Noncommutative Electrodynamics through symplectic embedding

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Outline

Motivations for NC field and gauge theory date back to many decades ago, but still valid:

- space-time discreteness emerging from various approaches to quantum gravity,
- composition of GR with Quantum Mechanics
- UV regularization of quantum field theories

There is no general consensus about the more appropriate formulation and many problems are yet to be solved; the approach chosen here wants to answer two questions

- get a formulation compatible with the commutative limit
- define gauge fields in such a way that gauge transformations compose nicely (i.e. they close a Lie algebra under star commutator)

In the standard approach noncommutative gauge and matter fields are described in terms of

- a noncommutative algebra $(\mathcal{A}, \star)$ representing space-time (it replaces $\mathcal{F}(M)$)
- a right $\mathcal{A}$-module, $\mathbb{M}$, representing matter fields (it replaces vector bundles);
- a group of unitary automorphisms of $\mathbb{M}$ acting on fields from the left, representing gauge transformations.

The dynamics of fields is described by means of a natural differential calculus based on derivations of the NC algebra;
The gauge connection is the standard noncommutative analogue of the Koszul connection (standard linear connection on vector bundles).
Therefore, the first problem to address is to have a well defined differential calculus, namely, an algebra of $\star$-derivations of $\mathcal{A}$ such that

$$D_a(f \star g) = D_a f \star g + f \star D_a g$$

For Moyal algebra $\mathbb{R}^{2n}_\theta$ a minimal set of derivations, $X \in Der(\mathbb{R}^{2n}_\theta)$, is given by $\frac{\partial}{\partial x^\mu}$

The case $(\mathcal{A}, \star) = \mathbb{R}^{2n}_\Theta$ Moyal plane

For QED the gauge group is $\widehat{U(1)}$, implying that charged matter fields are 1-dim complex vector fields (sections of 1-d complex vector bundle), namely a right module over $\mathcal{F}(\mathbb{R}^4)$

$\Longrightarrow$ The NC generalization is

- a 1-dim complex right module (one generator) over $\mathbb{R}^{2n}_\theta$

$$\mathcal{H} = \mathbb{C} \otimes \mathbb{R}^{2n}_\theta$$

with Hermitian structure h : $h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) = \psi_1^\dagger \star \psi_2$ In this case $\mathcal{H}$ has only one generator, $\mathbf{e}$ —> $\boldsymbol{\psi} = \mathbf{e}\psi, \psi \in R^{2n}_\theta$

• The connection is completely determined by its action on the module generator: $\nabla_X(\boldsymbol{\psi}) = \nabla_X(\mathbf{e})\psi + \mathbf{e}X(\psi)$

$\Longrightarrow$ The 1-form connection $\mathbf{A}$:

- $\mathbf{A} : X \to \mathbf{A}(X) := i\nabla_X(\mathbf{e}), \ \ \forall X \in \mathrm{Der}(\mathbb{R}^{2n}_\theta)$
- $\nabla_\mu(\mathbf{e}) =: -i\mathbf{A}(\partial_\mu) = -i\mathbf{e}A_\mu$
- so that
  $$\nabla_\mu \boldsymbol{\psi} := \nabla_\mu(\mathbf{e}\psi) = \mathbf{e}(\partial_\mu \psi - iA_\mu \star \psi)$$

• Gauge transformations can be identified with the unitaries $\mathcal{U}(\mathbb{R}^2_\theta) \ni g = \exp_\star(if)$

- gauge covariance: $(\nabla^A_\mu)^g(\psi) := g(\nabla^A_\mu(g^{-1}\psi)) = \nabla^{A^g}_\mu(\psi)$
  with
  $A^g_\mu = f_g \star A_\mu \star f_{g^{-1}} + i f_g \star \partial_\mu f_{g^{-1}}$
- Curvature:
  $\mathbf{F}_{\mu\nu} = ([\nabla^A_\mu, \nabla^A_\nu] - \nabla^A_{[\partial_\mu,\partial_\nu]}) = \mathbf{e}(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star)$
  $\mathbf{F}^g_{\mu\nu} = ([\nabla^A_\mu, \nabla^A_\nu] - \nabla^A_{[\partial_\mu,\partial_\nu]}) \overset{\text{check}}{=} \mathbf{e}(f_g \star F_{\mu\nu} \star f_{g^{-1}})$

Implying
$F^g_{\mu\nu} \star F^g{}_{\mu\nu} = f_g \star F_{\mu\nu} \star F_{\mu\nu} \star f_{g^{-1}}$

$\Longrightarrow$ The natural action $\int F_{\mu\nu} \star F_{\mu\nu}$ is gauge invariant thanks to the cyclicity of the product

