Phase transitions in a Φ^4 matrix model on a curved noncommutative space

Dragan Prekrat[1,2]

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Quantum gravity phenomenology in the multi-messenger approach

[1] University of Belgrade – Faculty of Physics
[2] University of Belgrade – Faculty of Pharmacy

Timeline

- gauge model on the truncated Heisenberg algebra
- phase transitions in matrix models on the NC space

- Faculty of Physics, Belgrade ’16 – Burić
- DIAS ’17, ’18 — COST STSM MP1405 — O’ Connor, Kováčik
- Faculty of Pharmacy, Belgrade ’19, ’22 — Vasović, Ranković
- Comenius University in Bratislava ’21, ’22 — Tekel, Kováčik

- arXiv:2002.05704 (JHEP)
- arXiv:2104.00657 (Phys.Rev.D)
- arXiv:2209.00592 (NEW)

Outline

1. GW model
2. Truncated Heisenberg space
3. Matrix model & Phase transitions
4. Analytical results
5. Conclusions

GW model

Noncommutative $\lambda\phi^4$

- Moyal $\star$-product:

$$f \star g = f e^{i/2 \overleftarrow{\partial} \, \theta \, \overrightarrow{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2} \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu} + \cdots$$

- NC coordinates:

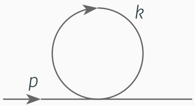
$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$$

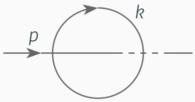
- the simplest model $\lambda\phi^4_\star$:

$$S = \int d^{2n}x \left(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)$$

UV/IR mixing

- propagator/mass corrections in the S_{eff}: $\sim \phi(p^2 + m^2 + \cdots)\phi$





$$\text{planar} = \frac{\lambda}{4!} \int \frac{d^2k}{(2\pi)^2} \frac{2}{k^2 + m^2}$$

$$\text{non-planar} = \frac{\lambda}{4!} \int \frac{d^2k}{(2\pi)^2} \frac{\exp(ik_\mu \theta^{\mu\nu} p_\nu)}{k^2 + m^2} = \frac{\lambda}{96\pi} \log \frac{\Lambda_{\text{eff}}^2}{m^2} + \cdots$$

$$\Lambda_{\text{eff}}^2 = \frac{1}{1/\Lambda^2 + |\theta^{\mu\nu} p_\nu|^2} \to \frac{1}{|\theta^{\mu\nu} p_\nu|^2}, \quad \Lambda \to \infty$$

GW model

$$S_{GW} = \int d^{2n}x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \boxed{\frac{1}{2}\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi)} + \right.$$
$$\left. + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)$$

$$\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu} x^\nu$$

- superrenormalizable in 2-dim (Grosse and Wulkenhaar '03)
- harm. osc. potential $\Leftrightarrow$ curvature (Burić and Wohlgenannt '10)

- renormalizability of the two-dimensional $\lambda\phi^4_\star$-model is restored by defining it as a $\Omega \to 0$ limit of the series of GW models in which Ω itself does not renormalize and serves as a series-label

Truncated Heisenberg space

Heisenberg algebra $\mathfrak{h}$

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu} = i\frac{\epsilon^{\mu\nu}}{\mu^2} \qquad \Rightarrow \qquad [\mu x^1, \mu x^2]_\star = i \qquad \Rightarrow \qquad [X, Y] = i$$

- infinite dim representation $\quad X \to + \quad Y/i \to -$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} & \pm 1 & & & \\ +1 & & \pm\sqrt{2} & & \\ & +\sqrt{2} & & \pm\sqrt{3} & \\ & & +\sqrt{3} & & \ddots \\ & & & \ddots & \end{bmatrix}$$

(Modified) truncated Heisenberg algebra $\mathfrak{h}_\epsilon^{\mathrm{tr}}$

$$[X, Y] = i\epsilon\,(1 - Z)\,, \qquad [X, Z] = i\epsilon\,\{Y, Z\}\,, \qquad [Y, Z] = -i\epsilon\,\{X, Z\}$$

$$\epsilon = \text{strength of noncomutativity}$$

$$\text{commutative limit} \quad \xleftarrow{\epsilon \to 0} \quad \mathfrak{h}_\epsilon^{\mathrm{tr}} \quad \xrightarrow{\epsilon \to 1} \quad \mathfrak{h}^{\mathrm{tr}} \quad \xrightarrow[N \xrightarrow{w} \infty]{Z \to 0} \quad \mathfrak{h}$$

