# The fuzzy BTZ black hole

Ilija Burić
University of Pisa

Quantum gravity phenomenology in the multi-messenger approach
September 3, 2022, Belgrade

based on work with Maja Burić, 2204.03673

# Noncommutative geometry

## Noncommutative coordinates at small scales

- Non-commuting coordinates: a part of many approaches to QG
- No point-like singularities $\rightarrow$ improved renormalisability [Grosse, Wulkenhaar]
- Sometimes: regularisation which respects symmetry

## Geometry from algebra

- Concepts of geometry formulated in terms of the algebra of functions
- Differential and (pseudo)-Riemannian aspects
- Inequivalent noncommutative generalisations
- Spectral properties and a classical limit

# Fuzzy spaces

## Frame formalism

- Noncommutative analogue of moving frames [Madore]
- Abstracted from matrix geometries, fuzzy sphere [Dubois-Violette, Kerner, Madore]
- Allows for formulation of gauge theories
- Rigid structure constrained by consistency
- Most examples highly symmetric [Wess, Zumino; Cerchiai, Fiore, Madore; Jurman, Steinacker; Burić, Wohlgennant; Latas, Nenadović...]

## Today

- Noncommutative model of a spacetime with interesting causal structure

## BTZ black hole [Banados, Teitelboim, Zanelli, Henneaux]

Characterised by mass and spin

Schwarzschild coordinates

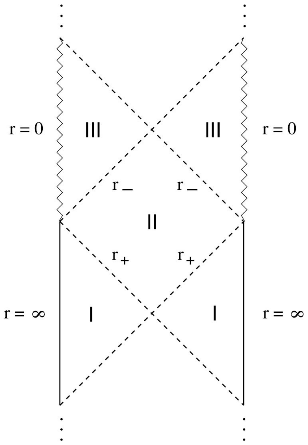
$$ds^2 = -N^2 dt^2 + \frac{1}{N^2}\, dr^2 + r^2 (N^\phi dt + d\phi)^2$$

$$N^2 = \frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2}\,, \quad N^\phi = -\frac{J}{2r^2}\,,$$

Inner and outer horizons

$$r_+ \pm r_- = \sqrt{M\ell^2 \pm J\ell}$$

Similar properties to Kerr in 4d



# Gravity in three dimensions

## Locally maximally symmetric space

- All solutions to vacuum Einstein equations locally isometric

$$ds^2 = \frac{-d\gamma^2 + d\beta^2 + dz^2}{z^2}$$

- Orthonormal (moving) frame $e^\mu_\alpha = e_\alpha(x^\mu), \quad \eta_{\alpha\beta} = g_{\mu\nu} e^\mu_\alpha e^\nu_\beta$

$$e_z = z\partial_z, \quad e_\beta = z\partial_\beta, \quad e_\gamma = z\partial_\gamma$$

- Local Killing vectors form the Lie algebra

$$\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$$

- BTZ black hole a discrete quotient of simply connected $AdS_3$

$$\mathrm{BTZ}_{M,J} = \widetilde{\mathrm{AdS}'}_3/\mathbb{Z}$$

- Points identified along the flow of a Killing vector

# Plan for the talk

# Noncommutative moving frames [Madore]

## Differential geometry

- Throughout: $\mathcal{A}$ = algebra generated by hermitian coordinates $x^\mu$
- Correspondence principle: momenta $p_\alpha \in \mathcal{A}$

$$\tilde{e}^\mu_\alpha = \tilde{e}_\alpha \tilde{x}^\mu \quad \longrightarrow \quad e^\mu_\alpha = [p_\alpha, x^\mu]$$

- Differential forms

$$\Omega^*(\mathcal{A}) = \langle x^\mu, \theta^\alpha \rangle, \qquad [x^\mu, \theta^\alpha] = 0, \qquad df = [p_\alpha \theta^\alpha, f]$$

