Non-commutative deformations from homotopy Lie algebras

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Physics at quantum gravity scales?

- Appropriate space-time probe is not a point particle.
- Concept of symmetry needs to be generalized.
- Field content needs to be extended.
- Wilsonian separation of scales (probably) fails in QG regime.

Different aspects made precise in string theory, holography, matrix/tensor models....

Symmetries for quantum gravity?

In string theory space-time probe is spatially extended, eg strings and branes

$\rightsquigarrow$ can wrap compactified spaces $\rightsquigarrow$ dualities in string theory, eg T-duality

$\rightsquigarrow$ couple to higher gauge potential $\rightsquigarrow$ higher gauge theories

$\rightsquigarrow$ low-energy effective dynamics $\rightsquigarrow$ non-commutative/non-associative deformation

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$\rightsquigarrow$ low-energy effective dynamics $\rightsquigarrow$ non-commutative/non-associative deformation

$\rightsquigarrow$ natural framework for the description of generalized symmetries in the quantum gravity regime $\rightsquigarrow$ homotopy Lie algebras, e.g., A_∞, L_∞, Stasheff '63, Stasheff, Schlesinger '77

In this talk

GOAL

Argue that homotopy Lie algebras are useful for construction of consistent non-commutative deformations (in field theory).

PLAN

- L_∞-algebra
- Drinfel'd twist of L_∞-algebra
- From kinematics to dynamics

On L_∞-algebras

L_∞-algebra $\rightsquigarrow$ generalizations of differential graded Lie algebras with possibly infinitely-many graded antisymmetric brackets satisfying higher versions of the Jacobi identity.

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- Quantization
  - BV formalism $\sim$ L_∞-algebra Zwiebach '92
  - Deformation quantization: formality thm $\sim$ L_∞ quasi-isomorphism Kontsevich '97;
    Poisson sigma model quantization Cattaneo, Felder, '99

On L_∞-algebras

L_∞-algebra $\rightsquigarrow$ generalizations of differential graded Lie algebras.

- Quantization $\rightsquigarrow$ BV-BRST, deformation quantization
- Geometry
  - Graded geometry: L_∞-algebra (cyclic) $\equiv$ Q(P) manifolds AKSZ '95
  - Generalized geometry of Courant, double field theory and exceptional algebroids
    Roytenberg, Weinstein '98; Deser, Saemann '16, LJ, Grewcoe '20; Cederwall, Palmkvist '18

On L_∞-algebras

L_∞-algebra $\rightsquigarrow$ generalizations of differential graded Lie algebras.

- Quantization $\rightsquigarrow$ BV-BRST, deformation quantization
- Graded and generalized geometry
- NC/NA field theory and gravity
  - $\star$-product: bootstraping nc gauge theories using L_∞ Blumenhagen et al '18, cf. Patrizia's talk
  - Drinfel'd twist and braided L_∞ Dimitrijević Ćirić et al '21, Nguyen, Schenkel, Szabo '21
  - HS in unfolded formalism Vasiliev

L_∞ - coalgebra formulation

Lada, Stasheff '92, Lada, Markl '94

An L_∞ algebra is a $\mathbb{Z}$-graded vector space

$$X = \bigoplus_{d \in \mathbb{Z}} X_d$$

with multilinear graded symmetric maps $b_i : X^{\otimes i} \to X$ of degree 1 such that the coderivation $D = \sum_{i=0} b_i$ is nilpotent.

$D^2 = 0 \rightsquigarrow$ homotopy relations for the maps b_i, e.g.

$$b_1(b_0) = 0 \ ,$$
$$b_2(b_0, x) + b_1^2(x) = 0 \ ,$$
$$b_3(b_0, x_1, x_2) + b_2(b_1(x_1), x_2) + (-1)^{|x_1||x_2|} b_2(b_1(x_2), x_1) + b_1(b_2(x_1, x_2)) = 0 \ .$$

Note that for $b_0 = 0$ we talk about *flat* L_∞-algebra and b_1 is differential.

L_∞ - coalgebra formulation

Extend the maps b_i on the whole graded symmetric tensor algebra over field $K =: S^0X$

$$\mathsf{S}(X) := \bigoplus_{n=0}^{\infty} S^n X \ ,$$

with graded symmetric tensor product $\vee$.

The maps $b_i : S^j X \to S^{j-i+1} X$ act as a coderivation:

$$b_i(x_1 \vee \ldots \vee x_j) = \sum_{\sigma \in \mathrm{Sh}(i,j-i)} \epsilon(\sigma; x) b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \ldots \vee x_{\sigma(j)} \ , j \geq i \ ,$$

where $\epsilon(\sigma; x)$ is the Koszul sign, and $\mathrm{Sh}(p, m-p) \in S_m$ denotes those permutations ordered as $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(m)$.

