Quantum mechanics using manifolds

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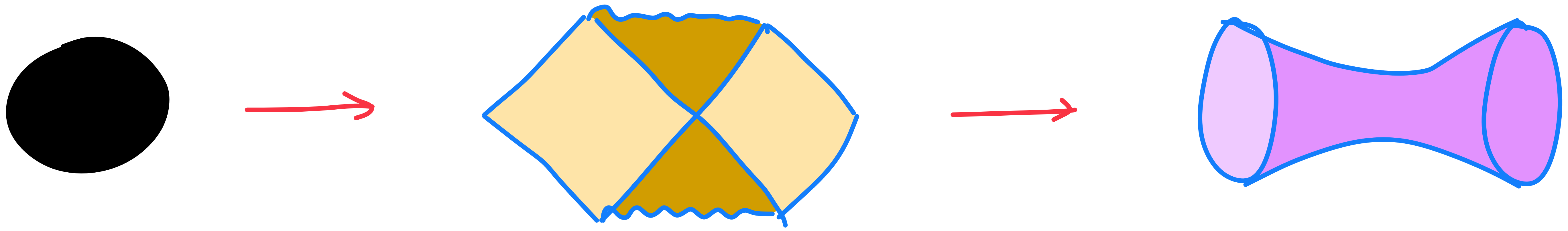
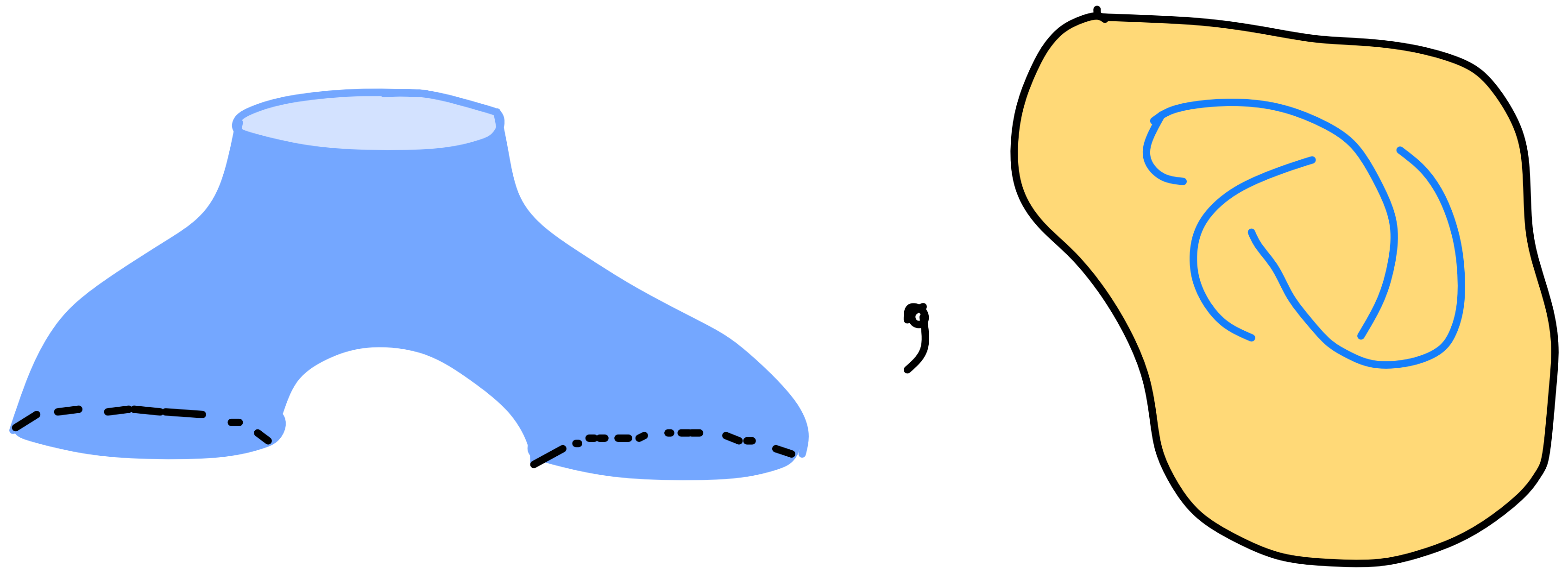
Geometry/topology is everywhere

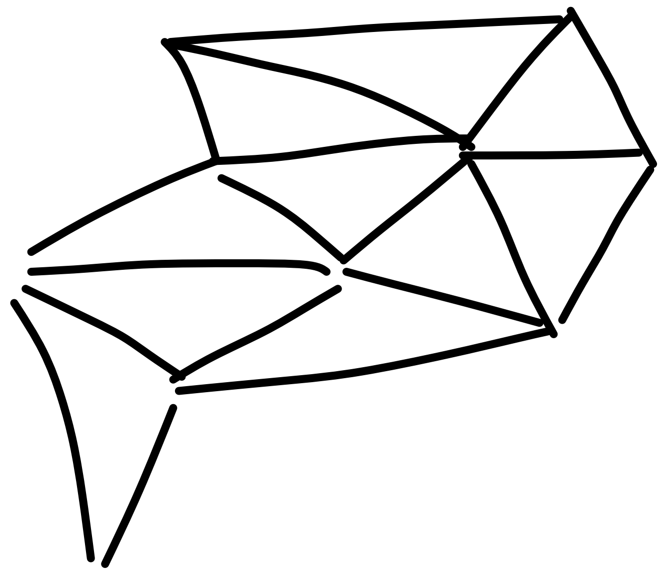
- Black Hole physics



- TQFT



- String Theory
- Loop Quantum Gravity



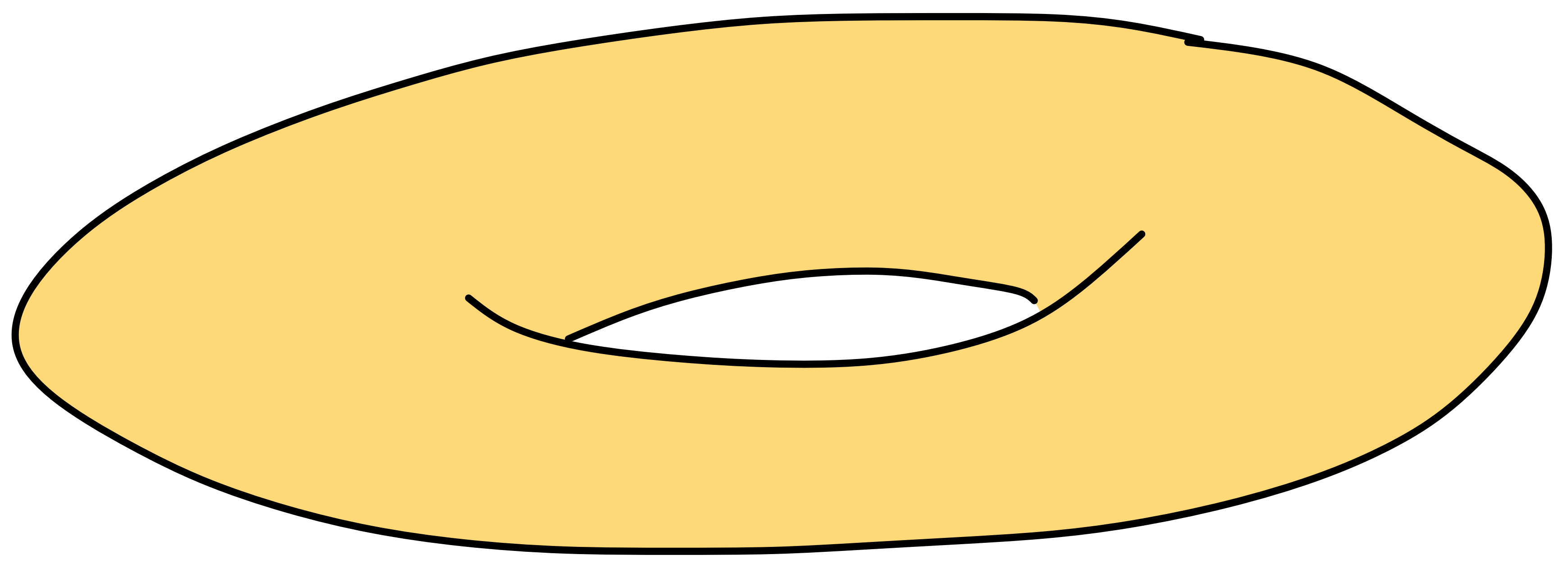
- ...