Coordinate-dependent noncommutativity

Given the star product of fields in the form

$$f \star g = f \cdot g + \frac{i}{2}\,\Theta^{ab}(x)\,\partial_a f \partial_b g + \ldots$$

ordinary derivations violate the Leibniz rule,

$$\partial_c(f \star g) = (\partial_c f) \star g + f \star (\partial_c g) + \frac{i}{2}\,\partial_c \Theta^{ab}(x)\,\partial_a f \partial_b g + \ldots$$

unless Θ is constant $\Longrightarrow$ star derivations are realised by star commutators

$$D_a f = (\Theta^{-1})_{ab}[x^b, f]_\star \xrightarrow{\Theta \to 0} \partial_a f$$

Lie algebra type star products, $[x^j, x^k]_\star = c_l^{jk} x^l$ do admit a generalisation according to

$$D_j f \propto [x^j, f]_\star$$

Gauge models have been constructed within the same approach but

- the limit $\theta \to 0$, does not yield the standard commutative result [[Géré-V.-Wallet, Wallet & coll.]
- the differential calculus not adequate to describe the dynamics

Summarising: Already for $[x^\mu, x^\nu]_\star = c^{\mu\nu}_\kappa x^\kappa$ we have problems

- the commutative limit ill defined

- the differential calculus does not fully describe the dynamics [this is the case of $\mathbb{R}^3_\lambda$ [Hammou-Lagraa-SheikhJabarhi ' 02] where the natural $3-d$ differential calculus does not capture radial dynamics]

Alternatives:

- modify the differential calculus (e.g. twist)

- modify the action functional

- ...

Approach presented in this talk

- Work in the semi-classical approximation [star-commutator replaced by Poisson bracket: $(\mathcal{A}, \star) \to (\mathcal{F}(M), \{.,.\}_\Theta)$]
- Reinterpret standard (commutative) gauge theory within symplectic geometry of (T^*M, ω_0) ("canonical embedding")
- Introduce a new symplectic form ω such that $\pi_* \omega^{-1} = \Theta$, $\Theta = \Theta^{\mu\nu}(x)\partial_\mu \wedge \partial_\nu$
  so to embed non trivial PB of M, $\{.,.\}_\Theta$
- Define gauge transformations, field strength and dynamics, within the modified symplectic embedding

Poisson gauge transformations should satisfy:

the closure of the gauge algebra $\quad [\delta_f, \delta_g]A = \delta\{f, g\}A$

reproduce the standard U(1) gauge transformation $\quad \delta_f A_\mu \overset{\Theta \to 0}{\to} \partial_\mu f$

Standard $U(1)$ gauge theory revisited

Goal: Show that the infinitesimal gauge transformation of the gauge potential $\delta_f A_\mu = \partial_\mu f$, descends from canonical PB over T^*M, locally $\{p_i, x^j\} = \delta_i^j$, $\{p_i, p_j\} = \{x^i, x^j\} = 0$. (M the space time)

Then define a new gauge transform for the case M endowed with a non-trivial space-time PB $\{x^i, x^j\} \neq 0$

Strategy:

- Recognize $A_\mu(x)$ as the fibre coordinate at x of a point in the cotangent space
- Associate the gauge transformation with a canonical transformation generated by the gauge parameter f
- compute the infinitesimal gauge transformation through Poisson bracket

Consider first the standard setting of $U(1)$ gauge theory, with $\Theta = 0$

- The cotangent space T^*M is endowed with the canonical symplectic form $\omega_0 = d\lambda_0$, λ_0 the Liouville one-form (locally $p_\mu dx^\mu$)
- The gauge field A is a local one-form on M –> it is associated with a local section of the cotangent space, $s_A : \mathcal{U} \to T^*\mathcal{U}$, through a local trivialisation

$$\psi_{\mathcal{U}}^{-1}(s_A(x)) = (x, A(x))$$

where $\mathcal{U}$ is a local chart on M.

- the local one-form

$$\xi_A = \lambda_0 - \pi^* A \quad (**)$$

vanishes locally, through the pullback s_A^*

$$s_A^*(\xi_A) = 0 \quad (i.e. p_\mu - A_\mu = 0)$$

because $s_A^*(\lambda_0) = A = (\pi \circ s_A)^*(A)$

[Remember that by definition, given a section $s : x \to u \in T^*M$ s.t. $\psi^{-1}(u) = (x, p)$ $(s^*\lambda_0)(x) = p$ - tautological one-form -]

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Thus

- ξ_A vanishes exactly on $\mathrm{im}(s_A) \subset T^*\mathcal{U} \Longrightarrow \mathrm{im}(s_A)$ is identified by the constraint $**$
- p, the fibre coordinate at x, gets fixed to its value $A(x)$ identified by the section s_A

