FACULTY OF PHYSICS
UNIVERSITY OF WARSAW

A generalization of Einstein's quadruple formula for radiated energy in de Sitter spacetime

Denis Dobkowski-Ryłko & Jerzy Lewandowski

arXiv:2205.09050

59. Winter School of Theoretical Physics and third COST Action CA18108 Training School *Gravity - Classical, Quantum and Phenomenology*

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- If one insists that the generalized $\mathscr{I}^+$ for de Sitter spacetime is a null surface, then a good candidate is the cosmological horizon.

Time changing matter source emitting gravitational radiation in de Sitter spacetime

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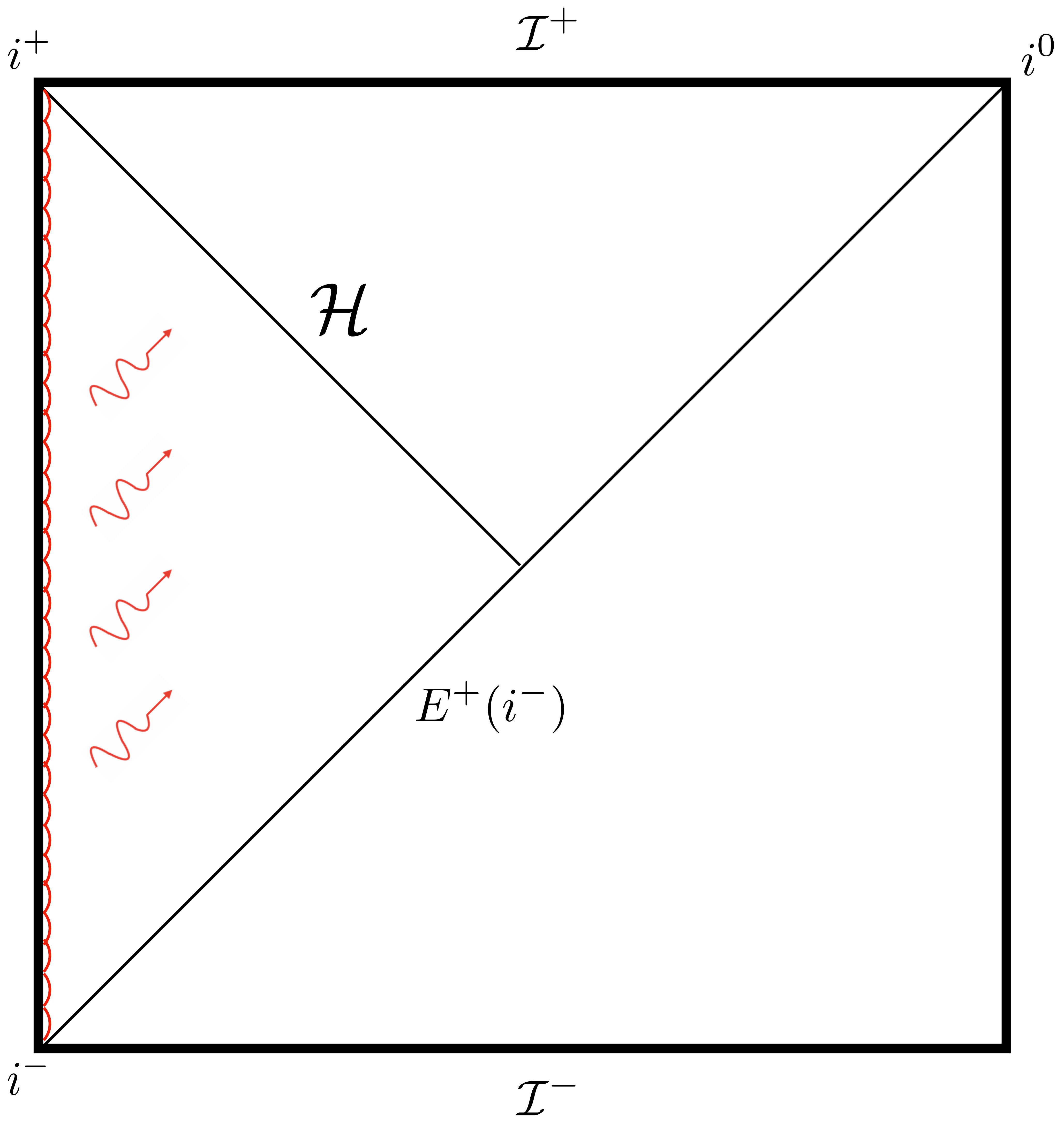

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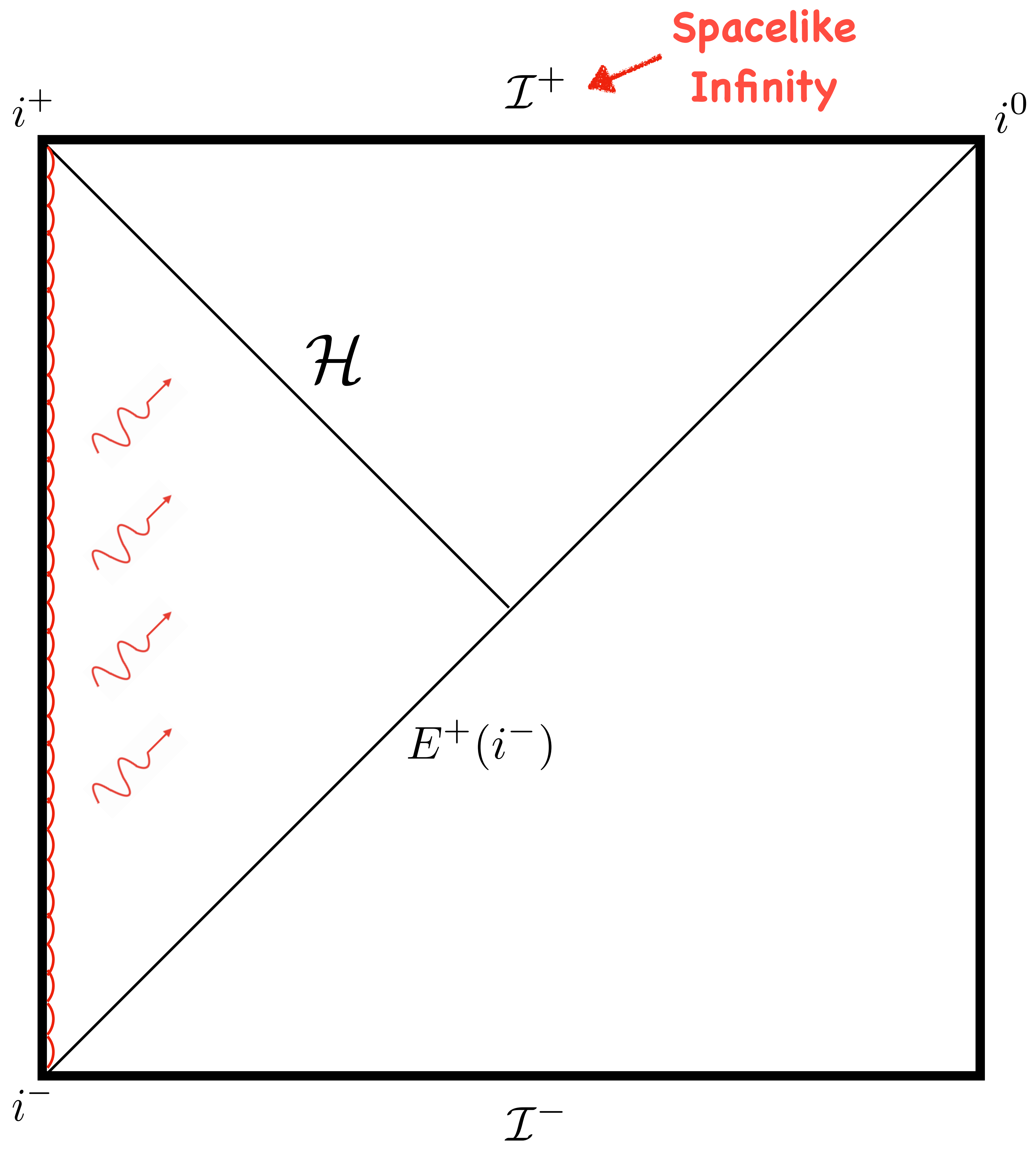