Let's start

- General relativity uses manifolds. For most physics students, this is the first encounter with differential geometry.
- Can we use manifolds to understand other theories (QFT for example)?

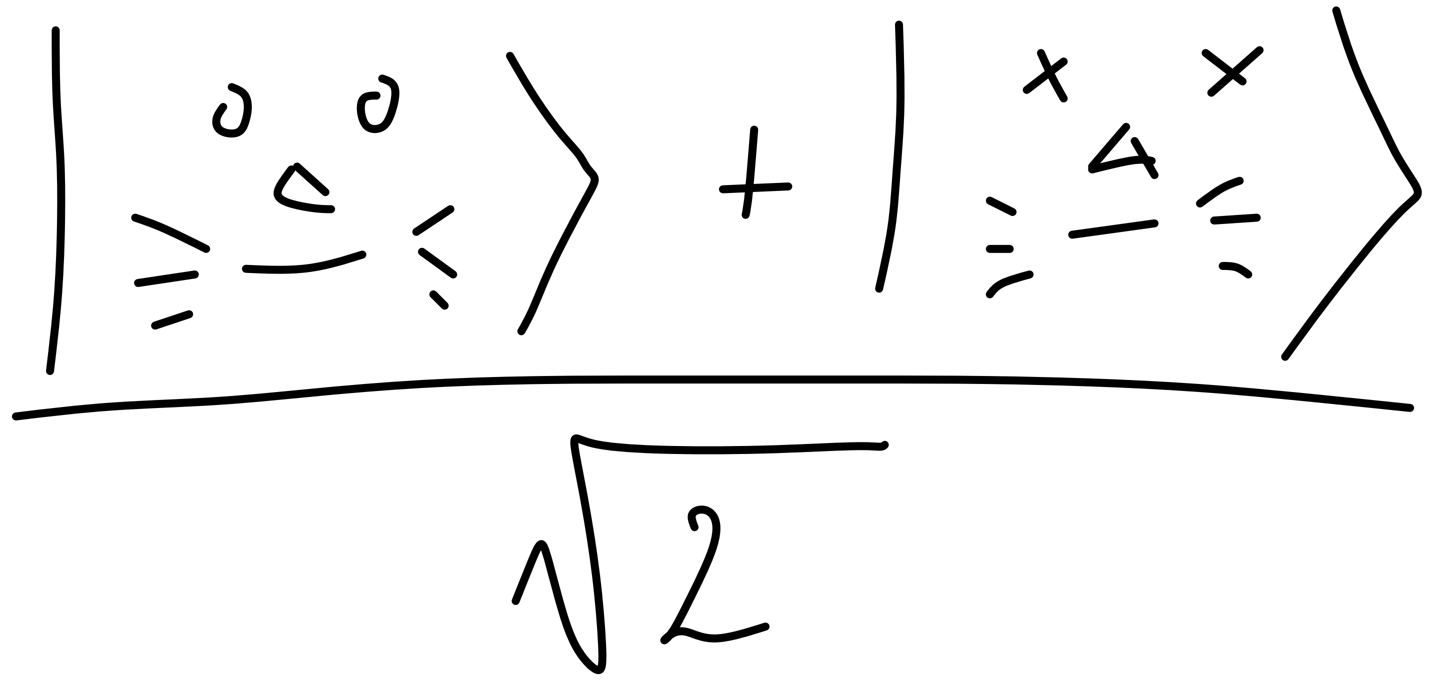
YES!

- Extended field theory vs Willsonian picture [Freed, Moore...]



Hierarchy of "knowledge"

- Hod much do we know about quantum gravity"?
- How much do we know about QFT ?
- How much do we know about quantum mechanics?



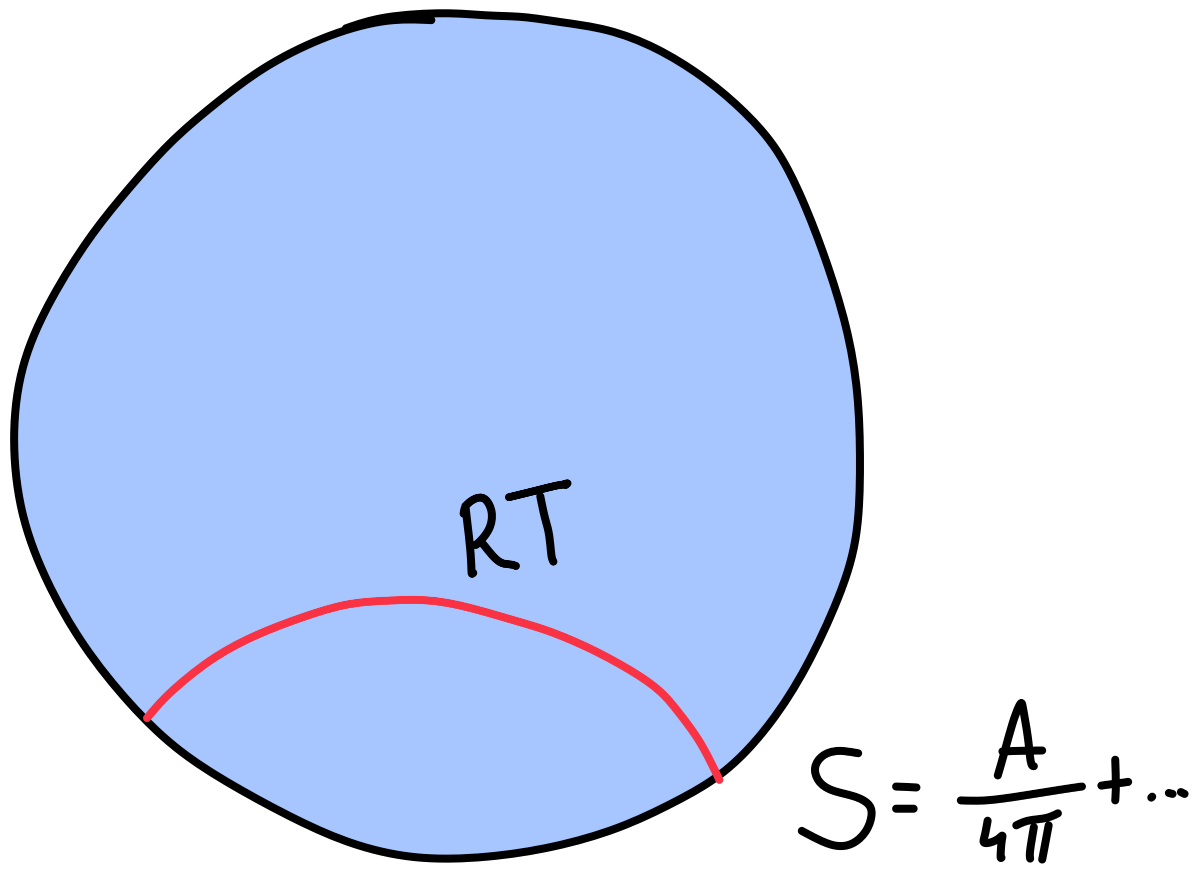
- Should we understand quantum mechanics first?

