Entropy of Kerr-Newman-AdS black holes with torsion

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Talk overview

The talk is based on the following papers

- M. Blagojević and B. Cvetković, Entropy in Poincaré gauge theory: Hamiltonian approach, Phys. Rev. D **99** (2019), 104058.
- M. Blagojević and B.Cvetković, Entropy in Poincaré gauge theory: Kerr–AdS solution, Phys.Rev.D **102** (2020), 064034.
- M. Blagojević and B.Cvetković, Entropy of Kerr–Newman–AdS black holes with torsion, Phys.Rev.D **105** (2022), 104014.

- Already in 1960s, Kibble and Sciama proposed a new theory of gravity, the Poincaré gauge theory (PG), based on gauging the Poincaré group of spacetime symmetries.
- PG is characterized by a Riemann-Cartan (RC) *geometry* of spacetime, in which both the torsion and the curvature are essential ingredients of the *gravitational dynamics*.
- Nowadays, PG is a well-established approach to gravity, representing a natural gauge-field-theoretic extension of general relativity (GR).
- In the past half century, many investigations of PG have been aimed at clarifying different aspects of both the geometric and dynamical roles of torsion. In particular, successes in constructing exact solutions with torsion naturally raised the question of how their *conserved charges* are influenced by the presence of torsion.

- The expressions for the conserved charges in PG were first found for asymptotically flat solutions. In the Hamiltonian approach to PG the conserved charges are represented by a boundary term, defined by requiring the variation of the canonical gauge generator to be a well-defined (differentiable) functional on the phase space.
- A covariant version of the Hamiltonian approach, introduced later by Nester, turned out to be an important step in understanding the conservation laws. This was clearly demonstrated by Hecht and Nester, in their analysis of the conserved charges for asymptotically flat or (A)dS.
- Despite an intensive activity in exploring the notion of conserved charges in the *generic* four-dimensional (4D) PG, systematic studies of black hole entropy in the presence of torsion have been largely neglected so far.

- In the 1990s, understanding of the *classical* black hole entropy reached a level that can be best characterized by Wald's words: "Black hole entropy is the Noether charge".
- The question we addressed is whether such a challenging idea can improve our understanding of black hole entropy in PG and a few years ago we proposed a general canonical approach to black hole entropy in PG.
- We constructed the canonical gauge generator in the first order formulation of PG, which improved form is used to obtain the variational equation for the asymptotic canonical charge, located at the spatial 2-boundary at infinity.
- Following the idea that "entropy is the canonical charge at horizon," we are led to define black hole entropy by the same variational equation, located at black hole horizon.
- The differentiability of the gauge generator guarantees the validity of the first law of black hole thermodynamics.

- Applying this approach to a number of black holes with or without torsion such we found somewhat unexpected result: in spite of many geometric and dynamic differences with respect to GR, entropy of black holes in PG *without matter*, as well as the associated first law, follow essentially the same pattern as in GR, up to a multiplicative constant.
- In the present talk, we extend our investigation of entropy by introducing *Maxwell field* as a matter source for gravity (PG-Maxwell system).
- The analysis is focussed on exploring thermodynamic properties of the generalized KN-AdS black hole, constructed by Baekler et al. in the late 1980s.

Notations and conventions

- Our conventions are as follows.
- The greek indices $(\mu, \nu, \ldots)$ refer to the coordinate frame, with a time-space splitting expressed by $\mu = (0, \alpha)$.
- The latin indices $(i, j, \ldots)$ refer to the local Lorentz frame.
- ϑ^i is the orthonormal tetrad (1-form), e_i is the dual basis (frame), with $e_i \rfloor \vartheta^k = \delta_i^k$, and the Lorentz metric is $\eta_{ij} = (1, -1, -1, -1)$.
- The volume 4-form is $\hat{\epsilon} = \vartheta^0\vartheta^1\vartheta^2\vartheta^3$, the Hodge dual of a form α is ${}^\star\alpha$, with ${}^\star 1 = \hat{\epsilon}$, and the totally antisymmetric tensor is defined by ${}^\star(\vartheta_i\vartheta_j\vartheta_m\vartheta_n) = \varepsilon_{ijmn}$, where $\varepsilon_{0123} = +1$.
- The exterior product of forms is implicit.