Then, the infinitesimal gauge transformation of the gauge potential A, with gauge parameter f, may be defined in terms of the canonical Poisson bracket ω_0^{-1} as follows

$$\delta_f A_\nu(x) = s_A^*\{\pi^* f, \xi_{A_\nu}\}_{\omega_0^{-1}} = \frac{\partial f}{\partial x^\mu}\frac{\partial(p_\nu - A_\nu)}{\partial p_\mu} = \partial_\nu f\,,$$

that is to say, one first performs the Poisson bracket in $T^*\mathcal{U}$, then goes to its local form through s_A, recovering the standard infinitesimal gauge variation of the potential

To summarize

- we have embedded the trivial PB $\{x^\mu, x^\nu\}$
- we have identified A with the fibre coordinate of a local section on T^*M
- we have computed PBs on T^*M
- and pulled back to M

We can add

$$\delta_f \delta_g A - \delta_g \delta_f A = 0$$

namely

$$[\delta_f, \delta_g] A = \delta_{\{f,g\}_0} A$$

it being $\{f(x), g(x)\}_0 = 0$

The algebra of gauge transformations closes wrt the PB on M

The conjugate momentum p is an auxiliary variable, disappeared from the gauge algebra

Generalization to non-trivial PB and symplectic embedding

The complicated procedure described above, certainly redundant in the canonical case $\Theta = 0$, is extremely useful and constructive for the case $\Theta \neq 0$
Symplectic embedding (SE) is a generalization of symplectic realizations [Weinstein '83]. In this context SE is due to [Kupriyanov & Szabo]
It goes as follows:

Define Λ as a deformation of Θ

$$\Lambda(x,p) = \theta^{\mu\nu}(x)\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} + \gamma^\mu_\nu(x,p)\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\nu}$$

satisfying Jacobi identity, provided Θ does; $\pi_* \Lambda = \Theta$
this produces an equation for the unknown γ

$$\gamma^\mu_\nu \frac{\partial}{\partial p_\nu}\gamma^\xi_\lambda - \gamma^\xi_\nu \frac{\partial}{\partial p_\nu}\gamma^\mu_\lambda + \theta^{\mu\nu}\frac{\partial}{\partial x^\nu}\gamma^\xi_\lambda - \theta^{\xi\nu}\frac{\partial}{\partial x^\nu}\gamma^\mu_\lambda - \gamma^\nu_\lambda \frac{\partial}{\partial x^\nu}\theta^{\mu\xi} = 0$$

Provided we find solutions for γ, gauge transformations are defined as in the canonical case:

$$\delta_f A_\nu(x) = s_A^*\{\pi^* f, \xi_{A_\nu}\}_\Lambda$$

with $\xi_A = \lambda_0 - \pi^* A \overset{\text{locally}}{\longrightarrow} p_\mu dx^\mu - A_\mu dx^\mu$

This yields

$$\begin{aligned} \delta_f A_a : &= s_A^* \{\pi^* f, \xi_{A_a}\}_{\omega^{-1}} = \Theta^{rs} \frac{\partial f}{\partial x^r} \frac{\partial \xi_{A_a}}{\partial x^s} + \gamma_s^r(x, A) \frac{\partial f}{\partial x^r} \frac{\partial \xi_{A_a}}{\partial p_s} \\ &= \{A_a, f\}_\Theta + \gamma_a^r(x, A) \frac{\partial f}{\partial x^r} \end{aligned}$$

Once the procedure understood, the result can be restated in a simpler language, by replacing the constraint with its local form

$$\Phi_a := p_a - A_a(x)$$

so to have directly

$$\delta_f A_a := \{f, \Phi_a\}_{\Phi_a = 0} = \{A_a(x), f(x)\}_\Theta + \gamma_a^r(A)\, \partial_r f(x)\,,$$

where the auxiliary variable p_a has been eliminated through the constraint notice the deformed derivative $\gamma_a^r\, \partial_r$

- The commutative limit $\Theta \to 0$, $\gamma^\mu_\nu \to \delta^\mu_\nu$ yields the classical result;
- Closure of the gauge algebra:

$$[\delta_f, \delta_g] A_\nu = -\{\{f,g\}_\Theta, A_\nu\}_\Theta + \gamma^\mu_\nu \frac{\partial}{\partial x^\nu}\{f,g\}_\Theta$$

namely $[\delta_f, \delta_g] A_\nu = \delta_{\{f,g\}_\Theta} A_\nu$

Notice that for the Moyal case $\Theta = const \Rightarrow \partial_\mu \Theta^{\rho\sigma} = 0$