In this talk we will not explore this fundamental question, but rather try to connect QM and manifolds in a more modest way.

Holography

- Our best understanding of quantum gravity: AdS/CFT
- Quantum dual to geometry
- We can calculate entanglement entropies from geometric data.

[RT,...]



- Holography played an important role in the past decade.
- ER=EPR?
- This all hints that there should be a connection between QM and manifolds.

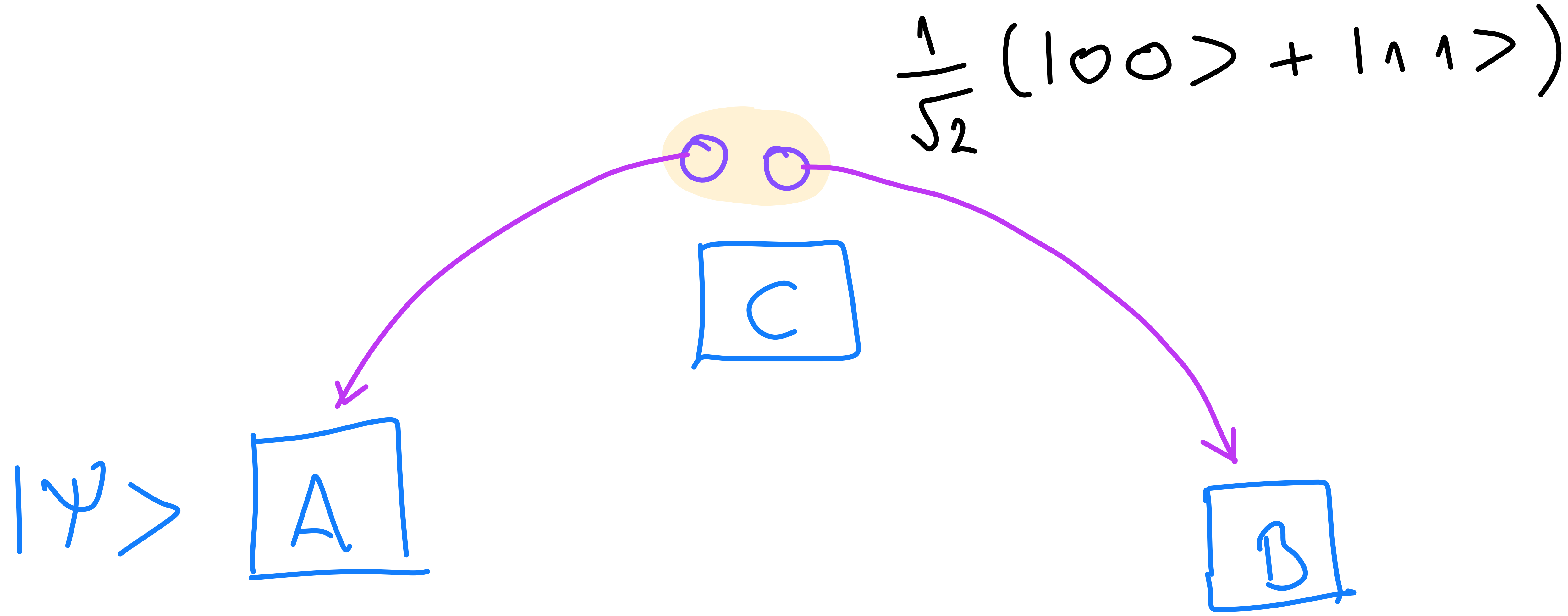
Think about QM

- We have Hilbert space, states, inner product, entanglement, measurement, operators, unitarity...
- Actually, Hilbert space is (almost) all we need. Once we decide to use it, other things are natural.
- Entanglement: Natural consequence of $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ (and vector spaces are linear)
- Do we need linearity to capture entanglement?
- Categorical quantum mechanics

Quantum teleportation

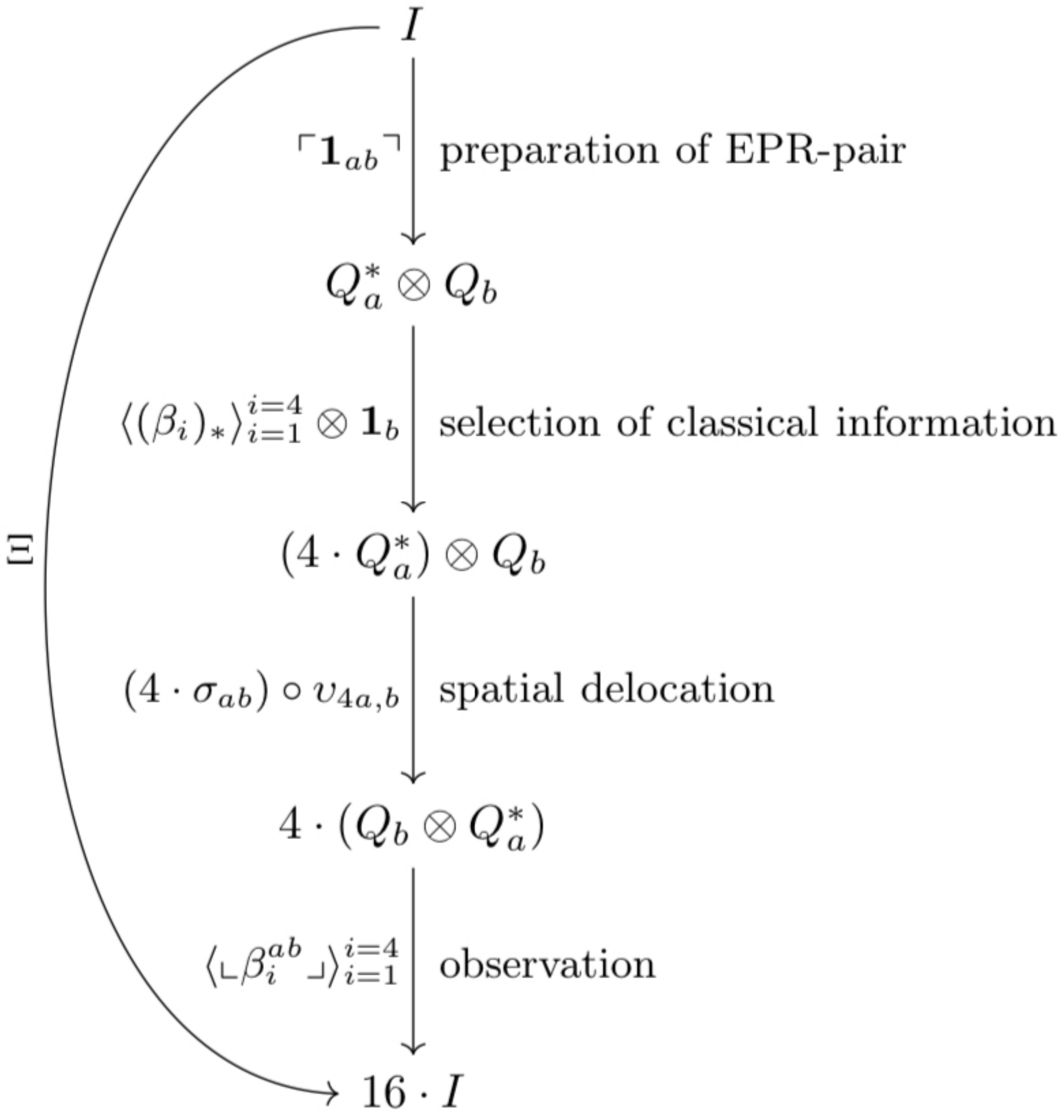
A quick review

- Sketch of a protocol:



- Information is transferred, not matter. No super-luminous travelling, we need classical communication.