Lagrangian formalism

- In PG the torsion $T^i = d\vartheta^i + \omega^i{}_k\vartheta^k$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_k\omega^{kj}$ (2-forms) are the gravitational field strengths, associated to the Poincaré gauge potentials, the tetrad ϑ^i and the Lorentz connection ω^{ij}, respectively. Our physical system contains also the Maxwell field characterised by the field strength $F = dA$ (2-form), where A is the electromagnetic gauge potential (1-form).
- Dynamical properties of the PG-Maxwell system are defined by the total Lagrangian

$$L = L_G + L_M\,,. \tag{2.1}$$

where $L_G = L_G(\vartheta^i, T^i, R^{ij})$ is a parity even PG Lagrangian, assumed to be at most quadratic in the field strengths, and $L_M = L_M(\vartheta^i, F)$ describes the Maxwell field interacting with gravity.

- After introducing the covariant field momenta, $H_i := \partial L_G/\partial T^i$ and $H_{ij} := \partial L_G/\partial R^{ij}$, and the associated energy-momentum and spin currents, $E_i := \partial L_G/\partial\vartheta^i$ and $E_{ij} := \partial L_G/\partial\omega^{ij}$, the gravitational field equations read

$$\delta\vartheta^i: \qquad \nabla H_i + E_i = -\tau_i\,, \tag{2.2a}$$
$$\delta\omega^{ij}: \qquad \nabla H_{ij} + E_{ij} = 0\,, \tag{2.2b}$$

  where $\tau_i := \partial L_M/\partial\vartheta^i$ is the Maxwell energy-momentum current , while the spin current vanishes, $\sigma_{ij} := \partial L_M/\partial\omega^{ij} = 0$.
- The variation of L with respect to the electromagnetic potential A yields the Maxwell equation,

$$\delta A: \qquad dH = 0\,, \tag{2.2c}$$

  where $H := \partial L_M/\partial A$ is the electromagnetic covariant momentum.

- The total Lagrangian has the form

$$
\begin{aligned}
L_G &= -{}^\star(a_0 R + 2\Lambda) + T^i \sum_{n=1}^{3} {}^\star(a_n\, {}^{(n)}T_i) + \frac{1}{2} R^{ij} \sum_{n=1}^{6} {}^\star(b_n\, {}^{(n)}R_{ij})\,, \\
L_M &:= 4a_1 \left(-\frac{1}{2} F\, {}^\star F\right)\,, \qquad F := dA\,. \qquad (2.3)
\end{aligned}
$$

- (a_0, Λ, a_n, b_n) are the gravitational coupling constants, and ${}^{(n)}T_i$, ${}^{(n)}R_{ij}$ are irreducible parts of the field strengths.
- The explicit formulas for the covariant momenta read

$$
\begin{aligned}
H_i &= 2\sum_{m=1}^{2} {}^\star(a_n\, {}^{(m)}T_i)\,, \quad H_{ij} = -2a_0\, {}^\star(\vartheta_i \vartheta_j) + 2\sum_{n=1}^{6} {}^\star(b_n\, {}^{(n)}R_{ij}) \\
H &= -4a_1\, {}^\star F\,. \qquad (2.4)
\end{aligned}
$$