- possible to define NC differential calculus on $\mathfrak{h}_\epsilon^{\mathrm{tr}}$
- differentiation via momenta $\partial_\alpha = [iP_\alpha, \cdot\,]$:

$$\epsilon P_1 = -Y, \qquad \epsilon P_2 = X, \qquad \epsilon P_3 = \frac{\mathbb{1}}{2} - Z$$

- curvature:

$$R = \frac{15}{2}\,\mathbb{1} - 2Z - 4\left(X^2 + Y^2\right) \approx -16\,\mathrm{diag}(1, 2, \ldots, N)$$

QFT on the truncated Heisenberg algebra

- scalar field coupled to curvature $\xi R\phi^2$
- $\Leftrightarrow$ renormalizable GW model when $N \to \infty$
- arXiv:0902.3408

- spinor field coupled to torsion $\mathrm{tr}(\psi\bar{\psi})(T_\alpha\gamma^\alpha)(\theta^\beta\gamma_\beta)$
- $\Leftrightarrow$ renormalizable Vignes-Tourneret model
- arXiv:1502.00761

- gauge field, background space defines derivatives
- nonrenormalizable: $1/\Box$ and $1/\Box^2$ non-local divergent terms
- arXiv:1610.01429

Matrix model & Phase transitions

Matrix model

- Weyl transform of the GW model:

$$\phi \longleftrightarrow \Phi, \qquad \int \longleftrightarrow \sqrt{\det 2\pi\theta}\ \mathrm{tr}$$

- matrix model on the truncated Heisenberg algebra:

$$\boxed{S_N = \mathrm{tr}\left(\Phi\mathcal{K}\Phi - c_r R\Phi^2 - c_2\Phi^2 + c_4\Phi^4\right), \qquad \mathcal{K} = [P_\alpha, [P_\alpha, \cdot\,]]}$$

- unscaled vs. scaled parameters:

$$\widetilde{c}_2 = g_2 = \frac{c_2}{N}, \qquad \widetilde{c}_4 = g_4 = \frac{c_4}{N}$$

Phase structure

- classical equation of motion:

$$2\mathcal{K}\Phi - c_r\{R,\Phi\} + 2\Phi\left(-c_2 + 2c_4\Phi^2\right) = \mathbb{0}$$

- classical vacuum solutions:

$$\Phi = \frac{\operatorname{tr}\Phi}{N}\,\mathbb{1}, \qquad \Phi = \mathbb{0}, \qquad \Phi^2 = \begin{cases} \mathbb{0} & \text{for } c_2 \leq 0 \\ \frac{c_2\,\mathbb{1}}{2c_4} & \text{for } c_2 > 0 \end{cases}$$

- 3 phases:
  - disordered/1-cut symmetric phase: $\Phi_{\updownarrow} = \mathbb{0}$
  - striped/2-cut symmetric phase: $\Phi_{\uparrow\downarrow} \propto U\,\mathbb{1}_{\pm}\,U^\dagger$
  - ordered/1-cut asymmetric phase: $\Phi_{\uparrow\uparrow} \propto \mathbb{1}$

- modified ordered phases:

$$\Phi_R^2 = \frac{c_2\,\mathbb{1} + c_r R}{2c_4}$$

Numerical simulations

- numerical treatment: Hybrid Monte Carlo, $N \leq 70$
- physics:

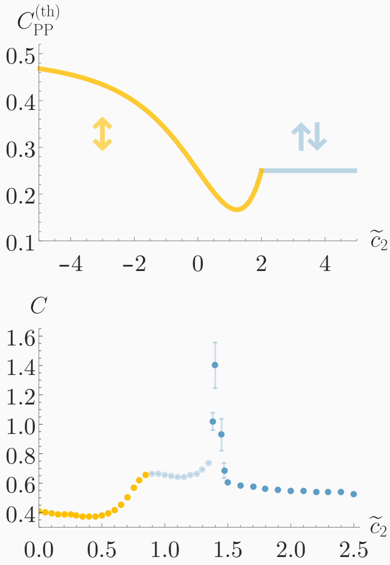
$$\langle \mathcal{O} \rangle = Z^{-1} \int d\Phi \, \mathcal{O} \, e^{-S}, \qquad \mathrm{Var}\, \mathcal{O} = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2$$

- control: Schwinger-Dyson identity

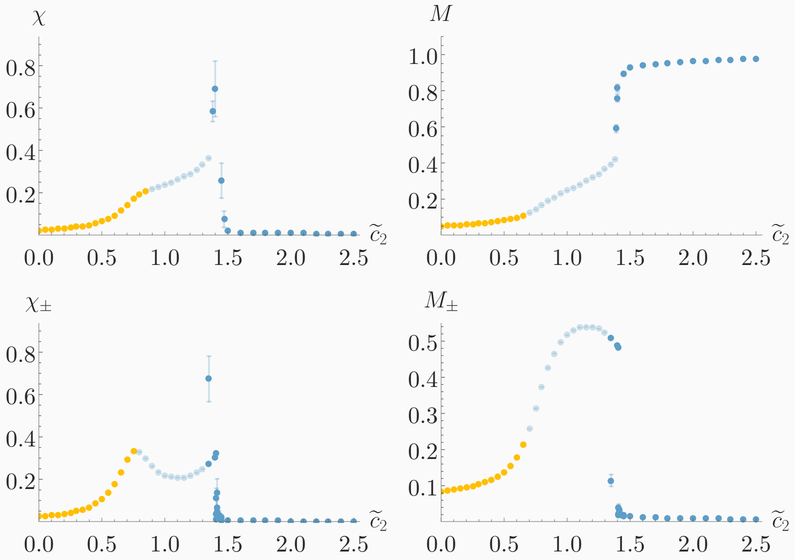
$$\left\langle \mathrm{tr} \left(2\Phi \mathcal{K} \Phi - 2c_r R \Phi^2 - 2c_2 \Phi^2 + 4c_4 \Phi^4 \right) \right\rangle = N^2$$

- thermodynamic quantities:
  - energy $E = \langle S \rangle / N^2, \quad E_\mathrm{k} = \langle S_\mathrm{k} \rangle / N^2$
  - heat capacity $C = \mathrm{Var}\, S / N^2$
  - magnetization $M = \langle |\, \mathrm{tr}\, \Phi | \rangle / N, \quad M_\pm = \langle |\, \mathrm{tr}\, \Phi \mathbb{1}_\pm \,| \rangle / N$
  - susceptibility $\chi = \mathrm{Var}\, M / N, \quad \chi_\pm = \mathrm{Var}\, M_\pm / N$
  - Binder cumulant $U = 1 - \left\langle |\, \mathrm{tr}\, \Phi |^4 \right\rangle / (3 \left\langle |\, \mathrm{tr}\, \Phi |^2 \right\rangle^2)$
- random field characteristics:
  - eigenvalue (ρ_λ) and trace (ρ_tr) probability distributions