- In category theory, it is usually importatnt to establish the commuativity of certain diagrams.
- Validity of quantum protocols can be expressed precisely in this way. [Abramski, Coecke]



Categories, finally

Def: A category consist of objects $(a, b, c, \ldots)$ and arrows $(f, g, h, \ldots)$, together with rules:

- For every arrow f, there exist objects $\mathrm{dom}(f)$, $\mathrm{cod}(f)$
- For arrows $f : A \to B$ and $g : B \to C$ there is an arrow $g \circ f : A \to C$
- For every object a there is an arrow $\mathbf{1_a} : A \to A$
- Composition is associative
- For every arrow $f : A \to B$ we have $f \circ \mathbf{1_a} = \mathbf{1_b} \circ f$

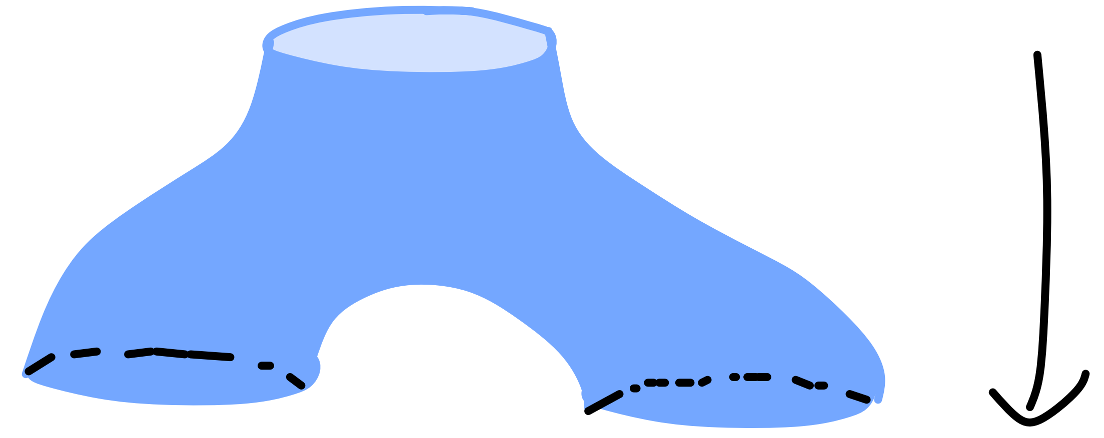
- Important example: $\mathbf{Vect}_{\mathbb{C}}$: objects are complex vector spaces, arrows are linear maps between vector spaces. We actually need **fdHilb**.

- We acually need dagger compact closed category with dagger biproducts. [Coecke, Abramski, Selinger]
- We will not give a precise definition of those categories, but only some intuition.
- Dagger: †, in QM used for adjoints of operators
- A compact closed category is a symmetric monoidal category in which every object a has its dual a^*: Dual vector spaces; will play a role in measurement $\otimes$
- Biproducts: measurements "branching" and basis

  (more about them later)
- But, where are promised **manifolds**?

Cobordisms

- We already mentioned in the context of TQFTs
- Best picture to have in mind:



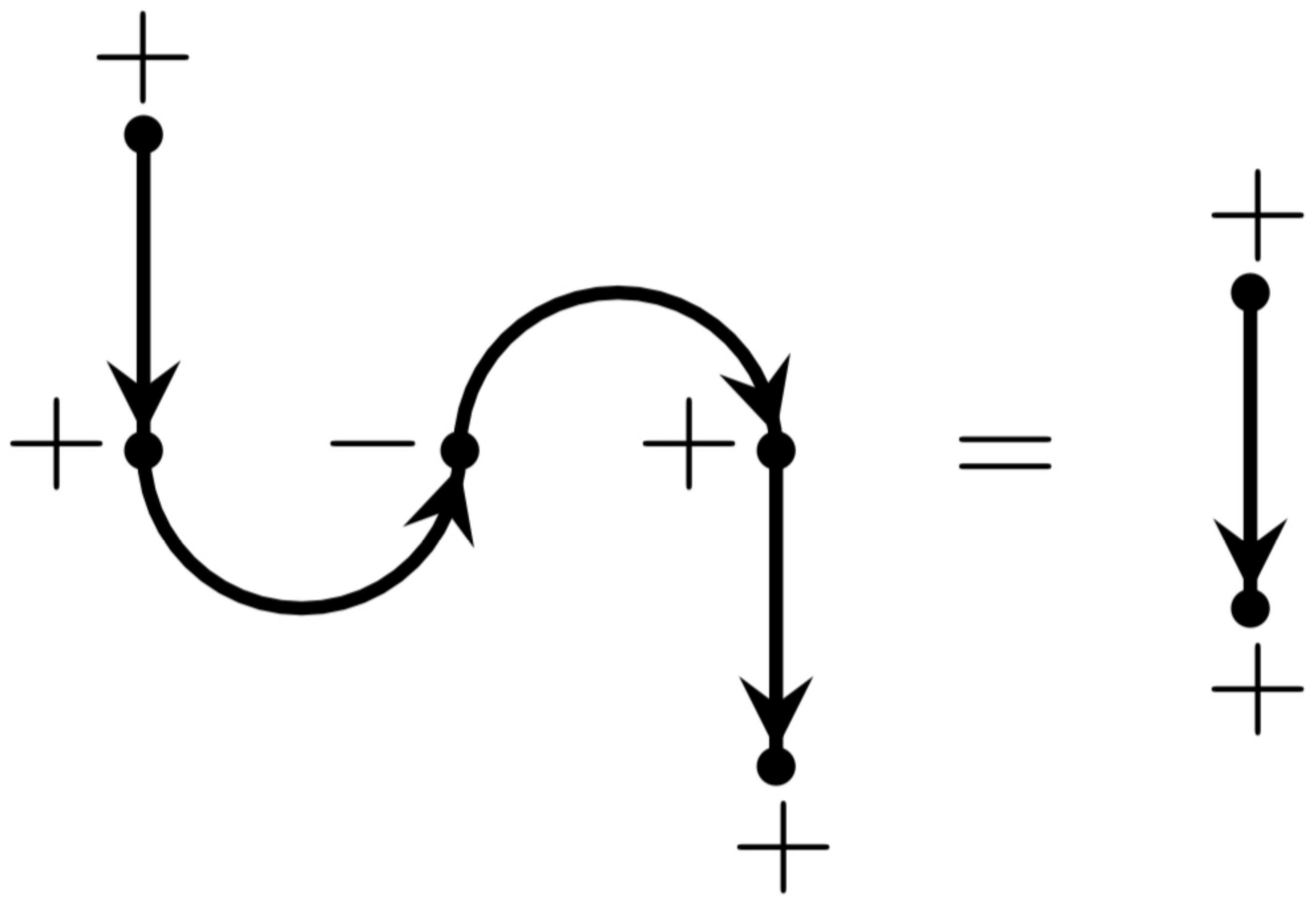
- Here we focuse on a category of one dimmensional cobordisms.
- For us 1-manifolds : one dimmensional compact oriented topological manifolds, possibly with boundary.
- Objects of $1\mathbf{Cob}$: finite sequence of points with orientation ($+$ or $-$).
- For objects a and b of 1Cob, a 1-cobordism from a to b is a triple $(\mathscr{M}, f_0 : a \to \mathscr{M}, f_1 : b \to \mathscr{M})$, where $\mathscr{M}$ is a 1-manifold and f_0, f_1 are embeddings.