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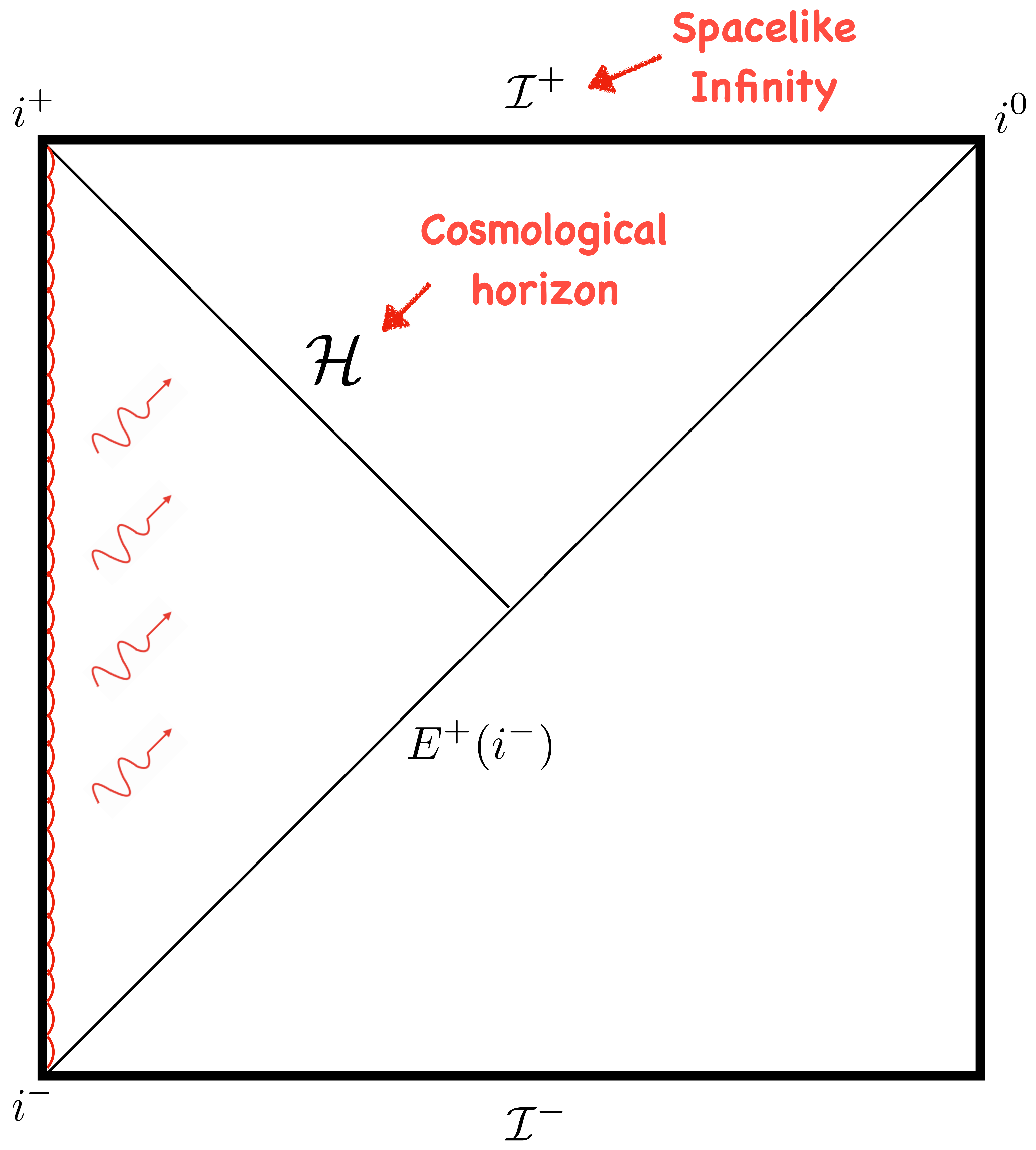

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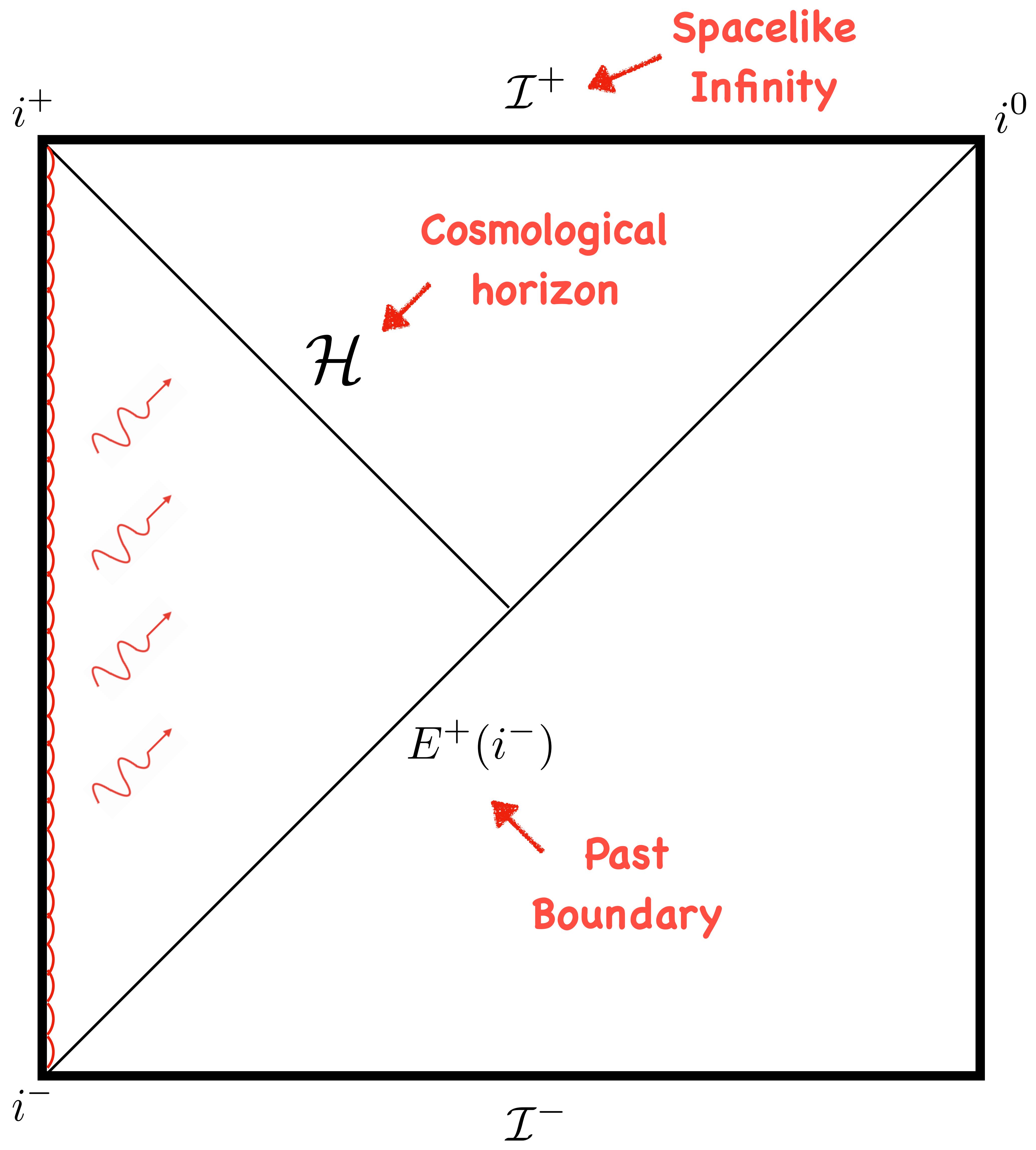