- Connection 1-forms and curvature 2-forms

$$D\theta^\alpha = -\omega^\alpha{}_\beta \otimes \theta^\beta = -\omega^\alpha{}_{\gamma\beta}\, \theta^\gamma \otimes \theta^\beta, \quad \Omega^\alpha_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \omega^\gamma{}_\beta = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} \theta^\gamma \theta^\delta$$

## Consistency $\rightarrow$ structure equations

$$2P^{\alpha\beta}{}_{\gamma\delta}\, p_\alpha p_\beta - F^\alpha{}_{\gamma\delta}\, p_\alpha - K_{\gamma\delta} = 0$$

# Momenta as symmetry generators

## Spaces based on Lie algebras

- Momenta form a Lie algebra

$$[p_\alpha, p_\beta] = F^\gamma_{\alpha\beta} p_\gamma$$

- Simple structure of differential forms

$$\theta^\alpha\theta^\beta = P^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma \otimes \theta^\delta = -\theta^\beta\theta^\alpha, \qquad d\theta^\alpha = -\frac{1}{2}F^\alpha{}_{\beta\gamma}\theta^\beta\theta^\gamma$$

- Momenta often taken among generators of the isometry group

## Generators of $SO(2,2)$

- Two copies of $\mathfrak{sl}(2,\mathbb{R})$

$$[H, E_+] = E_+, \qquad [H, E_-] = -E_-, \qquad [E_+, E_-] = 2H$$

- Momenta generators of translations and dilations

$$p_z = H + \bar{H}, \qquad p_\beta = E_+ + \bar{E}_+, \qquad p_\gamma = E_+ - \bar{E}_+$$

# Unitary irreducibles of $SO(2,2)$ [Bargmann]

## Representations

- Tensor products $\pi \otimes \bar{\pi}$ of $SL(2)$-representations
- Unitary irreducibles of $G = SL(2)$: principal, discrete and complementary series

## Discrete series

- Matrix elements square-integrable functions on $G$
- Lowest-weight representations $T_l^-$, labelled by $l < 0$
- Functions on $\mathbb{R}_+$, square integrable under

$$(\hat{F}_1, \hat{F}_2) = 2^{2l+1}\pi \int_0^\infty dx\, x^{2l+1}\overline{\hat{F}_1(x)}\hat{F}_2(x)$$

- Action of generators: $H = x\partial_x + l + 1, \quad E_+ = -ix$
- $E_+$ and $-E_-$ positive definite operators

# Coordinate operator algebra

## Coordinate operators

- Coordinate algebra $\mathcal{A} = \text{End}(T_l^- \otimes T_{\bar{l}}^-)$
- Poincaré coordinates

$$z = 2i\, E_+^a\, \bar{E}_+^{1-a} = 2x^a\, \bar{x}^{1-a}$$

$$\beta + \gamma = -2iE_+^{a-1}\, \bar{E}_+^{1-a} \left( H + \frac{a-1}{2} \right) = -2i \left( \frac{x}{\bar{x}} \right)^{a-1} \left( x\partial_x + l + \frac{a+1}{2} \right)$$

$$\beta - \gamma = -2iE_+^a\, \bar{E}_+^{-a} \left( \bar{H} - \frac{a}{2} \right) = -2i \left( \frac{x}{\bar{x}} \right)^a \left( \bar{x}\partial_{\bar{x}} + \bar{l} + 1 - \frac{a}{2} \right)$$

- Frame relations

$$[p_\gamma, \gamma] = z\,, \qquad [p_\beta, \beta] = z\,, \qquad [p_z, z] = z$$

- Momentum algebra

$$[p_z, p_\gamma] = p_\gamma, \qquad [p_z, p_\beta] = p_\beta, \qquad [p_\beta, p_\gamma] = 0$$