  ↑ orient. pres. ↪ orient rev

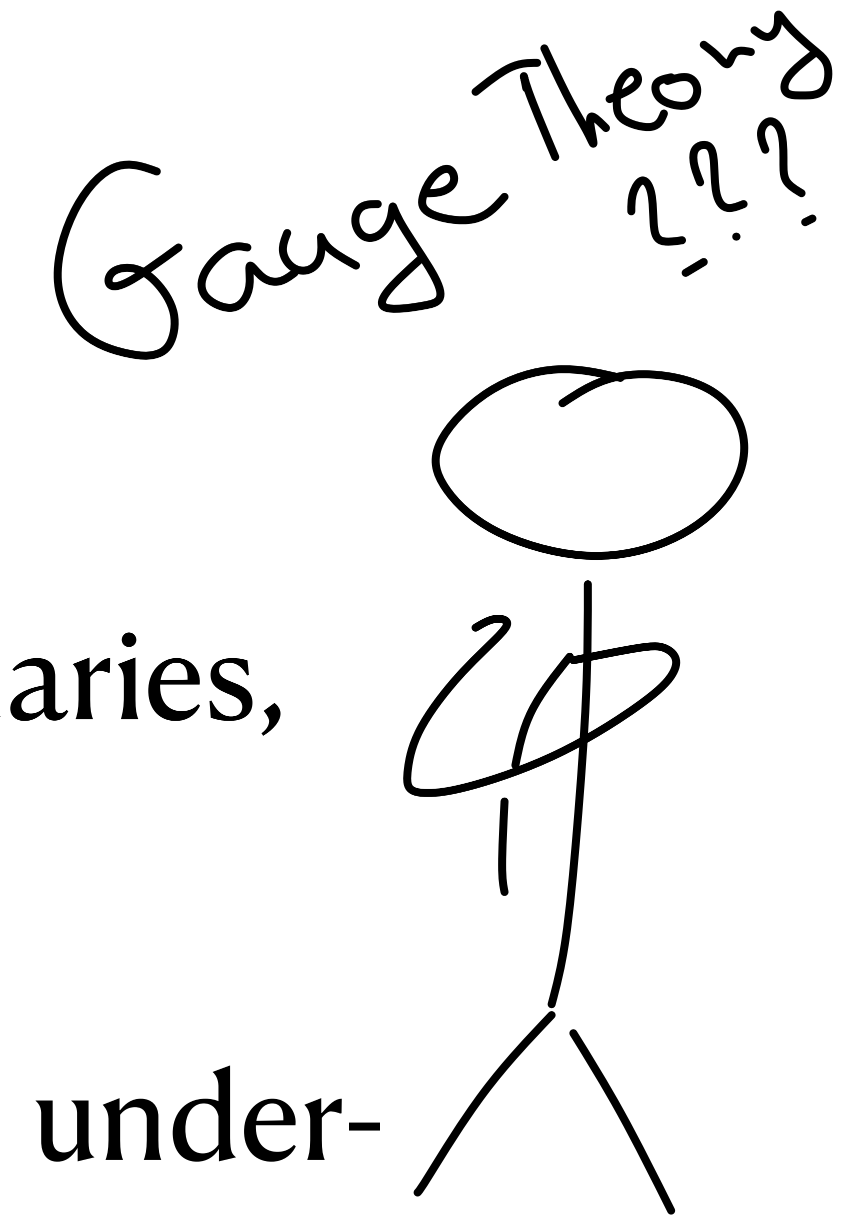
- Arrows of $1\mathbf{Cob}$ are equivalence clases of 1-cobordisms.

Coherence

- Not what you might expect when talking about quantum mechanics.
- We mean that there is a faithful functor from some freely generated category to our category that we use as a graphical language. [Petric, Zekic]
- For example, in a compact closed category, instead of using equalities that are valid, one can use pictures:



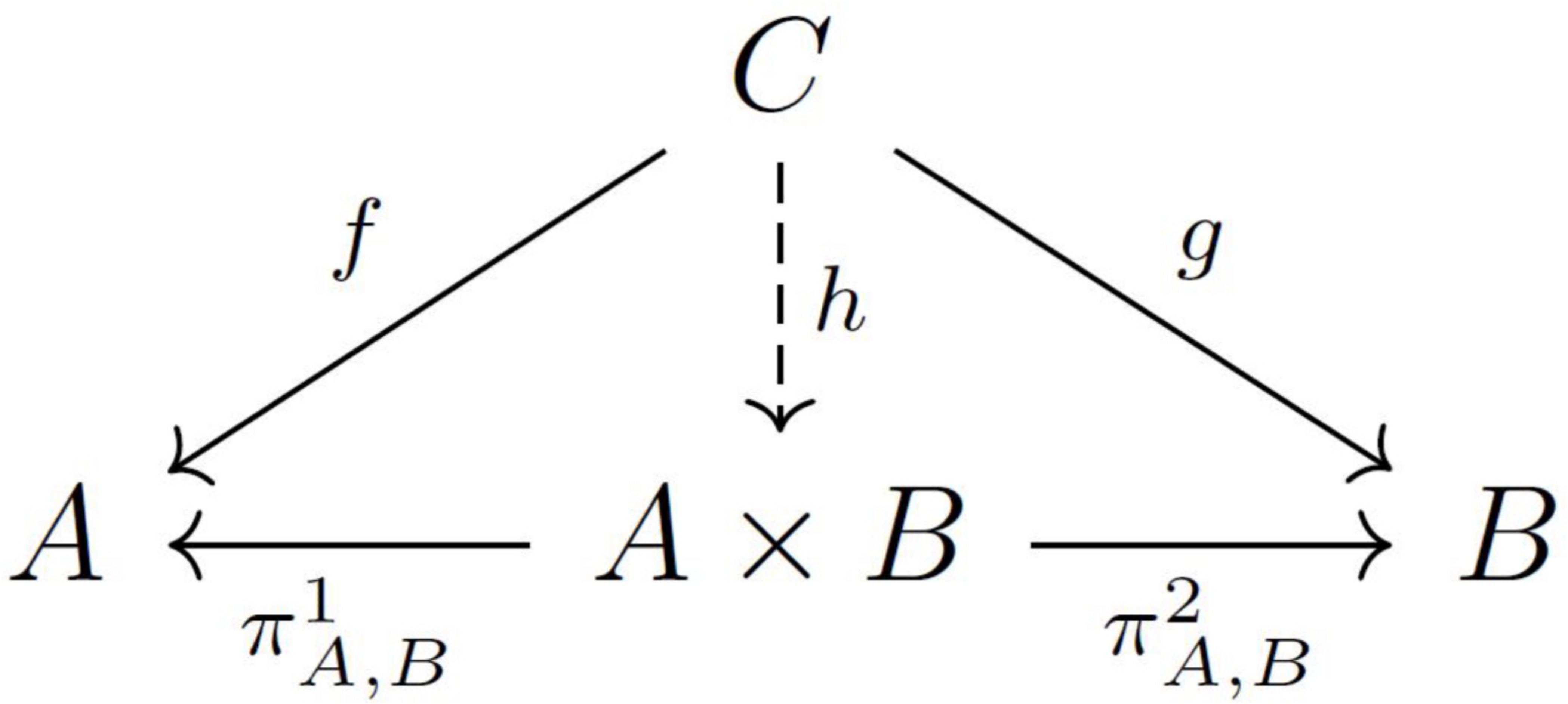
Category $\mathbf{1Cob}_{\mathscr{G}}$



- We start from a group $\mathscr{G}$ freely generated by Γ.
- A $\mathscr{G}$-component is a connected, oriented 1-manifold possibly with boundaries, together with an element of $\mathscr{G}$.
- A $\mathscr{G}$-cobordism from a to b is a finite collection of $\mathscr{G}$-components whose underlying manifold is $\mathscr{M}$, together with two embeddings $f_0 : a \to \mathscr{M}$ and $f_1 : b \to \mathscr{M}$ such that $(\mathscr{M}, f_0, f_1)$ is a 1-cobordism from a to b.
- Objects: as before, arrows: equivalence classes of $\mathscr{G}$-cobordism .