Introducing the permutation map $\tau^\sigma : X^{\otimes i} \to X^{\otimes i}$

$$b_i \circ \mathrm{id}^{\vee j} = \sum_{\sigma \in \mathrm{Sh}(i,j-i)} (b_i \vee \mathrm{id}^{\vee (j-i)}) \circ \tau^\sigma \ , \qquad j \geq i \ .$$

L_∞ - coalgebra formulation

Degree 1 coderivation $D : \mathsf{S}(X) \to \mathsf{S}(X)$ satisfies the co-Leibniz property:

$$\Delta \circ D = (1 \otimes D + D \otimes 1) \circ \Delta \ ,$$

with the coproduct map map $\Delta : \mathsf{S}(X) \to \mathsf{S}(X) \otimes \mathsf{S}(X)$

$$\Delta \circ \mathrm{id}^{\vee m} = \sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh}(p, m-p)} (\mathrm{id}^{\vee p} \otimes \mathrm{id}^{\vee (m-p)}) \circ \tau^{\sigma} \ , \ p, m \geq 0 \ ,$$

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$\rightsquigarrow$ L_∞-algebra as a coalgebra with coderivation and counit $\varepsilon : \mathsf{S}(X) \to K$, where $\varepsilon(1) = 1$ and $\varepsilon(x) = 0, \ x \in X$.

$L_\infty \rightsquigarrow$ Hopf algebra

Graded symmetric tensor algebra $\mathrm{S}(X)$ has an algebra structure given by the graded symmetric tensor product $\vee$ and a unit map $\eta : K \to \mathrm{S}(X)$, where $\eta(1) = 1$.

The algebra and coalgebra structure on $\mathrm{S}(X)$ make up a bialgebra, that admits a graded antipode map S

$$S(x_1 \vee \cdots \vee x_m) = (-1)^m (-1)^{\sum_{i=2}^{m} \sum_{j=1}^{i-1} |x_i||x_j|} x_m \vee \cdots \vee x_1 \ .$$

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$\rightsquigarrow$ the homotopy Lie algebra defined by the coalgebra structure on the graded symmetric tensor space $\mathsf{S}(X)$ extends to a cocommutative and coassociative Hopf algebra with compatible coderivation. Grewcoe, LJ, Kodzoman, Manolakos, '22

Non-commutative deformation

$\rightsquigarrow$ introduce non-(co)commutative deformation using Drinfel'd twist. cf. Andrzej's talk

Using invertible twist element $\mathcal{F} =: f^{\alpha} \otimes f_{\alpha} \in H \otimes H$

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} \ ,$$
$$(\epsilon \otimes id)\mathcal{F} = 1 \otimes 1 = (id \otimes \epsilon)\mathcal{F} \ ,$$

we obtain $(H^{\mathcal{F}}, \vee, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \epsilon)$, where $H^{\mathcal{F}}$ is the same as H as vector spaces and:

$$\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \ h \in H \ ,$$

and $S^{\mathcal{F}} = S$ for Abelian twist.

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$\rightsquigarrow$ twisted L_∞ or $(L_\infty^\mathcal{F}, \vee, \Delta^\mathcal{F}, S, \epsilon)$

Drinfel'd twist

In the spirit of deformation quantisation, while twisting Hopf algebra we simultaneously twist its modules. Taking Hopf algebra L_∞ as its own module
$\rightsquigarrow (L_\infty^\star, \vee_\star, \Delta_\star, S_\star, \epsilon)$:

$$\begin{aligned} x_1 \vee_\star x_2 &= \bar{f}^\alpha(x_1) \vee \bar{f}_\alpha(x_2) \ , \\ \Delta_\star(x) &= x \otimes 1 + \bar{R}^\alpha \otimes \bar{R}_\alpha(x) \ , \\ S_\star(x) &= -\bar{R}^\alpha(x)\bar{R}_\alpha \ . \end{aligned}$$

The $\mathcal{R}$-matrix $\mathcal{R} \in \mathsf{S}(X) \otimes \mathsf{S}(X)$ is an invertible matrix induced by the twist

$$\mathcal{R} = f_\alpha \bar{f}^\beta \otimes f^\alpha \bar{f}_\beta =: R^\alpha \otimes R_\alpha \ , \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha \ ,$$

The inverse $\mathcal{R}$-matrix controls noncommutativity of the $\vee_\star$-product and provides the representation of permutation group, e.g.,

$$\tau^\sigma_{\mathcal{R}}(x_1 \vee_\star x_2) = (-1)^{|x_1||x_2|} \bar{R}^\alpha(x_2) \vee_\star \bar{R}_\alpha(x_1) \ ,$$