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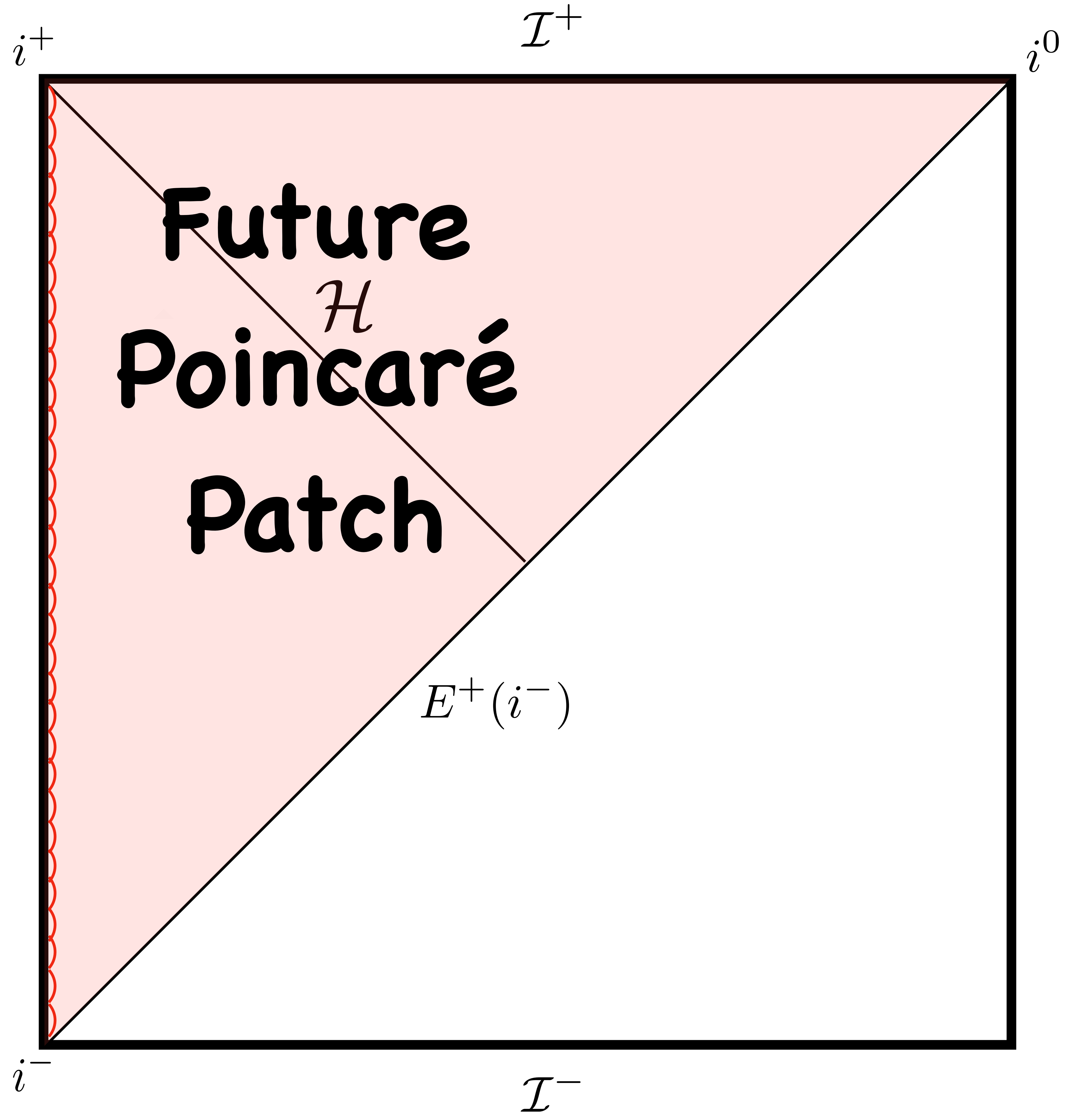

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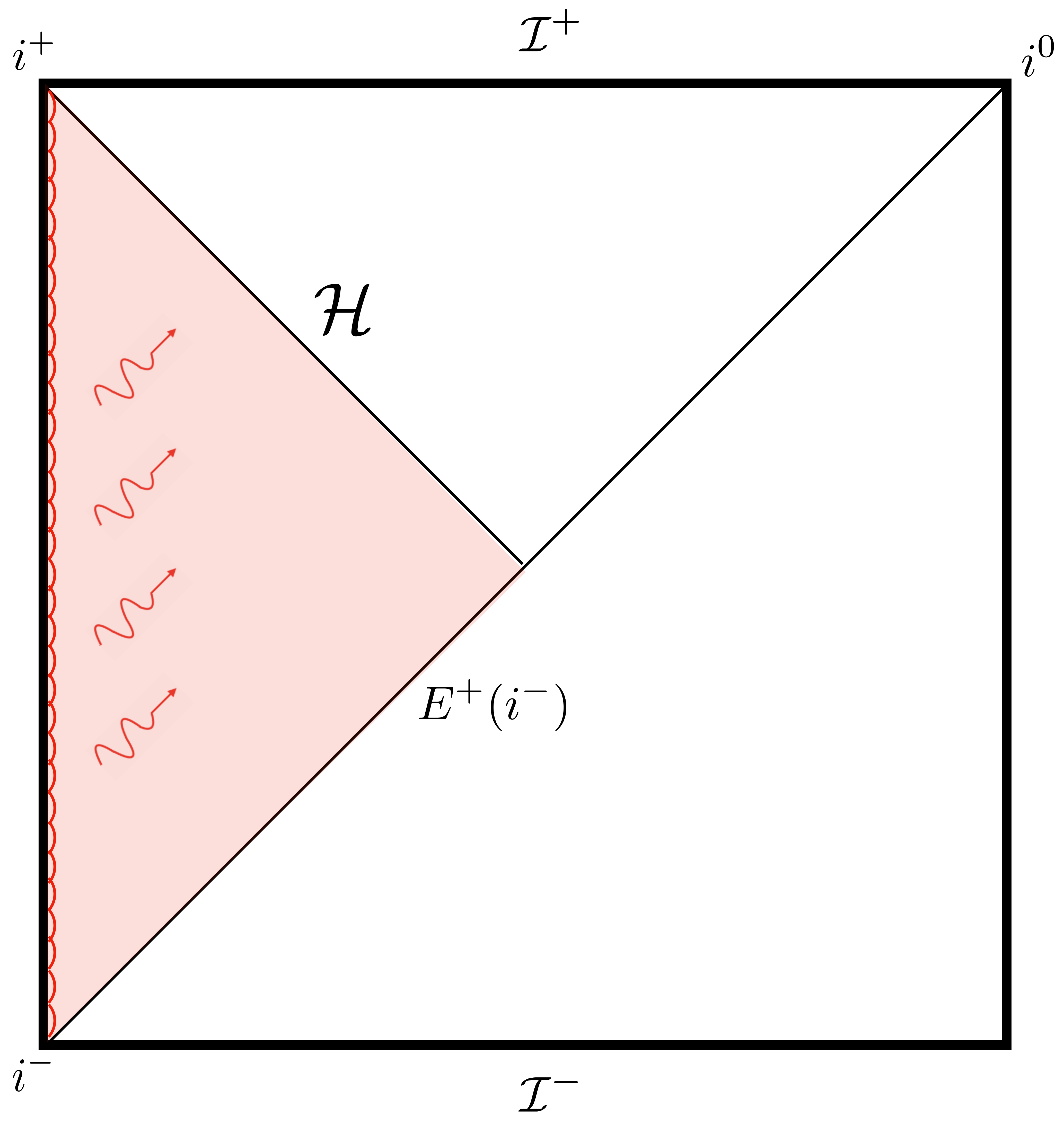

