

Thermodynamic equilibrium of black holes

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Outline

- ➊ The physics behind the first law of thermodynamics
- ➋ Conditions for global thermodynamic equilibrium
- ➌ Conditions for local thermodynamic equilibrium
- ➍ Applications to black holes and holography

The First Law of Thermodynamics

Perturbations of the **thermal equilibrium states** satisfy the first law of thermodynamics:

$$dU = TdS - pdV + \mu dN + \cdots = \sum_{a=1}^n I_a dE^a. \quad (1)$$

- $I_a = (T, -p, \mu, \dots)$ – intensive state quantities.
- $E^a = (S, V, N, \dots)$ – extensive state variables.
- $U = U(E^a)$ – the energy potential in its **natural variables**.

In equilibrium $U = U(E^a)$ is the **fundamental relation** and EoS:

$$I_a = \left. \frac{\partial U}{\partial E^a} \right|_{E^1, \dots, \hat{E}^a, \dots, E^n} \Rightarrow T = \left. \frac{\partial U}{\partial S} \right|_{V, N}, \quad -p = \left. \frac{\partial U}{\partial V} \right|_{S, N}, \dots \quad (2)$$

Quasi-homogeneous functions

U is **quasi-homogeneous** of degree r and type $(r_1, \dots, r_n)$ under dilatations by a scale factor $\lambda > 0$ if

$$U(\lambda^{r_1} S, \lambda^{r_2} V, \lambda^{r_3} N_1, \dots, \lambda^{r_n} N_k) = \lambda^r U(S, V, N_1, \dots, N_k). \quad (3)$$

A differentiable quasi-homogeneous function U satisfies a generalized **Euler identity** (necessary and sufficient):

$$\left(r_1 E^1 \frac{\partial}{\partial E^1} + r_2 E^2 \frac{\partial}{\partial E^2} + \dots + r_n E^n \frac{\partial}{\partial E^n} \right) U = rU. \quad (4)$$

In equilibrium Euler's identity becomes:

$$\sum_{a=1}^n r_a I_a E^a = rU. \quad (5)$$

This is the **Smarr relation** for black holes.

Global Thermodynamic Stability in U Representation

Global **minimum** of the energy potential (**globally convex**) iff

$$d^2U = dE^T.(Hess_E U).dE \geq 0 \quad \forall E, \quad [Hess_E U(E)]_{ab} = \frac{\partial^2 U(E)}{\partial E^a \partial E^b}. \quad (6)$$

- An eigenvalue problem: $\lambda_i \geq 0$,

$$Hess_E U(E) = M. \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix} . M^{-1}. \quad (7)$$

- Alternatively: **Sylvester's criterion** for positive semi-definite quadratic forms: $\Delta_k \geq 0$, $k = 1, \dots, n$. [[Phys. Rev. D **105**, no. 4, 044033 \(2022\)](#)].

Global Thermodynamic Stability in S Representation

Global **maximum** of the entropy potential (**fully concave**) iff

$$d^2 S = dE^T \cdot (Hess_E S) \cdot dE \leq 0 \quad \forall E, \quad [Hess_E S(E)]_{ab} = \frac{\partial^2 S(E)}{\partial E^a \partial E^b}. \quad (8)$$

- An eigenvalue problem: $s_i \leq 0$,

$$Hess_E S(E) = N \cdot \begin{pmatrix} s_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & s_n \end{pmatrix} \cdot N^{-1}. \quad (9)$$

- Alternatively: **Sylvester's criterion** for negative semi-definite quadratic forms: $(-1)^k \Delta_k \geq 0$. $k = 1, \dots, n$.

