= A_0(z) + \sum_{n=1}^{\infty}\frac{a_n(z)}{(kz_h)^{nd}}\sigma_- \end{aligned}$$

If $h_\pm(z)$ are two independent solutions, the general solution can be written in terms of the wronskian $w(z) = w(h_+, h_-, z)$

$$\begin{pmatrix} h(z) \\ h'(z) \end{pmatrix} = \underbrace{\begin{pmatrix} h_+(z) & h_-(z) \\ h'_+(z) & h'_-(z) \end{pmatrix}}_{=w(z)} \begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

which satisfies

$$\frac{d}{dz} w(z) = A(z) w(z)$$

Expanding

$$w(z) = w^{(0)}(z) \left(1 + \sum_{n=1}^{\infty} \frac{w^{(n)}(z)}{(kz_h)^{nd}} \right)$$

$$w^{(0)}(z) = w(h_+^{(0)}, h_-^{(0)}, z)$$

where

$$\begin{aligned} h_-^{(0)}(z) &= z^{1/2} K_\nu(kz) \\ h_+^{(0)}(z) &= z^{1/2} I_\nu(kz) \end{aligned}$$

solve the zeroth order ($T = 0$)

$$\frac{d}{dz} w^{(0)}(z) = A_0(z) w^{(0)}(z)$$

and corrections are $(n = 1, 2, \ldots)$

$$\frac{d}{dz} w^{(n)}(z) = a_n(z) B(z) + \sum_{l=1}^{n-1} a_{n-l}(z) B(z) w^{(l)}(z)$$

with

$$\begin{aligned} B(z) &= \left(w^{(0)}(z)\right)^{-1} \sigma_- w^{(0)}(z) \\ &= z \begin{pmatrix} I_\nu(kz) K_\nu(kz) & K_\nu^2(kz) \\ I_\nu^2(kz) & I_\nu(kz) K_\nu(kz) \end{pmatrix} \end{aligned}$$

To get $w^{(n)}(z)$ one needs to know all $w^{(k)}(z)$ for $k = 0, \ldots, n-1$

First correction to CFT: order T^4

The equation

$$\frac{d}{dz}w^{(1)}(z) = \left(\left(k^2+\omega^2\right)z^{d+1} + \left(\nu^2 - d^2/4\right)z^{d-1}\right) \times \begin{pmatrix} I_\nu(kz)K_\nu(kz) & K_\nu^2(kz) \\ I_\nu^2(kz) & I_\nu(kz)K_\nu(kz) \end{pmatrix}$$

$$\nu = \Delta - d/2$$

is easily solved

$$
\begin{aligned}
w^{(1)}(z) &= \begin{pmatrix} w^{(1)}_{0\,11} & w^{(1)}_{0\,12} \\ w^{(1)}_{0\,21} & w^{(1)}_{0\,22} \end{pmatrix} \\
&+ \int^z dz' \left(\left(k^2 + \omega^2\right) {z'}^{d+1} + \left(\nu^2 - d^2/4\right) {z'}^{d-1} \right) \\
&\times \begin{pmatrix} I_\nu(kz')K_\nu(kz') & K_\nu^2(kz') \\ I_\nu^2(kz') & I_\nu(kz')K_\nu(kz') \end{pmatrix}
\end{aligned}
$$

$w_{0\,ij}^{(1)}$ $(i,j=1,2)$ and the propagator are determined from

$$
\begin{aligned}
z \to 0 \quad &: \quad h_+(z) \to 0 \times z^{1/2-\nu} + z^{1/2+\nu}\left(1+\mathcal{O}(z)\right) \\
z \to 0 \quad &: \quad h_-(z) \to z^{1/2-\nu}\left(1+\mathcal{O}(z)\right) + G_2(k) z^{1/2+\nu}\left(1+\mathcal{O}(z)\right) \\
z \to \infty \quad &: \quad |h_-(z)| < \infty
\end{aligned}
$$

The result is

$$
\frac{G_2^{(1)}(k)}{G_2^{(0)}(k)} - 1 = \left(\frac{T}{k}\right)^4 \frac{2\pi^4}{15}\nu(\nu^2-1)(\nu^2-4)\left(4\frac{\omega^2}{k^2}-1\right)
$$

with

$$
G_2^{(0)}(k) = k^{2\nu}
$$

In x-space this is

$$\frac{G_2^{(1)}(x)}{G_2^{(0)}(x)} - 1 = \Delta \frac{T^4|x|^4\pi^4}{120}\left(4\frac{t^2}{|x|^2} - 1\right)$$

with

$$G_2^{(0)}(x) = \frac{1}{|x|^{2\Delta}}$$

This coincides exactly with the large Δ result!

Coincidence? To check at next order!

Second correction to CFT: order T^8

Here the work is still in progress. What we have is the complete $(T/k)^8(\omega/k)^4$ term:

$$\frac{G^{(2)}(k)}{G^{(0)}(k)} = \ldots + \left(\frac{T}{k}\right)^8 \left(\frac{\omega}{k}\right)^4 \frac{32}{225}\pi^8 \left(\frac{\Gamma(\Delta+1)}{\Gamma(\Delta-4)}\right)^2 + \ldots$$

In x space this produces the term

$$\frac{G^{(2)}(x)}{G^{(0)}(x)} = \ldots + \frac{T^8\pi^8x^4t^4}{1800}\Delta^2\frac{(\Delta-1)(\Delta-3)(\Delta-4)}{(\Delta-5)(\Delta-6)(\Delta-7)} + \ldots$$

which coincides with the literature (geodesic approximation) at large values of Δ

But for non-asymptotic Δ the two results do not coincide

$\rightarrow$

the equivalence at order T^4 was a coincidence!

Conclusions

- Thermal CFT propagators has been known so far in expansion of T/k only for $\Delta \to \infty$
- the recent exact solution of the problem is not appropriate for such expansion
- we performed such a calculation completely to order T^4 and partially to order T^8 for arbitrary values of Δ finding agreement at large Δ but getting the corrections for finite Δ at T^8 order
- the integrals needed are pretty evolved although we are seeing some peculiarities which may be useful at larger orders, work in progress