Braided L_∞-algebra

Extend the coproduct to whole tensor algebra:

$$\Delta_\star \circ \mathrm{id}^{\vee_\star m} = \sum_{\sigma\in \mathrm{Sh}(p,m-p)} (\mathrm{id}^{\vee_\star p} \otimes \mathrm{id}^{\vee_\star (m-p)}) \circ \tau_{\mathcal{R}}^{\sigma} \ , \ p,m \geq 0 \ .$$

The compatible coderivation $D_\star = \sum_{i=0}^{\infty} b_i^\star$ is defined in terms of braided graded symmetric maps $b_i^\star$

$$b_i^\star \circ \mathrm{id}^{\vee_\star j} = \sum_{\sigma\in \mathrm{Sh}(i,j-i)} (b_i^\star \vee_\star \mathrm{id}^{\vee_\star (j-i)}) \circ \tau_{\mathcal{R}}^{\sigma} \ , \ j \geq i \ ,$$

$$b_i^\star(x_1,\dots,x_m,x_{m+1},\dots,x_i) = (-1)^{|x_m||x_{m+1}|} b_i^\star(x_1,\dots,\bar{R}^\alpha(x_{m+1}),\bar{R}_\alpha(x_m),\dots,x_i) \ ,$$

and the condition $D_\star^2 = 0$ reproduces the deformed homotopy relations.

$\rightsquigarrow$ braided L_∞-algebra obtained in Dimitrijević Ćirić et al '21.

$L_\infty^\star$ vs. $L_\infty^{\mathcal{F}}$

As Hopf algebras $L_\infty^\star$ and $L_\infty^{\mathcal{F}}$ are isomorphic Aschieri et al '05, Schenkel '12
$\exists$ map $\varphi : L_\infty^\star \to L_\infty^{\mathcal{F}}$ such that

$$\varphi(x_1 \vee_\star x_2) = \varphi(x_1) \vee \varphi(x_2) \ ,$$
$$\Delta_\star = (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta^{\mathcal{F}} \circ \varphi \ ,$$
$$S_\star = \varphi^{-1} \circ S^{\mathcal{F}} \circ \varphi \ .$$

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On the other hand, we take $L_\infty^\star$-algebra as a module of $L_\infty^\mathcal{F}$ with an L_∞-action on an L_∞-algebra given by an L_∞-morphism Mehta, Zambon '12. Thus we obtain

$$D_\star = \varphi^{-1} D_\mathcal{F} \varphi \ .$$

From kinematics to dynamics

Hohm, Zwiebach ‘17; Jurčo et al. ‘18, ‘20; Giotopoulos, Szabo ’21

What do we have so far?

- $L_\infty \rightsquigarrow$ symmetry (gauge) algebra
- MC equations $\rightsquigarrow$ eoms

$$\sum_i \frac{1}{i!} b_i(x, \ldots, x) = 0 \ , x \in X_0$$

$$\delta_c x = \sum_i \frac{1}{i!} b_{i+1}(x, \ldots, x, c) \ , c \in X_{-1}$$

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To define a classical action and/or solution of classical master equation we need:

- Tensor product (of complexes)
- Inner product $\rightsquigarrow$ cyclic L_∞

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$$S_{\mathrm{MC}}[x] \equiv \sum_{i \geqslant 0} \frac{1}{(i+1)!} \langle x, b_i(x, \ldots, x) \rangle .$$

Outlook: From classical to quantum

- $Q = D^* \rightsquigarrow$ BRST operator

Evaluate b_i on basis of X $\rightsquigarrow$ structure constants of L_∞-algebra:

$$b_i(\tau_{\alpha_1}, ..., \tau_{\alpha_i}) = C^{\beta}_{\alpha_1 \dots \alpha_i} \tau_\beta$$

Use to define cohomological vector Q of degree 1

$$Q = \sum_{i=0}^{\infty} \frac{1}{i!} C^{\beta}_{\alpha_1 \dots \alpha_i} z^{\alpha_1} \cdots z^{\alpha_i} \frac{\partial}{\partial z^\beta}$$

with z^{α_i} basis of $X^\star$.

In Batalin-Vilkovisky formalism Q becomes BRST operator and z^{α_i} physical fields.

Outlook: From classical to quantum

- $Q = D^* \rightsquigarrow$ BRST operator
- Quantum homotopy algebra $\rightsquigarrow$ loop amplitudes Jurco, Macrelli, Saemann, Wolf '20
- Braided BV for theories with nc braided symmetries Dimitrijević Ćirić et al. '22
- Effective field theories from homotopy transfer e.g. Arvanitakis, Hohm, Hull, Lekeu '21

THANK YOU!