# Riemannian geometry

## Conformal boundary

- Flat commutative boundary

$$\left[\frac{p_\beta + p_\gamma}{2}, \frac{\beta+\gamma}{z}\right] = 1\,, \qquad \left[\frac{p_\beta - p_\gamma}{2}, \frac{\beta-\gamma}{z}\right] = 1$$

## Connection and curvature

- Frame is orthonormal: $g_{\alpha\beta} = \eta_{\alpha\beta}$
- Levi-Civita connection

$$D\theta^z = -\theta^\beta \otimes \theta^\beta + \theta^\gamma \otimes \theta^\gamma, \quad D\theta^\beta = \theta^\beta \otimes \theta^z, \quad D\theta^\gamma = \theta^\gamma \otimes \theta^z$$

- Curvature 2-forms

$$\Omega^z{}_\gamma = \theta^z\theta^\gamma, \quad \Omega^z{}_\beta = -\theta^z\theta^\beta, \quad \Omega^\beta{}_z = \theta^z\theta^\beta$$
$$\Omega^\beta{}_\gamma = \theta^\beta\theta^\gamma, \quad \Omega^\gamma{}_z = \theta^z\theta^\gamma, \quad \Omega^\gamma{}_\beta = \theta^\beta\theta^\gamma$$

- Einstein's equations $R_{\alpha\beta} = -2g_{\alpha\beta}$ hold

# Discrete identifications

## Classical action

- Action on Poincaré coordinates

$$z \mapsto z\, e^{-2\pi r_+}, \quad (\beta-\gamma) \mapsto (\beta-\gamma)\, e^{-2\pi(r_+ + r_-)}, \quad (\beta+\gamma) \mapsto (\beta+\gamma)\, e^{-2\pi(r_+ - r_-)}$$

## Quantum identifications

- Achieved by conjugation $f \mapsto UfU^{-1}$ with

$$U = e^{\alpha(H-\bar{H})}, \qquad \alpha = -2\pi r_-, \quad a = \frac{r_+ + r_-}{2r_-}$$

- In the representation, $x = \chi e^{\eta}, \quad \bar{x} = \chi e^{-\eta}$

$$(Uf)(\chi,\eta) = f(\chi,\eta+\alpha) \qquad \rightarrow \qquad \eta \sim \eta + \alpha$$

- Reduced Hilbert space $\mathcal{H}_{red} = L^2(\mathbb{R} \times S^1)$

$$\langle f_1, f_2\rangle = 2^{4l+2}\pi^2 \int d\eta\, d\chi\, \chi^{4l+3}\, \overline{f_1(\chi,\eta)} f_2(\chi,\eta)$$

# Radial eigenvalue problem

## Invariant coordinates

- Quantise coordinates invariant under identifications
- In Schwarzschild coordinates $(t, r, \phi) \to$ functions periodic in $\phi$
- Radial coordinate

$$B(r) = \frac{r^2 - r_+^2}{r_+^2 - r_-^2} = \frac{\beta^2 - \gamma^2}{z^2}$$

- Quantisation

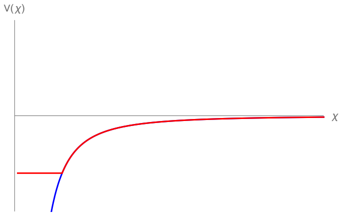
$$B = \frac{1}{4}\left(-\partial_\chi^2 - \frac{4l+3}{\chi}\,\partial_\chi - \frac{(2l+1)^2}{\chi^2} + \frac{1}{\chi^2}\,\partial_\eta^2\right)$$

- Commutes with the "mode operator" $H - \bar{H} = \partial_\eta$
- Solve for mode $n$ eigenfunctions $\psi(\chi, \eta) = f(\chi) e^{\frac{2\pi i n}{\alpha}\eta}$
- Canonical Schrödinger form = conformal quantum mechanics

$$H_B = -\partial_\chi^2 - \frac{n^2 r_-^{-2} + \frac{1}{4}}{\chi^2}$$

# Regularisation and spectrum

Coordinate $B$ bounded from below: $B \geq -\frac{r_+^2}{r_+^2 - r_-^2}$




Continuum of scattering states $\sqrt{\chi}\left(C_1\, H_{ic}^{(1)}(2\lambda\chi) + C_2\, H_{ic}^{(2)}(2\lambda\chi)\right)$