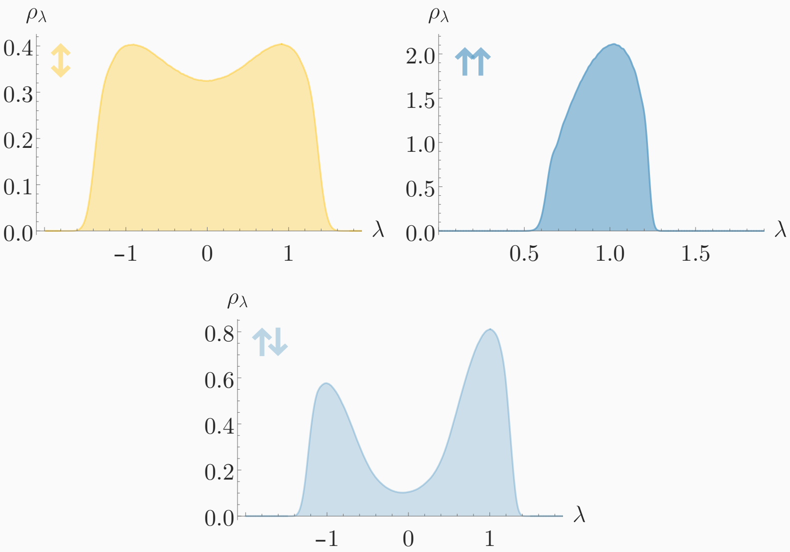
Thermodynamics



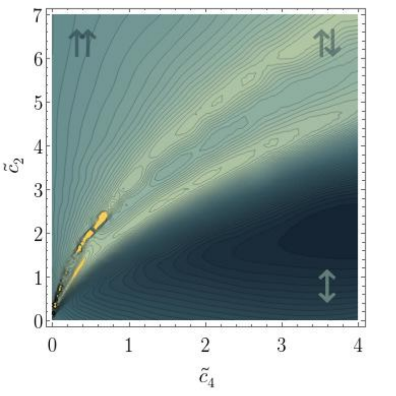
Thermodynamics

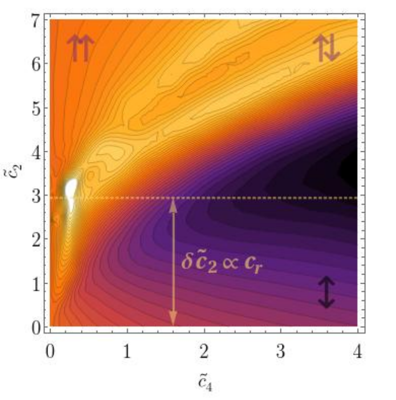


Eigenvalue distributions



Phase diagrams





$\lambda\phi_\star^4$

GW

$\mathrm{tr}\left(-c_r R \Phi^2\right) \Rightarrow \delta\widetilde{c}_2 \leq 16\widetilde{c}_r$

$\Phi_R \Rightarrow \delta\widetilde{c}_2 \geq 16\widetilde{c}_r$

Stripe phase removal

1. $\lambda\phi^4_\star$-renormalization:

$$\Omega \to 0 \quad \Rightarrow \quad \mathrm{GW} \to \lambda\phi^4_\star$$

2. parameter correspondence:

$$\Lambda^2 \sim N, \quad \Omega \sim \frac{1}{\log N} \quad \Rightarrow \quad c_r = \frac{\Omega^2/8}{1-\Omega^2/2} \sim \frac{1}{\log^2 N}$$

3. triple point T shift:

$$\widetilde{\delta c_2} \propto c_r \quad \Rightarrow \quad \delta c_2 \propto N c_r \quad \Rightarrow \quad \boxed{c_2(T) \sim \frac{N}{\log^2 N} \to \infty}$$

4. mass renormalization shift:

$$\delta m^2_{\mathrm{ren}} = \frac{\lambda}{12\pi(1+\Omega^2)} \log \frac{\Lambda^2\theta}{\Omega} \quad \Rightarrow \quad \boxed{|\delta c_2^{\mathrm{ren}}| \sim \log N < c_2(T)}$$

Analytical results

Eigenvalue distribution equation

$$S(\Phi) = N\,\mathrm{Tr}\Big(\Phi\widetilde{\mathcal{K}}\Phi - g_r\widetilde{R}\Phi^2 - g_2\Phi^2 + g_4\Phi^4\Big)$$

$$\Phi = U\Lambda U^\dagger, \qquad \langle\mathcal{O}(\Lambda)\rangle = Z^{-1}\int \mathrm{d}\Phi\,\mathcal{O}(\Lambda)\,e^{-S(\Phi)}, \qquad \int \mathrm{d}U?$$

$$S(\Lambda, U) = N\,\mathrm{Tr}\Big((U\Lambda U^\dagger)\widetilde{\mathcal{K}}(U\Lambda U^\dagger) - g_r\widetilde{R}U\Lambda^2U^\dagger - g_2\Lambda^2 + g_4\Lambda^4\Big)$$