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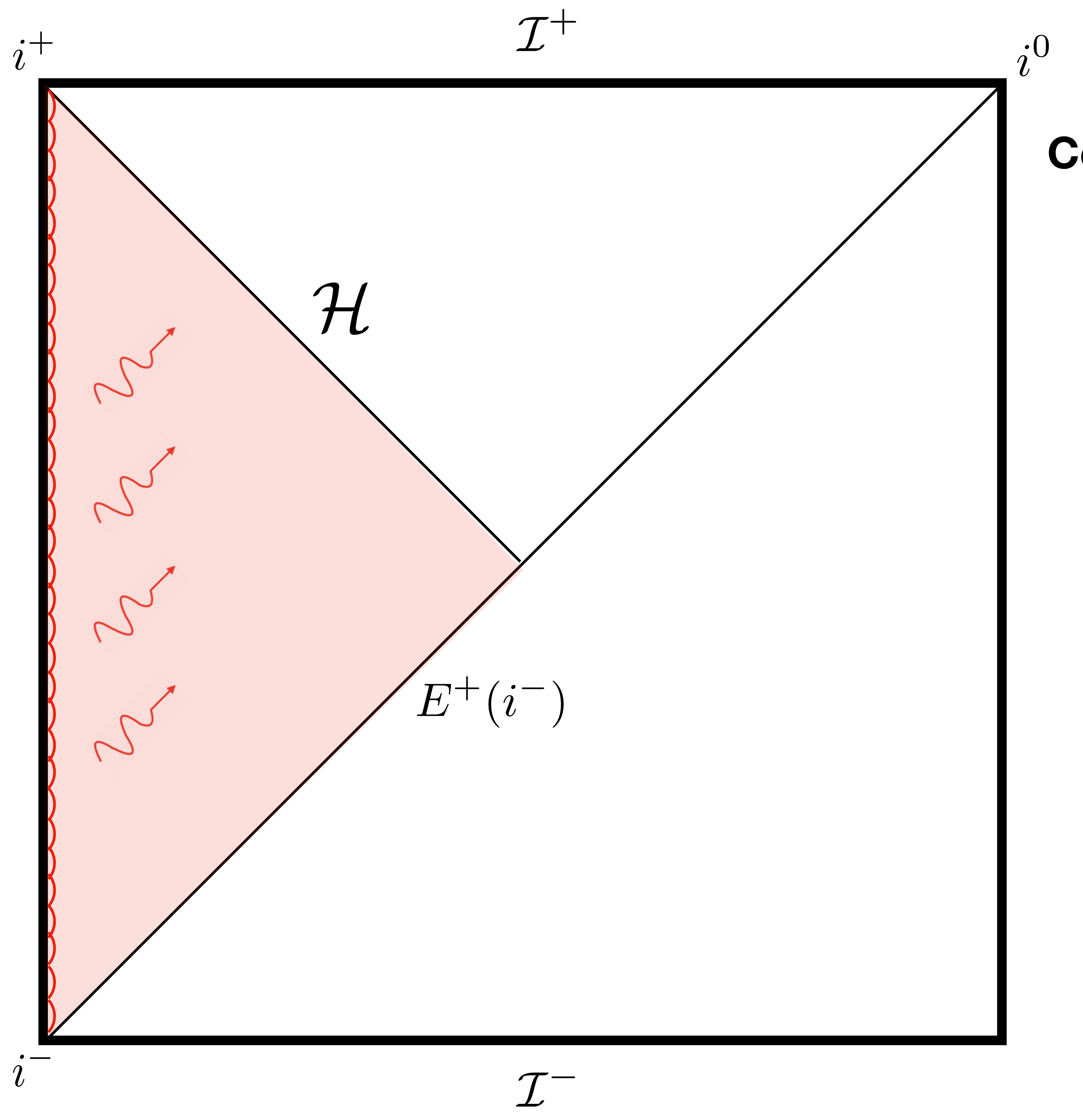

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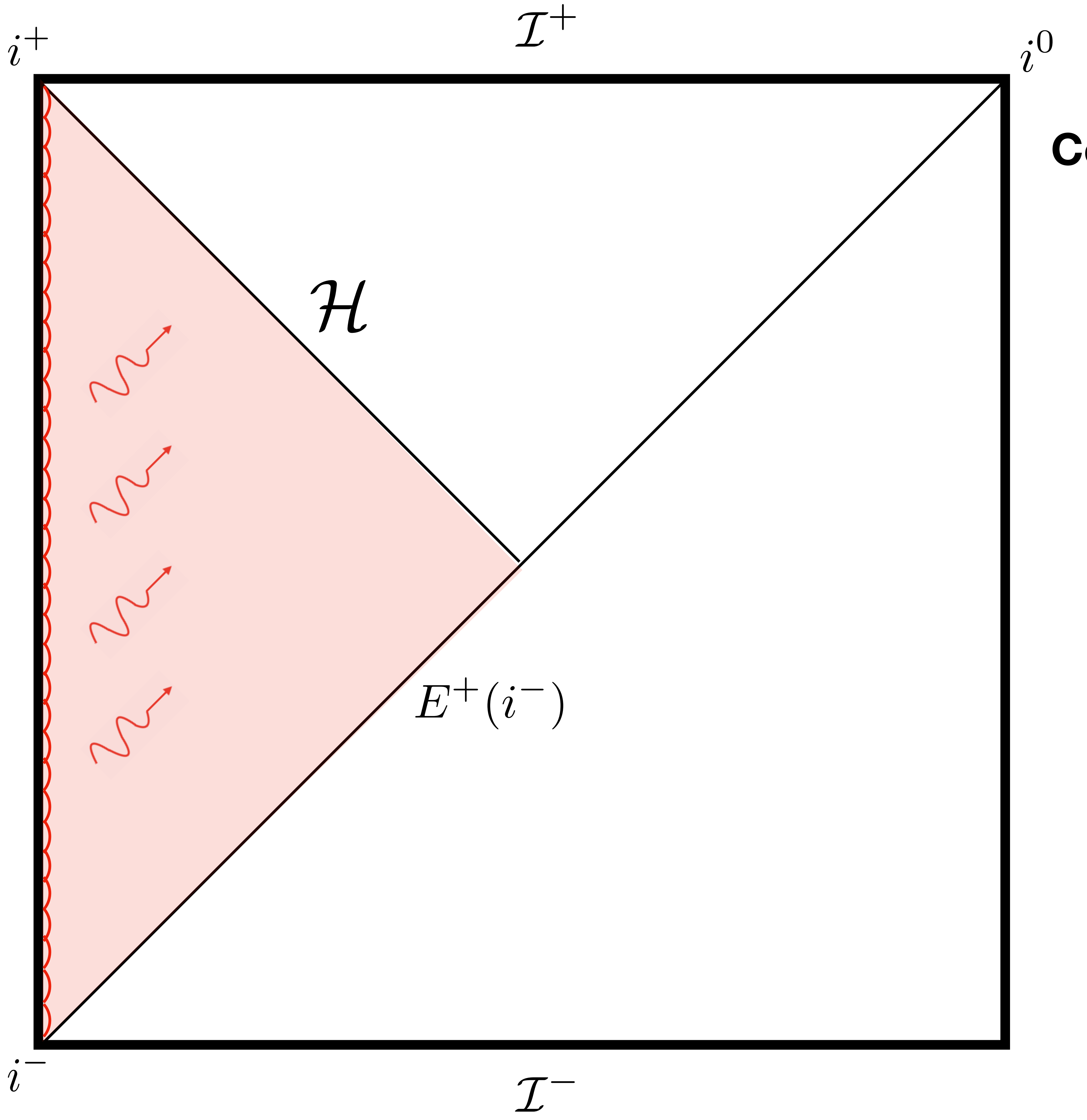