Thermodynamics

- The Hamiltonian approach to black hole entropy in PG is based on the ideas developed originally in GR, according to which the asymptotic charges as well as entropy, can be defined by certain boundary terms.
- Consider a stationary black hole spacetime whose spatial section Σ has a two-component boundary, one component at infinity and the other at horizon, $\partial\Sigma = S_\infty \cup S_H$.
- Asymptotic charges and entropy of a PG-Maxwell black hole are determined by the boundary integral

$$\delta\Gamma_\infty = \oint_{S_\infty} \delta B(\xi)\,, \qquad \delta\Gamma_H = \oint_{S_H} \delta B(\xi)\,, \tag{2.5a}$$

$$\begin{aligned} \delta B(\xi) := {} & (\xi \rfloor \vartheta^i)\delta H_i + \delta\vartheta^i(\xi \rfloor H_i) + \frac{1}{2}(\xi \rfloor \omega^{ij})\delta H_{ij} + \frac{1}{2}\delta\omega^{ij}(\xi \rfloor \delta H_{ij}) \\ & + (\xi \rfloor A)\delta H + (\delta A)(\xi \rfloor H)\,. \end{aligned} \tag{2.5b}$$

- δB is obtained from the canonical generator.
- The Killing vectors ξ ($\xi = \partial_t,\ \partial_\varphi$ or their linear combination) are chosen so that the boundary integrals (Γ_∞, Γ_H) could be physically interpreted in terms of the *asymptotic charges*, *black hole entropy*, and *Maxwell term*.
- We require the operation δ to satisfy the following rules:
  - **(r1)** On S_∞, the variation δ acts on the parameters of a black hole solution, but not on the parameters of the background.
  - **(r2)** On S_H, the variation δ must keep surface gravity constant.
  - **(r3)** When the boundary terms are δ-*integrable* and *finite*, they can be given the usual thermodynamic interpretation.
- The regularity of the generator reveals the relation

$$\delta\Gamma_\infty - \delta\Gamma_H = 0\,, \tag{2.6}$$

which is equivalent to the *first law* of black hole thermodynamics. The Maxwell contribution to δB is an essential part of the first law.

Metric and tetrad

- We shall now analyse basic properties of KN-AdS black holes as solutions of the PG-Maxwell system.
- The metric of a KN-AdS black hole in Boyer-Lindquist (BL) coordinates has the form

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt+\frac{a}{\alpha}\sin^2\theta d\varphi\right)^2-\frac{\rho^2}{\Delta}dr^2-\frac{\rho^2}{f}d\theta^2-\frac{f}{\rho^2}\sin^2\theta\left[adt+\frac{(r^2+a^2)}{\alpha}d\varphi\right]^2 \tag{3.1a}$$

$$\begin{aligned}&\Delta(r) := (r^2+a^2)(1+\lambda r^2)-2(mr-q^2)\,, \qquad \alpha := 1-\lambda a^2\,,\\ &\rho^2(r,\theta) := r^2+a^2\cos^2\theta\,, \qquad f(\theta) := 1-\lambda a^2\cos^2\theta\,.\end{aligned} \tag{3.1b}$$

- Here, m, a and q are the parameters characterising energy, angular momentum and electric charge of the solution, and $\lambda = -\Lambda/3a_0$.

- The orthonormal tetrad associated to the metric is chosen in the form

$$\vartheta^0 = N\Big(dt + \frac{a}{\alpha}\sin^2\theta\, d\varphi\Big)\,, \quad \vartheta^1 = \frac{dr}{N}\,, \quad \vartheta^2 = Pd\theta\,,$$
$$\vartheta^3 = \frac{\sin\theta}{P}\Big[a\,dt + \frac{(r^2+a^2)}{\alpha}d\varphi\Big]\,, \quad N = \sqrt{\Delta}/\rho\,, \quad P = \rho/\sqrt{f}\,. \tag{3.2}$$

- The larger root of $\Delta(r) = 0$ defines the outer horizon, and the angular velocity and surface gravity have the GR form

$$\omega_+ = \frac{a\alpha}{r_+^2+a^2}\,, \qquad \Omega_+ := \omega_+ + \lambda a = \frac{a(1+\lambda r_+^2)}{r_+^2+a^2}\,, \tag{3.3}$$
$$\kappa = \frac{r_+^2 + 3\lambda r_+^4 + \lambda a^2 r_+^2 - a^2 - 2q^2}{2r_+(r_+^2+a^2)}\,, \tag{3.4}$$

and the area of the horizon is $A_H = \int_{r_+}\vartheta^2\vartheta^3 = \frac{4\pi(r_+^2+a^2)}{\alpha}$.