Biproducts

- We have to deal with measurements: “branching”.
- To do so, we introduce biproducts.
- A zero-object is an object which is both initial and terminal.
- Products:



- Coproducts: injections
- Biproducts: products and coproducts, with certain “expected” equalities (assumed zero object).

Category $\mathbf{1Cob}^{\oplus}_{\mathscr{G}}$

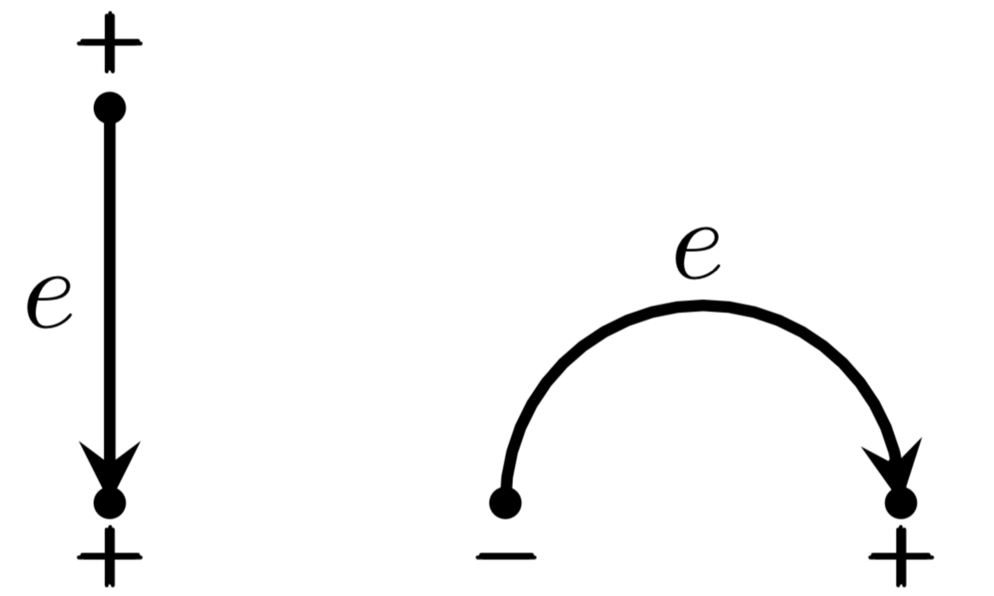
- Objects are finite (possible empty) sequences $(a_0, a_1, \ldots)$ of objects $\mathbf{1Cob}_{\mathscr{G}}$.
- Arrows from $(a_0, \ldots, a_{n-1})$ to $(b_0, \ldots, b_{m-1})$ are $m \times n$ matrices whose ij entry is a formal sum of arrows of $1\mathbf{Cob}$ from a_j to b_i.
- Th: The category $\mathbf{1Cob}^{\oplus}_{\mathscr{G}}$ has the structure of strict compact closed category with biproducts. The group of automorphisms of the object + in this category is isomorphic to $\mathscr{G}$. Moreover, † is definable in $\mathbf{1Cob}^{\oplus}_{\mathscr{G}}$, which makes it dagger strict compact closed category with dagger biproducts, while the automorphisms of + are unitary.
- Summary: this is the desired category!

Free category

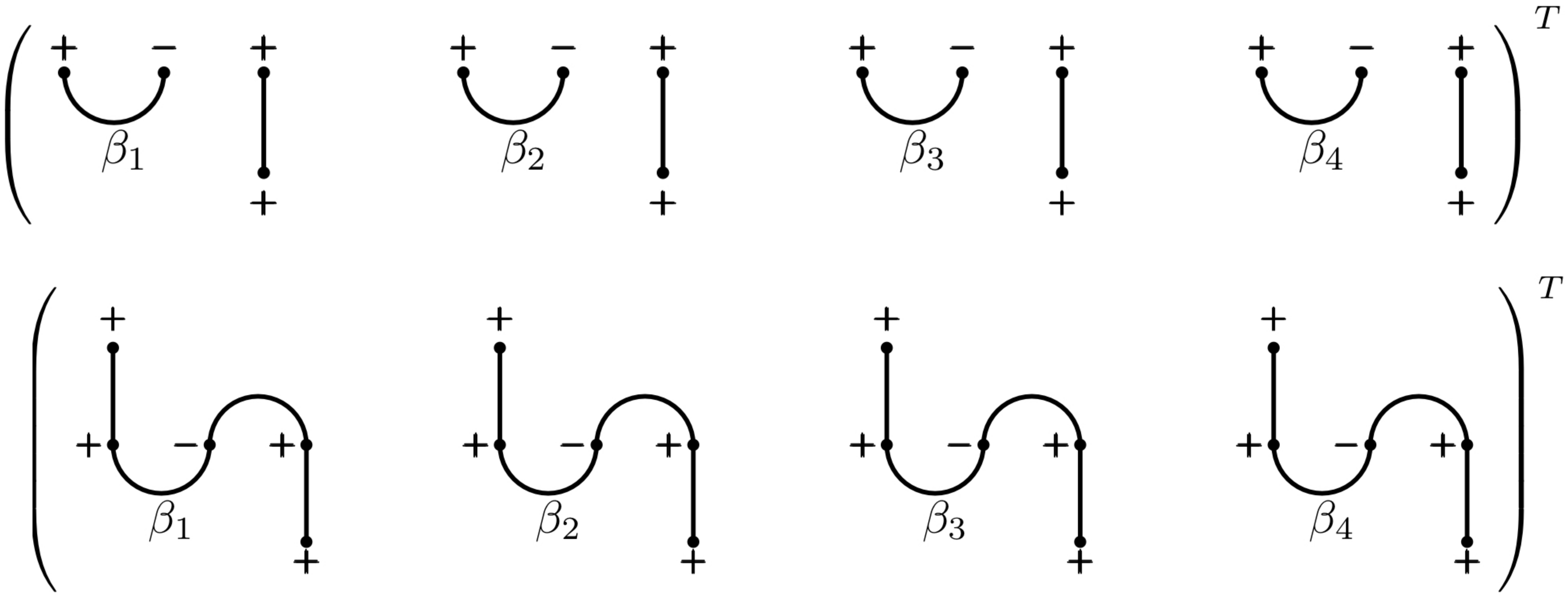
- We now come to a point to deal with coherence.
- We construct a category $\mathscr{F}$ from the introduction (but we don't describe it here).
- Then, we can find a functor H that has just the right properties, and show:
- Th: The functor $H : \mathscr{F} \to \mathbf{1Cob}^{\oplus}_{\mathscr{G}}$ is faithful.
- Moral: we can use $\mathbf{1Cob}^{\oplus}_{\mathscr{G}}$ to check the validity of quantum protocols.