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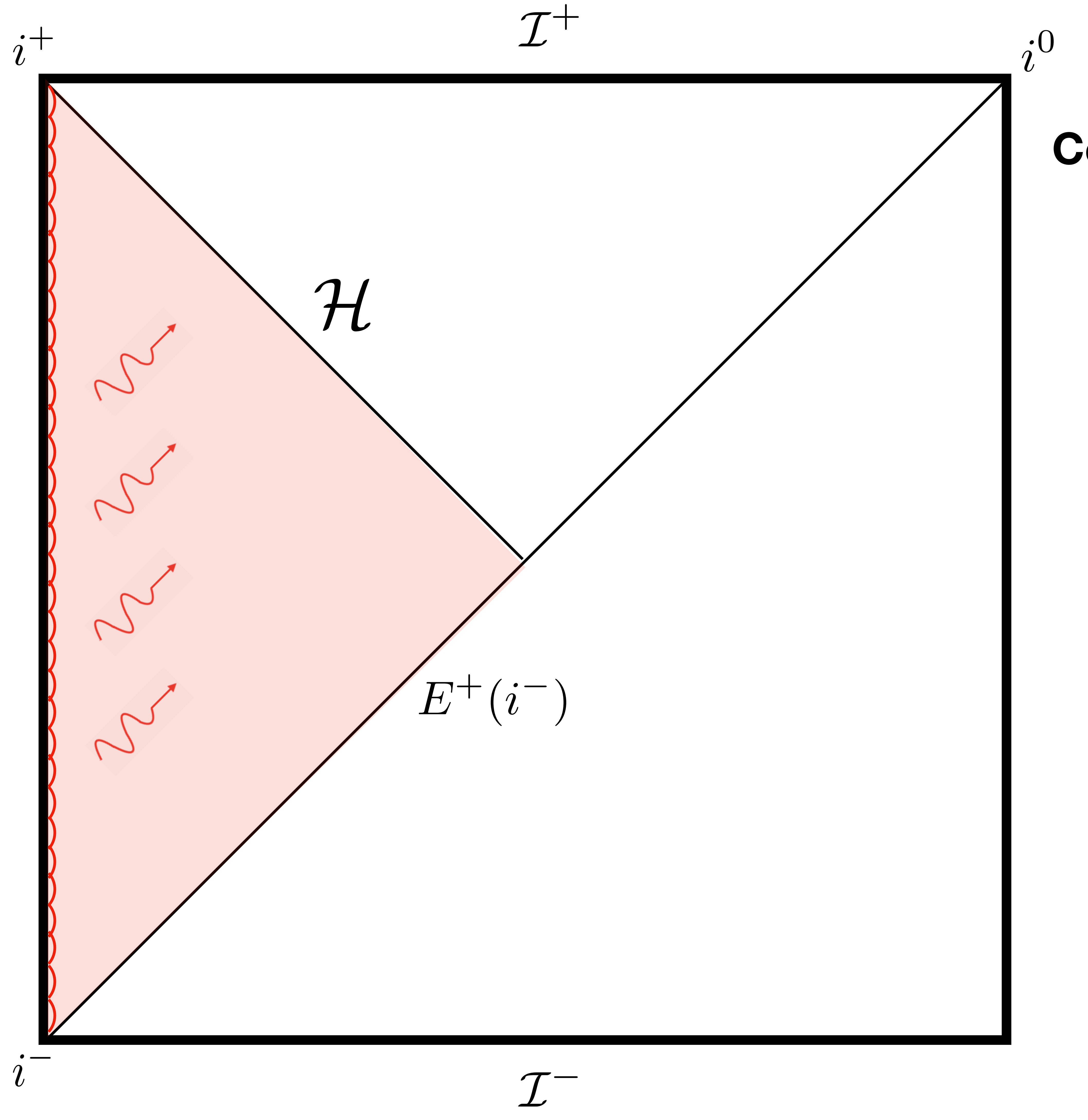