Torsion, connection and curvature

- Riemann-Cartan geometry of PG is characterized by a nonvanishing torsion. For KN-AdS black holes, the ansatz for torsion is formally the same as for the Kerr-AdS case

$$
\begin{aligned}
T^0 = T^1 &= \frac{1}{N}\Big[-V_1\vartheta^0\vartheta^1 - 2V_4\vartheta^2\vartheta^3\Big] + \frac{1}{N^2}\Big[V_2\vartheta^-\vartheta^2 + V_3\vartheta^-\vartheta^3\Big]\,, \\
T^2 &:= \frac{1}{N}\Big[V_5\vartheta^-\vartheta^2 + V_4\vartheta^-\vartheta^3\Big]\,, T^3 := \frac{1}{N}\Big[-V_4\vartheta^-\vartheta^2 + V_5\vartheta^-\vartheta^3\Big], \qquad (3.5) \\
V_1 &= \frac{1}{\rho^4}\left[(mr-2q^2)r - ma^2\cos^2\theta\right]\,, V_2 = -\frac{1}{\rho^4 P}(mr-q^2)a^2\sin\theta\cos\theta\,, \\
V_3 &= \frac{1}{\rho^4 P}(mr-q^2)ra\sin\theta\,, V_4 = \frac{1}{\rho^4}(mr-q^2)a\cos\theta\,, V_5 = \frac{1}{\rho^4}(mr-q^2)r\,,
\end{aligned}
$$

where $\vartheta^- = \vartheta^0 - \vartheta^1$ and the metric function N and the torsion functions V_n are modified by the presence of the nonvanishing electric charge parameter q^2.

- Having introduced torsion, the Riemann-Cartan connection can be expresses as

$$\omega^{ij} = \tilde{\omega}^{ij} + K^{ij} \,, \tag{3.6}$$

where $\tilde{\omega}^{ij}$ is Levi-Chivita (Riemannian) connection and K^{ij} is the contortion 1-form, implicitly defined by the relation $T^i = K^i{}_k b^k$.

- The curvature 2-form $R^{ij} = d\omega^{ij} + \omega^i{}_k \omega^{kj}$ has only two nonvanishing irreducible parts:

$$^{(6)}R^{ij} = \lambda \vartheta^i \vartheta^j \,, \qquad ^{(4)}R^{Ac} = \frac{\lambda}{\Delta}(mr - q^2)\vartheta^- \vartheta^c \,. \tag{3.7}$$

The quadratic invariants (Euler, Pontryagin and Nieh-Yan) are given by

$$\begin{aligned} I_E &:= (1/2)\varepsilon_{ijmn} R^{ij} R^{mn} \equiv {}^\star R_{mn} R^{mn} = 12\lambda^2 \hat{\epsilon} \,, \\ I_P &:= R^{ij} R_{ij} = 0 \,, \qquad I_{NY} = T^i T_i - R_{ij} b^i b^j = 0 \,. \end{aligned} \tag{3.8}$$

PG-Maxwell field equations

- The covariant momenta H_i and H_{ij}, are given by

$$H_i = 2a_1 {}^\star({}^{(1)}T_i - 2\,{}^{(2)}T_i)\,, \quad H_{ij} = -2(a_0 - \lambda b_6){}^\star(\vartheta_i\vartheta_j) + 2b_4{}^\star\,{}^{(4)}R_{ij}\,.$$