- For us, qubit (Q) is $+$. In general, this will not be enough te really simulate all the properties from Hilbers state picture. [Baez]
- For example, one should insist on a unitary morphism $I \oplus I \to Q$.
- This does not prevent us to check the validity of quantum protocols.
- For our purpose, we need four unitary operations (2×2 matrices in the Hilbert space picture).
- We therefore consider a free group $\mathscr{G}$ with four generators

Teleportation protocol

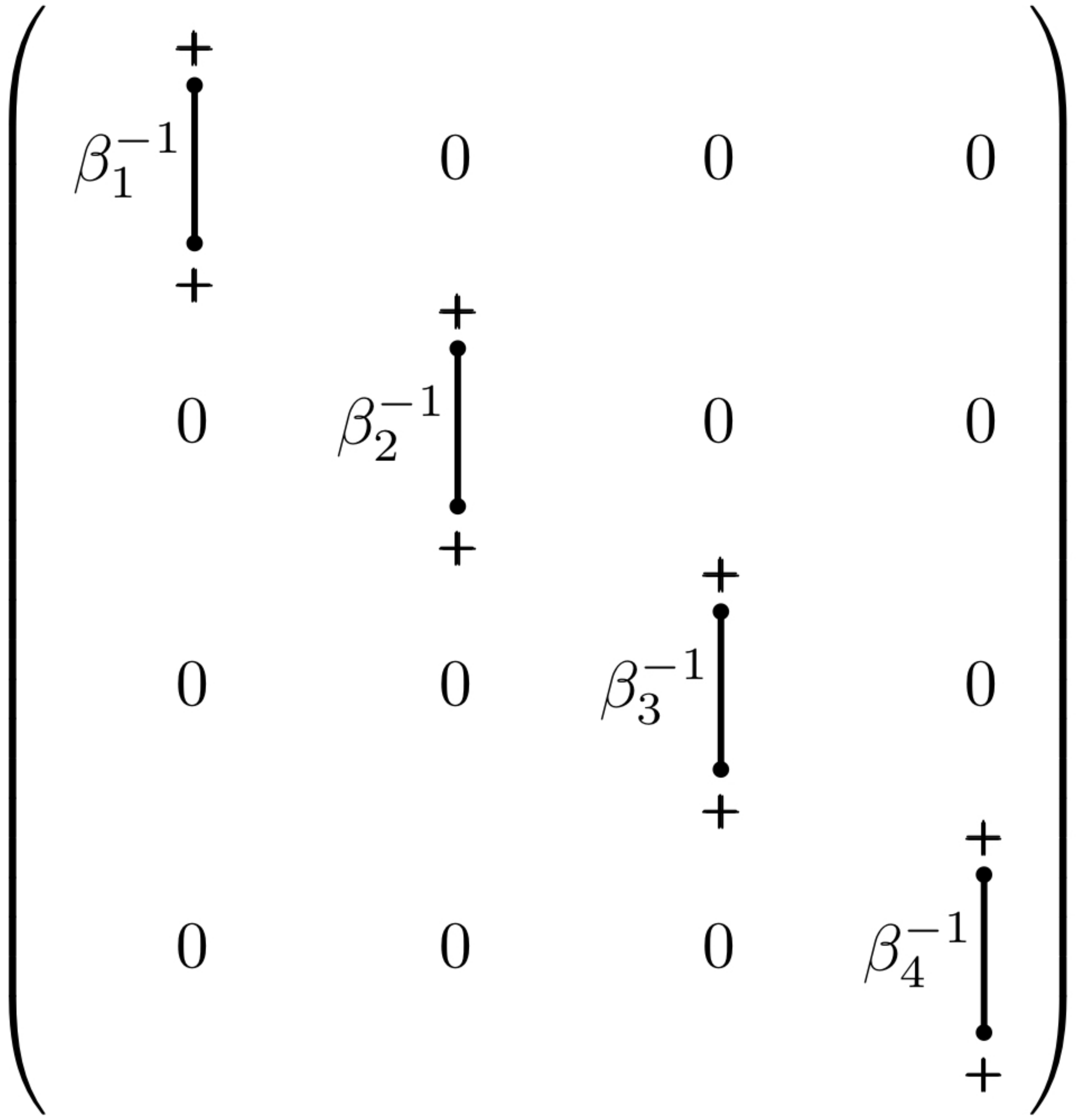


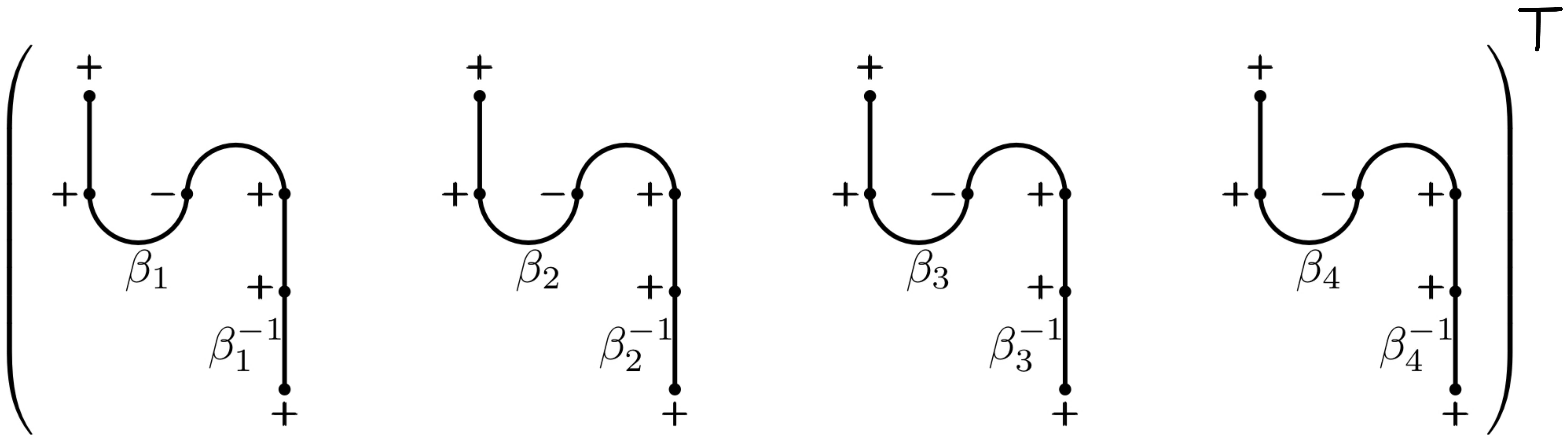
\\particle creation as in Feynmann diagrams