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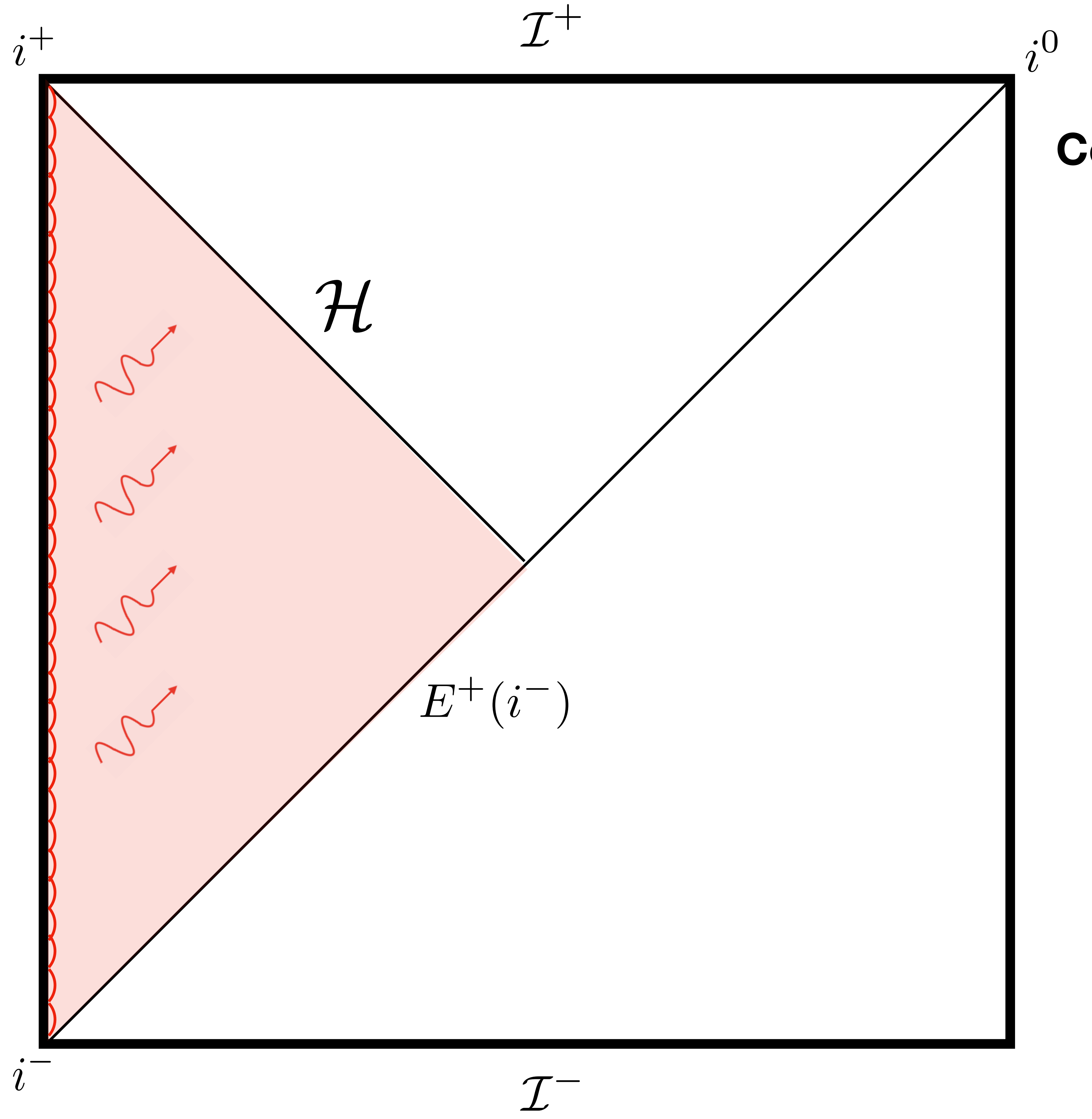

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smallness parameter

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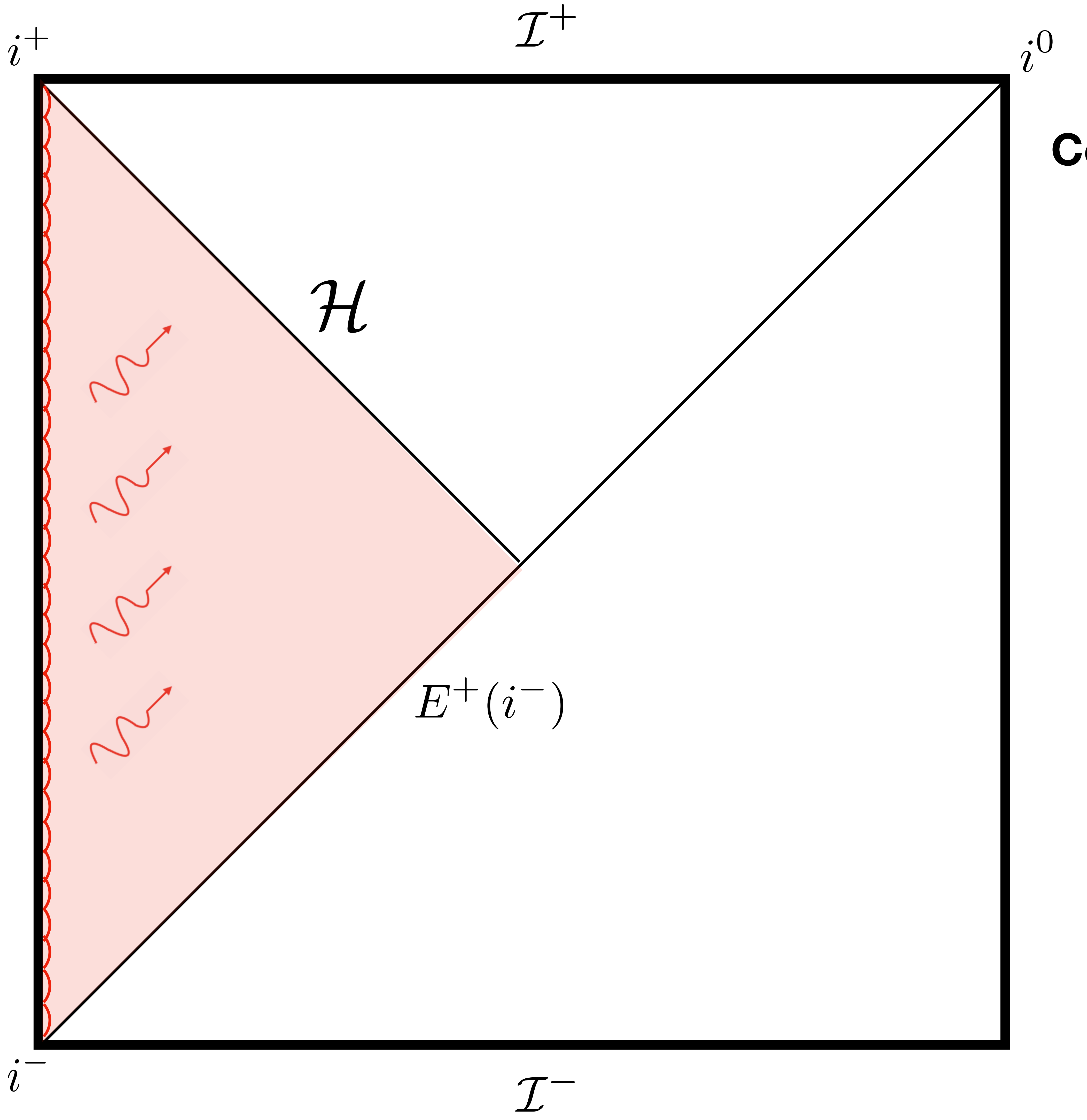