Legendre Transformation

Let $\mathcal{P}(I) = \{\mathcal{P}_1(I), \mathcal{P}_2(I), \dots\} = \{\{I_a\}, \{I_a, I_b\}, \dots\}$ is the power set of all intensive variables. Define a dot product of power sets:

$$\mathcal{P}(I) \cdot \mathcal{P}(E) := \{\{I_a\}, \{I_a, I_b\}, \dots\} \cdot \begin{Bmatrix} \{E^a\} \\ \{E^a, E^b\} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} I_a E^a \\ I_a E^a + I_b E^b \\ \vdots \end{Bmatrix}, \quad (10)$$

$$\mathcal{P}_k(I) \cdot \mathcal{P}_k(E) = I_a E^a + I_b E^b + I_c E^c + \dots \quad (11)$$

- Legendre transformation to a new potential Φ :

$$\Phi = \mathcal{L}_{\mathcal{P}_k(E)} U(E) := U(E) - \mathcal{P}_k(I) \cdot \mathcal{P}_k(E). \quad (12)$$

- Masseur-Plank potentials Ψ :

$$\Psi = \mathcal{L}_{\mathcal{P}_k(E)} S(E) := S(E) - \mathcal{P}_k(I) \cdot \mathcal{P}_k(E). \quad (13)$$

Energy/Entropy Derived Potentials

Energy derived:

$$\mathcal{L}_S U = U - TS = F \text{ (Helmholtz)}, \quad (14)$$

$$\mathcal{L}_V U = U + PV = H \text{ (Enthalpy)}, \quad (15)$$

$$\mathcal{L}_{S,V} U = U - TS + PV = G \text{ (Gibbs)}, \quad (16)$$

$$\mathcal{L}_{S,V,N} U = U - TS + PV - \mu N = \Omega \text{ (Grand)}. \quad (17)$$

Entropy derived (Massieu-Planck):

$$\mathcal{L}_U S = S - \frac{U}{T} = \Phi \text{ (Massieu)}, \quad (18)$$

$$\mathcal{L}_V S = S - \frac{PV}{T}, \quad (19)$$

$$\mathcal{L}_{U,V} S = S - \frac{U}{T} - \frac{PV}{T} = \Xi \text{ (Planck)}, \quad (20)$$

$$\mathcal{L}_{U,V,N} S = S - \frac{U}{T} - \frac{PV}{T} + \frac{\mu N}{T}. \quad (21)$$

Local Thermodynamic Equilibrium

Let X^m is a set of m fix state variables and Y^{m+1} is a set of $m + 1$ arbitrary state quantities (also containing elements of X^m):

- Specific heats ([Davies'77](#), [Mansoori & Mirza'14](#)):

$$C_X = T \left(\frac{\partial S}{\partial T} \right) \Big|_X = T \frac{\{S, X\}_Y}{\{T, X\}_Y}. \quad (22)$$

- **Local stability conditions** and phase transitions ([Davies'77](#)):

$$C_X = \begin{cases} > 0, \text{ locally stable,} \\ < 0, \text{ locally unstable,} \\ = 0, \text{ phase transition,} \\ \rightarrow \pm\infty \text{ phase transition.} \end{cases} \quad (23)$$

Applications to Black Holes and Holography

VA, HD, MR, RR and TV (upcoming 2022)

System	Globally stable	Locally stable
Schwarzschild	No	No
Reisner-Nordstrom	No	Yes
Kerr	No	Yes
Kerr-Neuman	No	Yes
Kerr-Neuman-AdS	Yes	Yes

Phys. Rev. D **105**, no.4, 044033 (2021).

Phys. Rev. D **99**, no.12, 126007 (2019).

Induced Hessians and Information Geometry

- Generalized **pushforward** from e -space to E -space (Avramov, Dimov, Iliev, Radomirov, Rashkov, Vetsov upcoming 2022):

$$g_{ab}^{(W)}(E) = \frac{\partial^2 U(E)}{\partial E^a \partial E^b} = \frac{\partial e^\alpha}{\partial E^a} \left(\frac{\partial^2 U(e)}{\partial e^\alpha \partial e^\beta} - \Gamma_{\alpha\beta}^\gamma(e) \frac{\partial U(e)}{\partial e^\gamma} \right) \frac{\partial e^\beta}{\partial E^b}. \quad (24)$$