Infinite discrete set of bound states $\sqrt{\chi}\, K_{ic}(2\kappa\chi)$

Zero eigenvalue corresponds to the horizon $r = r_+$

# Semi-classical states

## States labelled by classical points [Perelomov. . . ]

- Generated by group transformations

$$\xi = \lambda^{-\tilde{p}_z} e^{-b\tilde{p}_\beta} e^{-c\tilde{p}_\gamma} \xi_0 \quad \rightarrow \quad |\xi\rangle = \lambda^{p_z} e^{bp_\beta} e^{cp_\gamma} |0\rangle$$

- Expectation values of coordinates = classical values

$$\xi = (\lambda^{-1} z_0, \beta_0 - bz_0, \gamma_0 - cz_0) = \Big(\langle\xi|z|\xi\rangle, \langle\xi|\beta|\xi\rangle, \langle\xi|\gamma|\xi\rangle\Big)$$

## Classical limit

- Relative uncertainty

$$\Delta_\xi X = \frac{\langle\xi|X^2|\xi\rangle - \langle\xi|X|\xi\rangle^2}{\langle\xi|X^2|\xi\rangle}$$

- In the limit $l \to \infty$, uncertainties vanish

$$\lim_{l\to\infty} (\Delta_\xi z) = 0\,, \quad \lim_{l\to\infty} (\Delta_\xi \beta) = 0\,, \quad \lim_{l\to\infty} (\Delta_\xi \gamma) = 0$$

# Summary of results

1. A noncommutative model of local $AdS_3$ geometry
2. Proposal to define BTZ through a discrete quotient
3. Spectrum of the regularised radial coordinate:
   exterior $\leftrightarrow$ scattering states, interior $\leftrightarrow$ bound states
4. Notion of the classical limit

# Future directions and open questions

1. Additional operators in terms of Kaluza-Klein modes

$$\mathcal{A}_{red} = \mathrm{End}(\mathcal{H}_{red}) \sim C\left(\mathbb{R} \times S^1 \times \mathbb{R} \times S^1\right)$$

2. Scalar field theory: start with quantum mechanical Laplacian

$$\Delta = -\chi^2\partial_\chi^2 + \left(4\chi^2 + \frac{3}{4}\right)$$

$\rightarrow$ solve for eigenfunctions of the field-theoretic Laplacian. . .

3. Geometry from semi-classical states
$\rightarrow$ extension to the BTZ quotient
$\rightarrow$ overlaps, matrix elements of geometric operators

4. Gravitational perturbations, entropy . . .

Thank you!

# Laplace-Beltrami operator

Laplacian on functions

$$\Delta f = -[p_z,[p_z,f]] - [p_\beta,[p_\beta,f]] + [p_\gamma,[p_\gamma,f]] + 2[p_z,f]$$

Laplacian on 1-forms

$$\Delta(f_z\theta^z + f_\beta\theta^\beta + f_\gamma\theta^\gamma) = (\Delta f_z - [p_\beta,f_\beta] + [p_\gamma,f_\gamma])\,\theta^z$$

$$+ (\Delta f_\beta - 2[p_z,f_\beta] + 3[p_\beta,f_z])\,\theta^\beta + (\Delta f_\gamma - 2[p_z,f_\gamma] + 3[p_\gamma,f_z])\,\theta^\gamma$$

Lowest-weight vector

$$|0\rangle = C^2(x\bar{x})^{-2l-1}e^{-x-\bar{x}}\,, \qquad C^{-2} = 2^{4l+1}\pi\,\Gamma(-2l)$$