Teleportation protocol

- We then apply unitary corrections

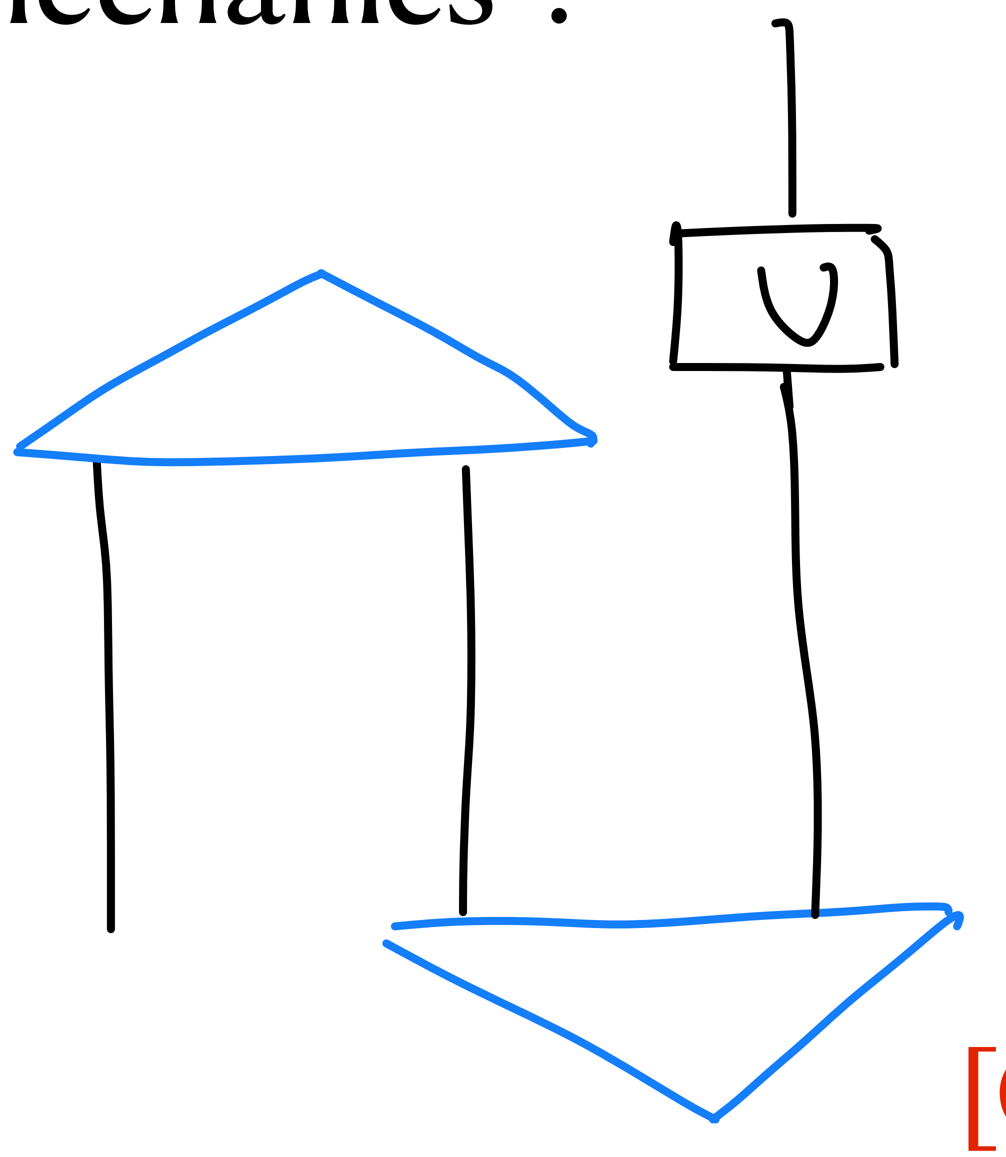




- We can stretch elements of this matrix, thus obtaining the identity element in every matrix entry, thus proving the validity of the protocol.

Another story

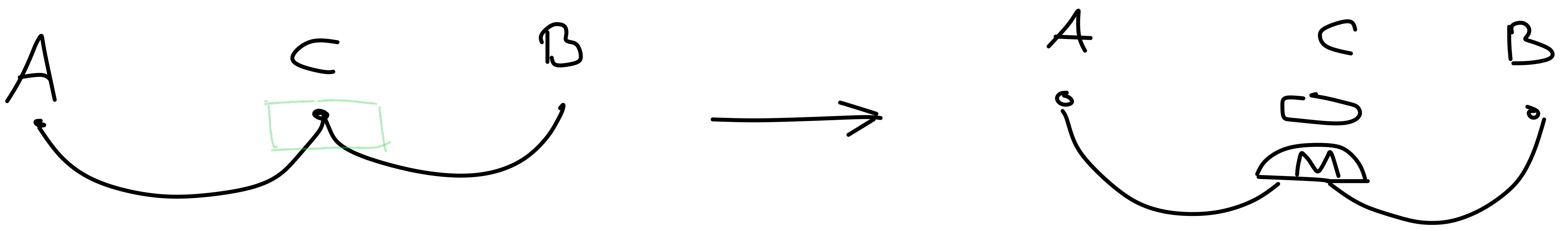
- There are many other graphical languages for categorical quantum mechanics.
- ZX calculus
- “Kindergarten quantum mechanics”:



[Coecke]

Entanglement swapping

- There are similar porposals: tensor networks, augmented space

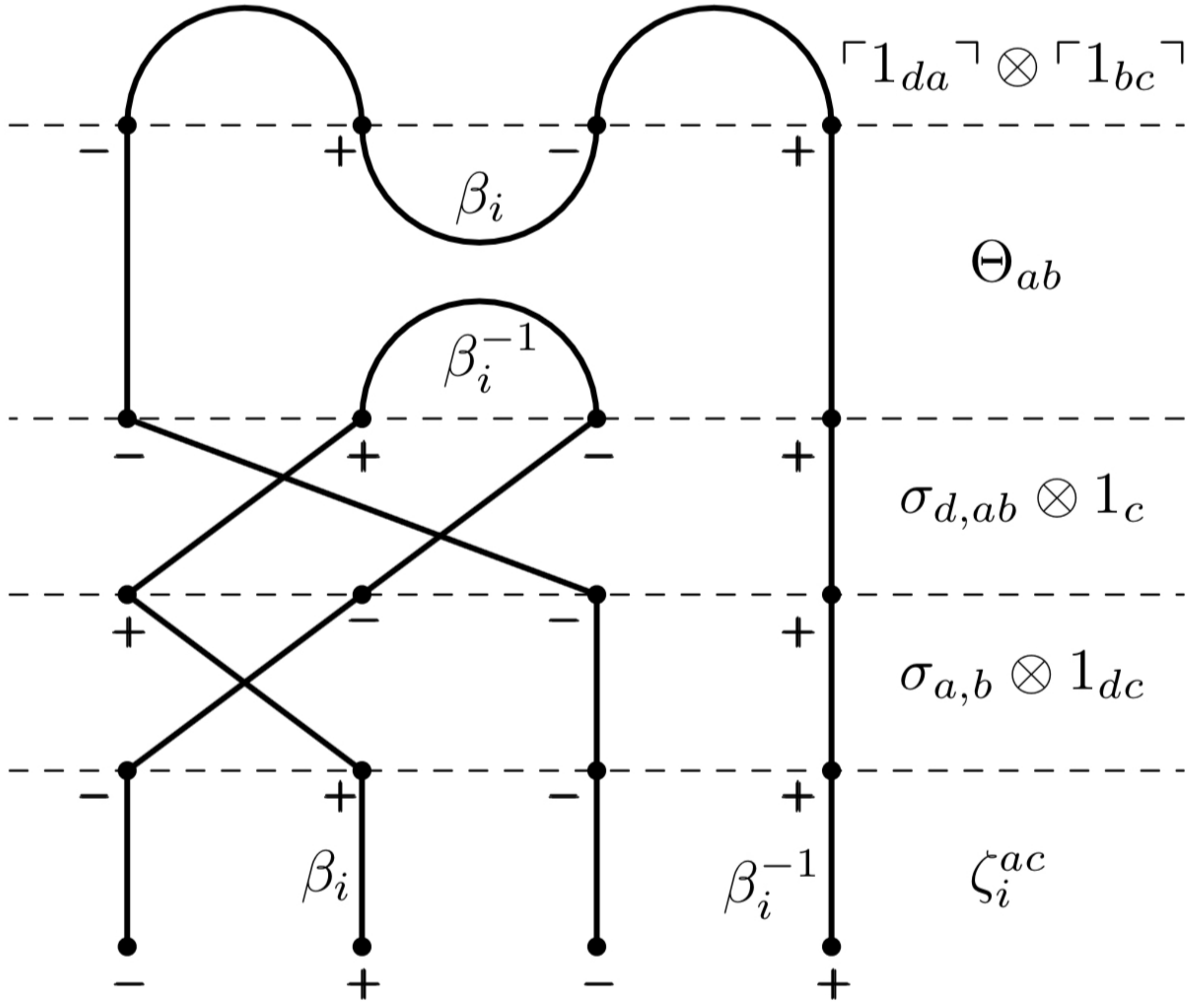


[Kauffman, Mehrota]

- One does not include matrices of pictures in this approach, but rather consider case by case scenario, that is perfectly fine in many cases.
- In a way, our approach is more fundamnetal, but in principle our theorem is much stronger then the claim we use to check the validity of QM protocols.

Entanglement swapping

- Goal: interchange entanglement between two pairs of qubits:



- End result: same as top of this picture.

Conclusion

- We can use category $\mathbf{1Cob}_{\mathscr{G}}^{\oplus}$ to check the validity of a certain class of quantum protocols.
- This introduces another way in which manifolds can be relevant for quantum mechanics, and therefore a possible direction to think in quantum gravity.
- This is stil not fully satisfactory picture of an interplay between QM and manifolds, it seems as we can improve it in the future.
- Increase dimensionality of manifolds?

Thank you for your attention!