- Generalized **pullback** from E -space to e -space:

$$g_{\alpha\beta}^{(W)}(e) = \frac{\partial^2 U(e)}{\partial e^\alpha \partial e^\beta} = \left(\frac{\partial e^\alpha}{\partial E^a} \right)^{-1} \left(\frac{\partial^2 U(E)}{\partial E^a \partial E^b} \right) \left(\frac{\partial e^\beta}{\partial E^b} \right)^{-1} + \Gamma_{\alpha\beta}^\gamma(e) \frac{\partial U(e)}{\partial e^\gamma}. \quad (25)$$

- Induced metric:

$$h_{\alpha\beta}(e) = \delta_{ab} \frac{\partial E^a}{\partial e^\alpha} \frac{\partial E^b}{\partial e^\beta}, \quad \Gamma_{\alpha\beta}^\gamma(e) = \frac{1}{2} h^{\gamma\delta} (\partial_\beta h_{\delta\alpha} + \partial_\alpha h_{\delta\beta} - \partial_\delta h_{\alpha\beta}). \quad (26)$$

Conjectures and Other Relations

- Fluctuation theory ([G. Ruppeiner '75](#))
- Linear stability of black holes = thermodynamic stability ([R. Wald '12](#))
- Geometrothermodynamics ([H. Quevedo '07](#))
- Probing quantum gravity ([G. Ruppeiner '10](#))
- Holographic complexity ([D. Stanford, L. Susskind '14](#))

Probing Quantum Gravity

G. Rupeiner '10:

$$R = \begin{cases} > 0, & \text{Elliptic geometry} \rightarrow \text{repulsive interactions,} \\ < 0, & \text{Hyperbolic geometry} \rightarrow \text{attractive interactions,} \\ = 0, & \text{Flat geometry} \rightarrow \text{free system,} \\ \rightarrow \pm\infty & \text{Phase transition.} \end{cases} \quad (27)$$

What about higher order **algebraic invariants**?

And even more about the plethora of **differential invariants**?

Relations to Nambu Brackets

VA, HD, MR, RR and TV (2022):

$$\frac{\partial^2 E}{\partial S^2} \Big|_{J,V,\alpha} = \frac{\partial T}{\partial S} \Big|_{J,V,\alpha} = \frac{\{T, J, V, \alpha\}_{T,\Omega,P,\alpha}}{\{S, J, V, \alpha\}_{T,\Omega,P,\alpha}}. \quad (28)$$

$$\frac{\partial^2 E}{\partial J^2} \Big|_{S,V,\alpha} = \frac{\partial \Omega}{\partial J} \Big|_{S,V,\alpha} = \frac{\{\Omega, S, V, \alpha\}_{T,\Omega,P,\alpha}}{\{J, S, V, \alpha\}_{T,\Omega,P,\alpha}}. \quad (29)$$

$$\frac{\partial^2 E}{\partial J \partial S} \Big|_{V,\alpha} = \frac{\partial}{\partial J} \left(\frac{\partial E}{\partial S} \Big|_{J,V,\alpha} \right) \Big|_{S,V,\alpha} = \frac{\partial T}{\partial J} \Big|_{S,V,\alpha} = \frac{\{T, S, V, \alpha\}_{T,\Omega,P,\alpha}}{\{J, S, V, \alpha\}_{T,\Omega,P,\alpha}}, \quad (30)$$

or equivalently

$$\frac{\partial^2 E}{\partial S \partial J} \Big|_{V,\alpha} = \frac{\partial}{\partial S} \left(\frac{\partial E}{\partial J} \Big|_{S,V,\alpha} \right) \Big|_{J,V,\alpha} = \frac{\partial \Omega}{\partial S} \Big|_{J,V,\alpha} = \frac{\{\Omega, J, V, \alpha\}_{T,\Omega,P,\alpha}}{\{S, J, V, \alpha\}_{T,\Omega,P,\alpha}}. \quad (31)$$

What If...? Summary

- How to: **Geometrization** of Nonequilibrium thermodynamics and statistical physics?

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