$\Longrightarrow \gamma^\mu_\nu = \delta^\mu_\nu$

yielding back

$$\delta_f A_\nu = \partial_\nu f - \{f, A_\nu\}_\Theta$$

Also notice that $\gamma^\mu_\nu = \delta^\mu_\nu$ also solution of the full noncommutative case yielding $\delta_f A_\nu = \partial_\nu f - [f, A_\nu]\star$

Solutions for γ^μ_ν have been found for Lie algebra type noncommutativity [Kupriyanov-Kurkov-Vitale '20-'22]; perturbative solutions have been found for general x dependence [Kupriyanov '19]; non uniqueness of solution and relation with Seiberg-Witten map discussed in [Kupriyanov-Kurkov-Vitale '22]

The field strength

What about the field strength?
Let us return to the classical case $\Theta = 0$
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_n u A_\mu$ may be rewritten in terms of canonical PB as

$$\begin{aligned} F_{\mu\nu} &= s_A^*\{\xi_{A_\mu}, \xi_{A\nu}\}_{\Lambda_0} = -\frac{\partial}{\partial x^\rho}(p_\mu - A_\mu)\frac{\partial}{\partial p_\rho}(p_\nu - A_\nu) + \frac{\partial}{\partial x^\rho}(p_\nu - A_\nu)\frac{\partial}{\partial p_\rho}(p_\mu - A_\mu) \\ &= \partial_\mu A_\nu - \partial_n u A_\mu \end{aligned}$$

$\Longrightarrow$ a natural generalization when $\{x^\mu, x^\nu\} \neq 0$ would be to use the symplectic embedding again

$$\tilde{F}_{\mu\nu} = s_A^*\{\xi_{A_\mu}, \xi_{A\nu}\}_\Lambda = \{A_\mu, A_\nu\}_\Theta - \gamma^\rho_\sigma(A)\frac{\partial A_\mu}{\partial x^\rho}\frac{\partial p_\nu}{\partial p_\sigma} + \gamma^\rho_\sigma(A)\frac{\partial A_\nu}{\partial x^\rho}\frac{\partial p_\mu}{\partial p_\sigma}$$

namely

$$\tilde{F}_{\mu\nu} = \{A_\mu, A_\nu\}_\Theta + \gamma^\rho_\mu \partial_\rho A_\nu - \gamma^\rho_\nu \partial_\rho A_\mu$$

and compute its gauge transformation according to the definition

$$\delta_f \tilde{F}_{\mu\nu} = s_A^*\{\pi^* f, \pi^* \tilde{F}_{\mu\nu}\}_\Lambda$$

However, this definition can be checked not to be covariant
$\delta_f \tilde{F}_{\mu\nu} = -\{f, \tilde{F}_{\mu\nu}\}_\Theta + \text{unwanted terms}$
$\Longrightarrow$ The definition of F has to be modified

In [Kupriyanov-Szabo '21] a solution is searched in the form $F_{\mu\nu} = R^{\rho\sigma}_{\mu\nu}\tilde{F}_{\rho\sigma}$; this yields an equation for $R^{\rho\sigma}_{\mu\nu}$
To maintain the interpretation in terms of symplectic embedding a solution may be found [Kupriyanov-Kurkov-Vitale '22] by performing a non-linear transformation

$$\Phi_\mu \to \Phi'_\mu = \rho^\nu_\mu \Phi_\nu$$

with ρ a non-degenerate matrix to be determined by the covariance request. The non degeneracy ensures that $\Phi_\mu = 0 \Leftrightarrow \Phi'_\mu = 0$
We find

- $\delta_f F_{\mu\nu} = \{F_{\mu\nu}, f\}$
  if $\gamma^j_b \partial_p \rho^i_a + \rho^b_a \partial_p \gamma^j_b + \Theta^{jb}\partial_b \rho^i_a = 0$
  for arbitrary PB a solution is found in form of a perturbative series
- for Lie algebra type noncommutativity simple solutions have been found; e.g. λ-Minkowski (or angular noncommutativity) [Dimitrijevic-Konjik-Samsarov '17] $\mathbb{R}^3_\lambda$ (or $SU(2)$ like) ...
- The corresponding gauge invariant Maxwell Lagrangian has been introduced

I have presented a new approach to NC gauge theories based on symplectic embedding, originally started by Kupriyanov and Kupriyanov-Szabo; the approach is related to L_∞ bootstrap [Blumenhagen-Brünner-Kupriyanov-Lüst '18, Kupriyanov-V. '20]
With Max Kurkov we have worked out applications to κ-Minkowski space-time, $\mathbb{R}^3_\lambda$, λ-Minkowski and other Lie algebra-type noncommutativity

To do:

- Matter sector
- Study the dynamics, exhibit solutions
- Go to the full NC regime

Thank you!