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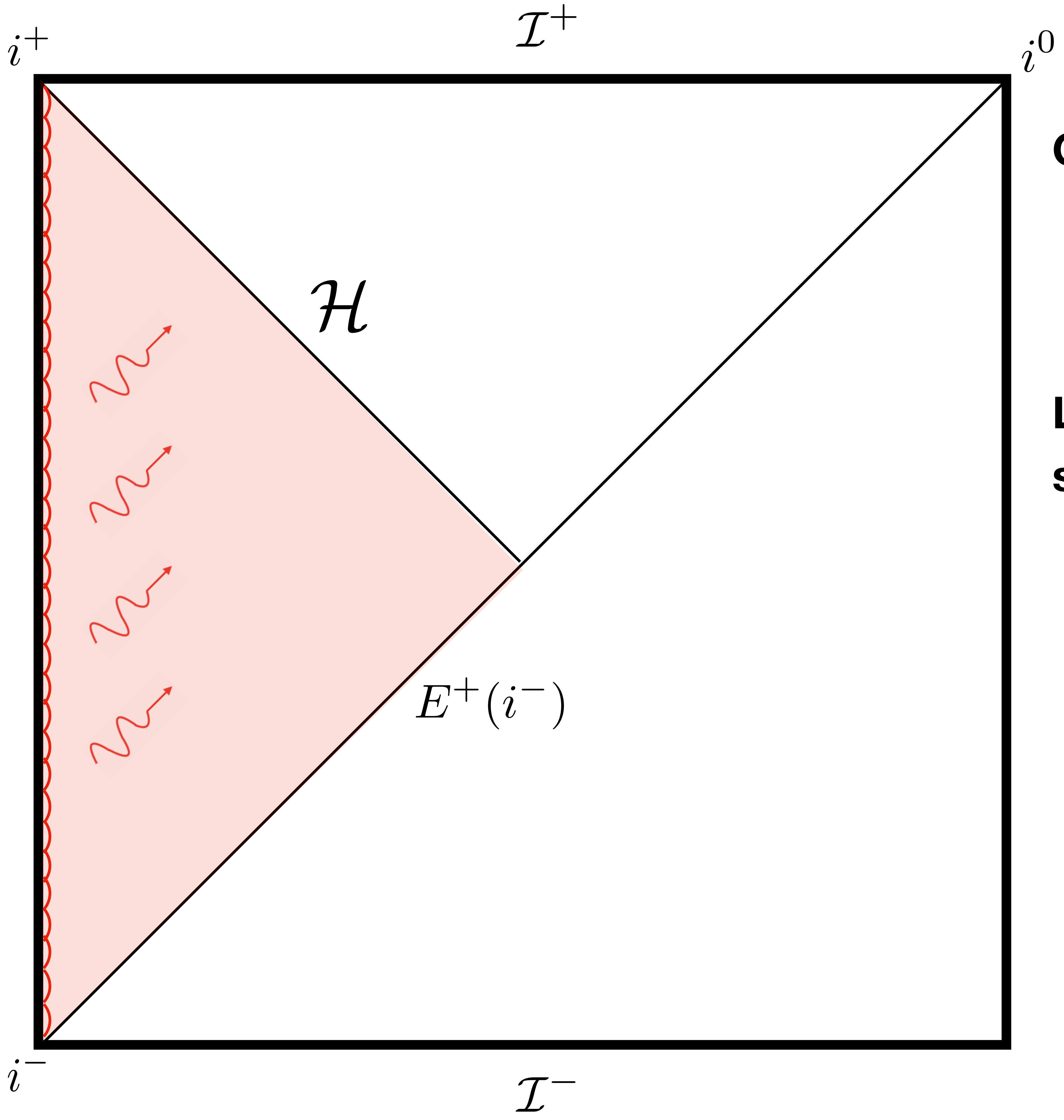

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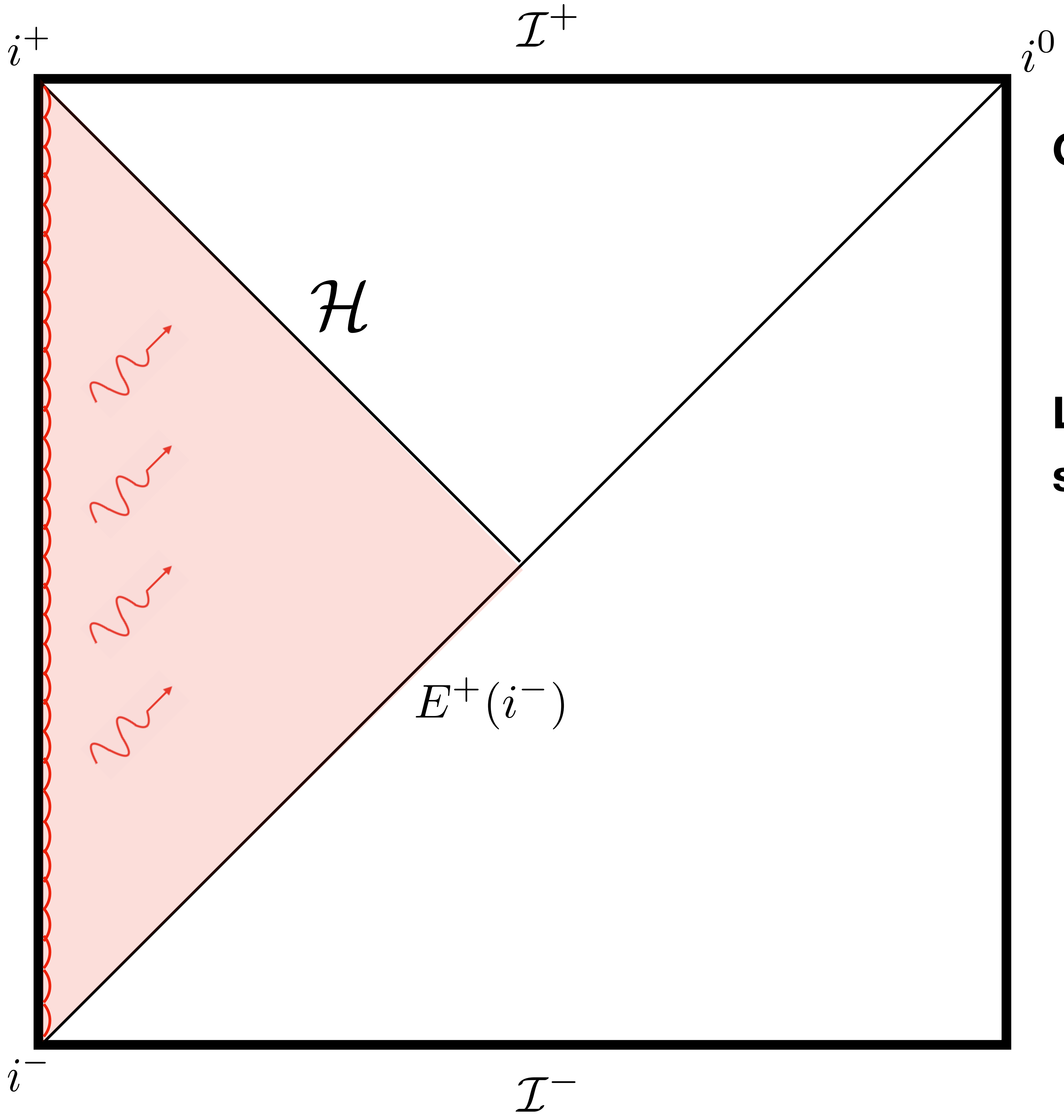

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$$\bar{\Box}\,\bar{\gamma}_{\alpha\beta} - 2\,\bar{\nabla}_{(\alpha}\,\bar{\nabla}^{\mu}\bar{\gamma}_{\beta)\mu} + \bar{g}_{\alpha\beta}\,\bar{\nabla}^{\mu}\,\bar{\nabla}^{\nu}\bar{\gamma}_{\mu\nu} - \frac{2}{3}\Lambda(\bar{\gamma}_{\alpha\beta} - \bar{g}_{\alpha\beta}\bar{\gamma}) = -\,16\pi T_{\alpha\beta}$$