$$\langle\mathcal{O}(\Lambda)\rangle = Z^{-1}\int \mathrm{d}\Lambda\,\mathcal{O}(\Lambda)\,e^{-S_{\text{eff}}(\Lambda)},$$

$$S_{\text{eff}}(\Lambda) = N\,Tr\big(? - g_2\Lambda^2 + g_4\Lambda^4\big) - \sum_{m\neq n}\log|\lambda_m - \lambda_n|$$

$$? - g_2\lambda + 2g_4\lambda^3 = \int_{\text{support}} \mathrm{d}\lambda'\,\frac{\rho(\lambda')}{\lambda - \lambda'}$$

Harish-Chandra-Itzykson-Zuber (HCIZ) integral

D. Prekrat, D. Ranković, N. K. Todorović-Vasović, S. Kováčik, J. Tekel, *Approximate treatment of noncommutative curvature in quartic matrix model*, arXiv:2209.00592 [hep-th] (NEW)

$$I = \int_{U(N)} dU \, \exp\left(t \, \mathrm{tr} \, AUBU^\dagger\right) = \frac{c_N}{t^{N(N-1)/2}} \frac{\det \exp\left[t a_i b_j\right]}{\Delta(A)\Delta(B)}$$

$$c_N = \prod_{k=1}^{N-1} k!, \qquad \Delta(A) = \prod_{1 \leq i < j \leq N} (a_j - a_i)$$

$$I = \sum_{n=0}^{\infty} \frac{t^n}{n!} I_n = \exp\left(-\sum_{n=1}^{\infty} \frac{t^n}{n!} S_n\right)$$

(A very sketchy) expansion sketch

$$\begin{pmatrix} e^{ta_1 b_j} \\ e^{ta_2 b_j} \\ \vdots \\ e^{ta_N b_j} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + t \begin{pmatrix} a_1 b_j \\ a_2 b_j \\ \vdots \\ a_N b_j \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} a_1^2 b_j^2 \\ a_2^2 b_j^2 \\ \vdots \\ a_N^2 b_j^2 \end{pmatrix} + \cdots$$

$$\begin{pmatrix} e^{ta_1 b_1} & e^{ta_1 b_2} & \dots & e^{ta_1 b_N} \\ e^{ta_2 b_1} & e^{ta_2 b_2} & \dots & e^{ta_2 b_N} \\ \vdots & \vdots & & \vdots \\ e^{ta_N b_1} & e^{ta_N b_2} & \dots & e^{ta_N b_N} \end{pmatrix} =$$

$$= \sum_{k_i \geq 0} t^{k_1 + k_2 + \cdots + k_N} \frac{b_1^{k_1} b_2^{k_2} \dots b_N^{k_N}}{k_1! k_2! \dots k_N!} \begin{vmatrix} a_1^{k_1} & a_1^{k_2} & \cdots & a_1^{k_N} \\ a_2^{k_1} & a_2^{k_2} & \cdots & a_2^{k_N} \\ \vdots & \vdots & & \vdots \\ a_N^{k_1} & a_N^{k_2} & \cdots & a_N^{k_N} \end{vmatrix}$$

HCIZ expansion

$$I_0 = 1\,, \qquad\qquad I_1 = \frac{\operatorname{tr} A \operatorname{tr} B}{N}$$

$$I_2 = \frac{(\operatorname{tr}^2 A + \operatorname{tr} A^2)(\operatorname{tr}^2 B + \operatorname{tr} B^2)}{2N(N+1)} + \frac{(\operatorname{tr}^2 A - \operatorname{tr} A^2)(\operatorname{tr}^2 B - \operatorname{tr} B^2)}{2N(N-1)}$$

$$\begin{aligned} I_3 = &\frac{(\operatorname{tr}^3 A + 3 \operatorname{tr} A \operatorname{tr} A^2 + 2 \operatorname{tr} A^3)(\operatorname{tr}^3 B + 3 \operatorname{tr} B \operatorname{tr} B^2 + 2 \operatorname{tr} B^3)}{6N(N+1)(N+2)} \\ &+ \frac{(\operatorname{tr}^3 A - 3 \operatorname{tr} A \operatorname{tr} A^2 + 2 \operatorname{tr} A^3)(\operatorname{tr}^3 B - 3 \operatorname{tr} B \operatorname{tr} B^2 + 2 \operatorname{tr} B^3)}{6N(N-1)(N-2)} \\ &+ \frac{2(\operatorname{tr}^3 A - \operatorname{tr} A^3)(\operatorname{tr}^3 B - \operatorname{tr} B^3)}{3N(N-1)(N+1)} \end{aligned}$$