- The Maxwell potential in a KN-AdS spacetime is

$$A := -\frac{q_e r}{\rho\sqrt{\Delta}}\vartheta^0 \equiv -\frac{q_e r}{\rho^2}\left(dt + \frac{a}{\alpha}\sin^2\theta d\varphi\right), \tag{3.9}$$

  where q_e is the electromagnetic charge parameter. This expression is a natural generalization of the spherically symmetric form $A = -(q_e/r)dt$.
- The related field strength and the covariant momentum are

$$F = -\frac{q_e}{\rho^4}\left[(r^2 - a^2\cos^2\theta)\vartheta^0\vartheta^1 + 2ar\cos\theta\,\vartheta^2\vartheta^3\right],$$

$$H = -4a_1\frac{q_e}{\rho^4}\left[(r^2 - a^2\cos^2\theta)\vartheta^2\vartheta^3 - 2ar\cos\theta\,\vartheta^0\vartheta^1\right].$$

- When all the previous results taken into account, the explicit calculation shows that basic dynamical variables $(\vartheta^i, \omega^{ij}, A)$ of a KN-AdS black hole solve the PG-Maxwell field equations if the Lagrangian parameters (a_n, b_n, Λ) and the solution parameters (λ, q, q_e) satisfy the relations

$$\begin{aligned} &2a_1 + a_2 = 0\,, \qquad && a_0 - a_1 - \lambda(b_4 + b_6) = 0\,, \\ &3\lambda a_0 + \Lambda = 0\,, \qquad && q_e^2 = 2q^2\,. \end{aligned} \tag{3.10}$$

- Thus, according to our conventions, the electromagnetic charge parameter q_e differs from the metric charge parameter q. However, none of them coincides with the asymptotic Maxwell charge, as will be shown.

- The asymptotic values of energy and angular momentum are defined by the boundary term $\delta B(\xi)$ in (2.5).
- Let us mention that Carter and Henneaux and Teitelboim demonstrated that the asymptotic metric of Kerr-AdS spacetimes cannot be properly described in BL coordinates. They found a new set of coordinates in which this deficiency is brought under control. However, our variational approach allows a simpler procedure, in which only the subset (t, φ) of the BL coordinates is transformed

$$T = t\,, \qquad \phi = \varphi - \lambda a t\,. \tag{4.1a}$$

- Consequently. the components of metric transform as

$$\begin{aligned} g_{T\varphi} &= g_{t\varphi} + g_{\varphi\varphi}\,, \\ \Omega_+ &:= \left(\frac{g_{T\varphi}}{g_{\varphi\varphi}}\right)_{r_+} = \omega_+ + \lambda a = \frac{a(1+\lambda r_+^2)}{r_+^2 + a^2}\,. \end{aligned} \tag{4.1b}$$

Angular momentum

- It is interesting to note that the contribution of the Maxwell field in the expression $\delta B(\xi)$, yields vanishing boundary terms at infinity, but not at horizon.
- The angular momentum is defined by $\delta E_\varphi := \delta\Gamma_\infty(\partial_\varphi)$. The nonvanishing contributions are

$$\omega^{13}{}_\varphi \delta H_{13} + \delta\omega^{13} H_{13\varphi} = 2a_1 \delta\left(\frac{ma}{\alpha^2}\right) d\Omega' \,,$$

$$b^0{}_\varphi \delta H_0 + \delta b^0 H_{0\varphi} = 4a_1 \delta\left(\frac{ma}{\alpha^2}\right) d\Omega' \,,$$

$$d\Omega := \sin\theta d\theta d\varphi \to 4\pi \,, \qquad d\Omega' := \sin^3\theta d\theta d\varphi \to \frac{2}{3} 4\pi \,.$$

- Summing up the two terms, one obtains

$$\delta E_\varphi = 16\pi a_1 \delta\left(\frac{ma}{\alpha^2}\right) . \tag{4.2}$$