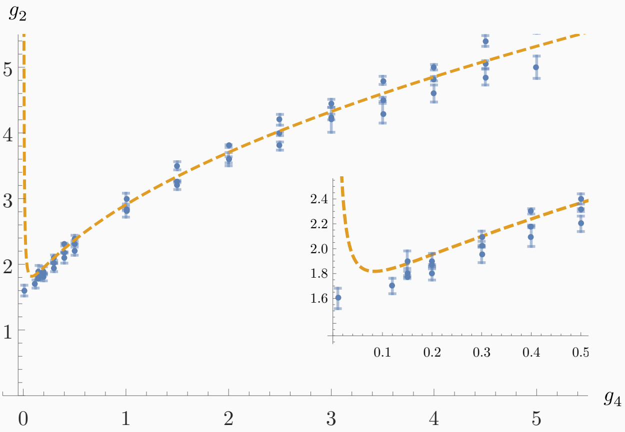
Effective action

$$S_1 = -\frac{N+1}{2}\operatorname{tr}\Lambda^2\,, \qquad S_2 = \frac{1}{12}\operatorname{tr}^2\Lambda^2 - \frac{N}{12}\operatorname{tr}\Lambda^4\,, \qquad S_3 = 0$$

$$S_4 = \frac{1}{120}\left(N\operatorname{tr}\Lambda^8 + 3\operatorname{tr}^2\Lambda^4 - 4\operatorname{tr}\Lambda^6\operatorname{tr}\Lambda^2\right)$$

$$\boxed{\begin{aligned} S(\Lambda) = N\operatorname{tr}\left(-(g_2 - 8g_r)\Lambda^2 + \left(g_4 - \frac{32}{3}g_r^2\right)\Lambda^4 + \frac{1024}{45}g_r^4\operatorname{tr}\Lambda^8\right) + \\ + \frac{32}{3}g_r^2\operatorname{tr}^2\Lambda^2 + \frac{1024}{15}g_r^4\operatorname{tr}^2\Lambda^4 - \frac{4096}{45}g_r^4\operatorname{tr}\Lambda^6\operatorname{tr}\Lambda^2 - \log\Delta^2(\Lambda)\end{aligned}}$$

Phase diagram



$$g_2 = 2\sqrt{g_4} + 8g_r + \frac{32}{3}\frac{g_r^2}{\sqrt{g_4}} + \frac{256}{15}\frac{g_r^4}{g_4\sqrt{g_4}}$$

Conclusions

Model comparison

space	model	mass shift	triple point	start. phase	renorm.
$\mathbb{R}^2_\theta$, $\mathfrak{h}^{tr}$	$\lambda\phi^4_\star$	UV/IR	0	$\uparrow\downarrow$	no
$\mathbb{R}^2_\theta$, $\mathfrak{h}^{tr}$	GW	$\log N$	N	$\updownarrow$	yes
$\mathbb{R}^2_\theta$, $\mathfrak{h}^{tr}$	$\lambda\phi^4_{GW}$	$\log N$	$N \big/ \log^2 N$	$\updownarrow$	yes
$\mathfrak{h}^{tr}_\epsilon$	$U(1)$		one phase?	$\uparrow\downarrow$?	no
S^2_N	$\lambda\phi^4_\star$	$\log N$	$N^{3/2}$	$\updownarrow$	yes

Conclusions

- hypothesis: no $\uparrow\downarrow$-phase $\Leftrightarrow$ renormalizability
- demonstrated: connection between the GW model renormalizability and its phase structure
- extendable to other models
- numerical simulation of the phase transitions could tell us in advance about the renormalization properties of new models

The End

Thank you for the attention.