Energy

- Going over to energy, we calculate the nonvanishing contributions to $\delta E_t = \delta\Gamma_\infty(\partial_t)$,

$$\delta\omega^{12}H_{12t} + \delta\omega^{13}H_{13t} = 2a_1 m\delta\left(\frac{1}{\alpha}\right)d\Omega\,,$$
$$b^0{}_t\delta H_0 = 4a_1\delta\left(\frac{m}{\alpha}\right)d\Omega\,.$$

- Hence,

$$\delta E_t = 16\pi a_1\left[\frac{m}{2}\delta\left(\frac{1}{\alpha}\right) + \delta\left(\frac{m}{\alpha}\right)\right]\,.$$

- The result is not δ-integrable but, as we mentioned above, it can be corrected by moving to the untwisted coordinates

$$\delta E_T = \delta E_t + \lambda a\delta E_\varphi = 16\pi a_1\delta\left(\frac{m}{\alpha^2}\right)\,. \qquad (4.3)$$

- First we analyse the PG part of the boundary term at horizon, $\delta\Gamma_H$, where the Killing vector ξ is given by

$$\xi := \partial_T - \Omega_+\partial_\phi = \partial_t - \omega_+\partial_\varphi \,. \tag{5.1}$$

- This part defines the black hole entropy. After very lengthy calculation we get

$$\begin{aligned}(\delta\Gamma_H)^{PG} &= 8\pi a_1 \kappa\delta\Big(\frac{r_+^2 + a^2}{\alpha}\Big) = T\delta S\,,\\ S &:= 16\pi a_1 \frac{\pi(r_+^2 + a^2)}{\alpha}\,,\end{aligned} \tag{5.2}$$

where $T := \kappa/2\pi$ is the temperature.

- Thus, entropy is as the conserved charges proportional to the GR value.

Maxwell boundary term and the first law

- The asymptotic electric charge Q can be defined byl

$$Q = -\int_{S_\infty} H = 4a_1 \int_{S_\infty} \frac{q_e}{\rho^4}(r^2 - a^2\cos^2\theta)b^2 b^3 = 16\pi a_1 \frac{q_e}{\alpha}\,. \tag{5.3}$$

- The electric potential Φ is defined by

$$\Phi := A_\xi\Big|_{r_+}^{\infty} = -\frac{q_e r_+}{\rho_+^2 N} b^0{}_\xi\Big|_{r_+}^{\infty} = \frac{q_e r_+}{r_+^2 + a^2}\,. \tag{5.4}$$

- Then, the Maxwell contribution on horizon has the form

$$(\delta\Gamma_H)^M = A_\xi \delta H + (\delta A)H_\xi = A_\xi \delta H = \Phi\,\delta Q\,. \tag{5.5}$$

- Combining this relation with the already obtained results, one can immediately conclude that the first law $\delta\Gamma_H = \delta\Gamma_\infty$ takes the form

$$T\delta S + \Phi\delta Q = \delta E_T - \Omega_+ \delta E_\varphi\,. \tag{5.6}$$

- We introduced the natural extension of the canonical approach to black hole entropy to *non-vacuum* solutions, by including Maxwell field as a matter source of gravity.
- We studied the thermodynamic properties of KN-AdS black holes, encoded in the boundary terms at infinity and horizon, $\delta\Gamma_\infty$ and $\delta\Gamma_H$, respectively.
- Analysing energy and angular momentum as the boundary terms at infinity, we found that their KN-AdS values are exactly the same as for the uncharged Kerr-AdS solution This is in agreement with the fact that the asymptotic Maxwell contribution vanishes. Moreover, these asymptotic charges are proportional to the related GR expressions.
- The boundary term at horizon produces entropy and an external, Maxwell term, such that both of them are also proportional to the corresponding GR expressions. Then, the first law is described by the general relation $\delta\Gamma_\infty = \delta\Gamma_H$.