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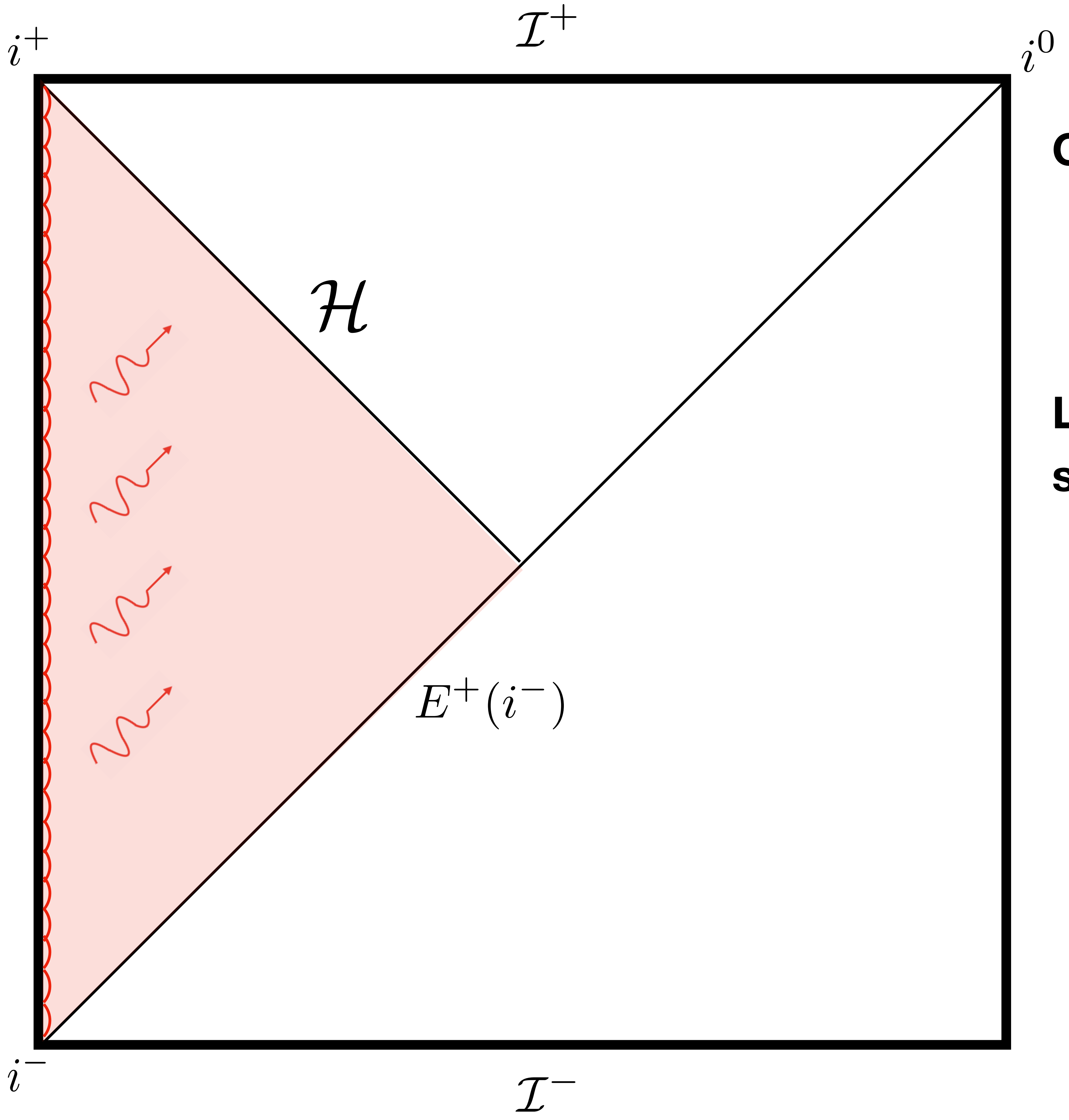

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Trace-reversed metric perturbation:

$\bar{\gamma}_{\alpha\beta} := \gamma_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}\,\gamma$

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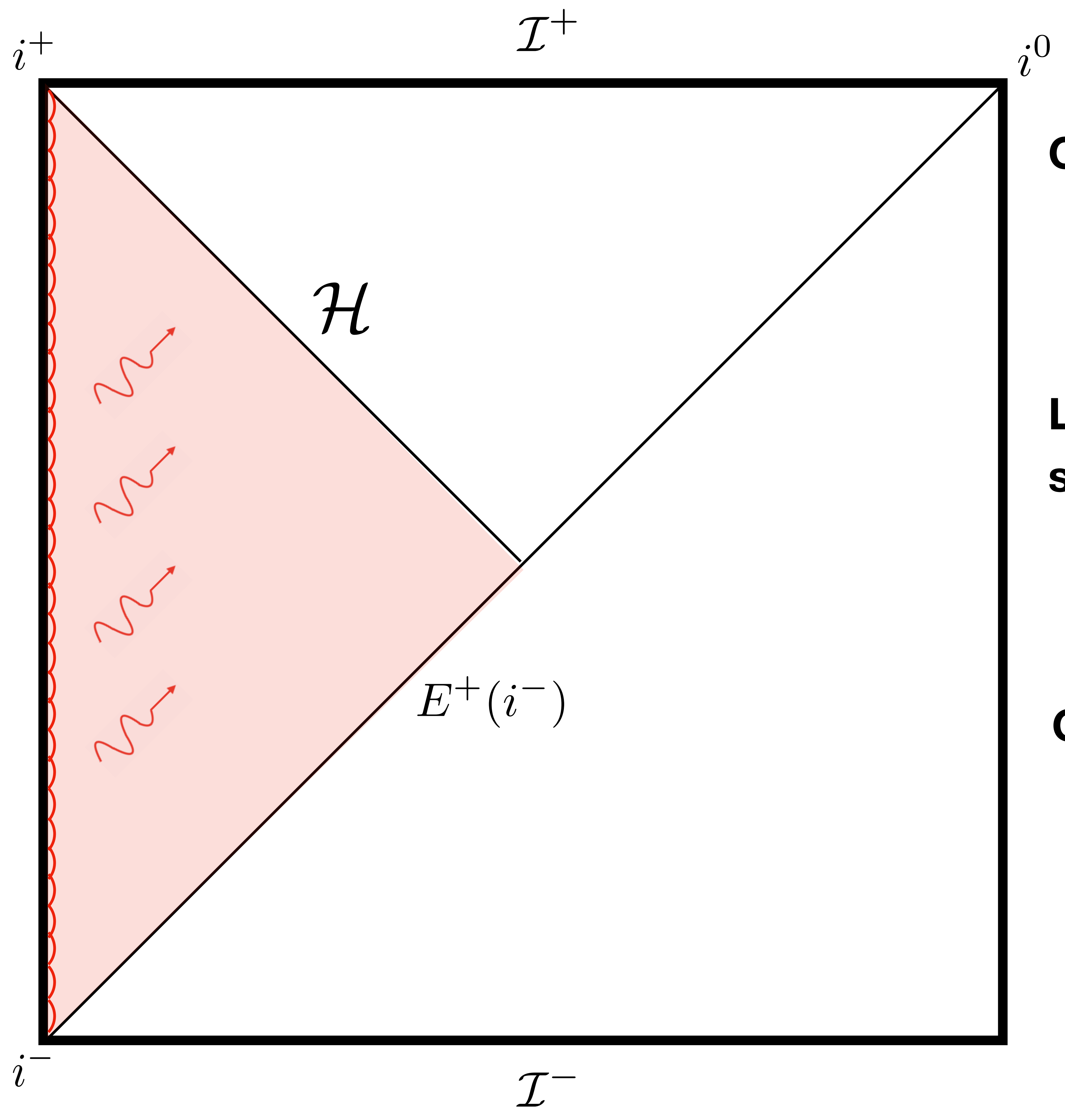

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Gauge condition: $\bar{\nabla}^\alpha\bar{\gamma}_{\alpha\beta} = 2Hn^\alpha\bar{\gamma}_{\alpha\beta}$ **, where** $n^\alpha\partial_\alpha = -H\eta\partial_\eta$

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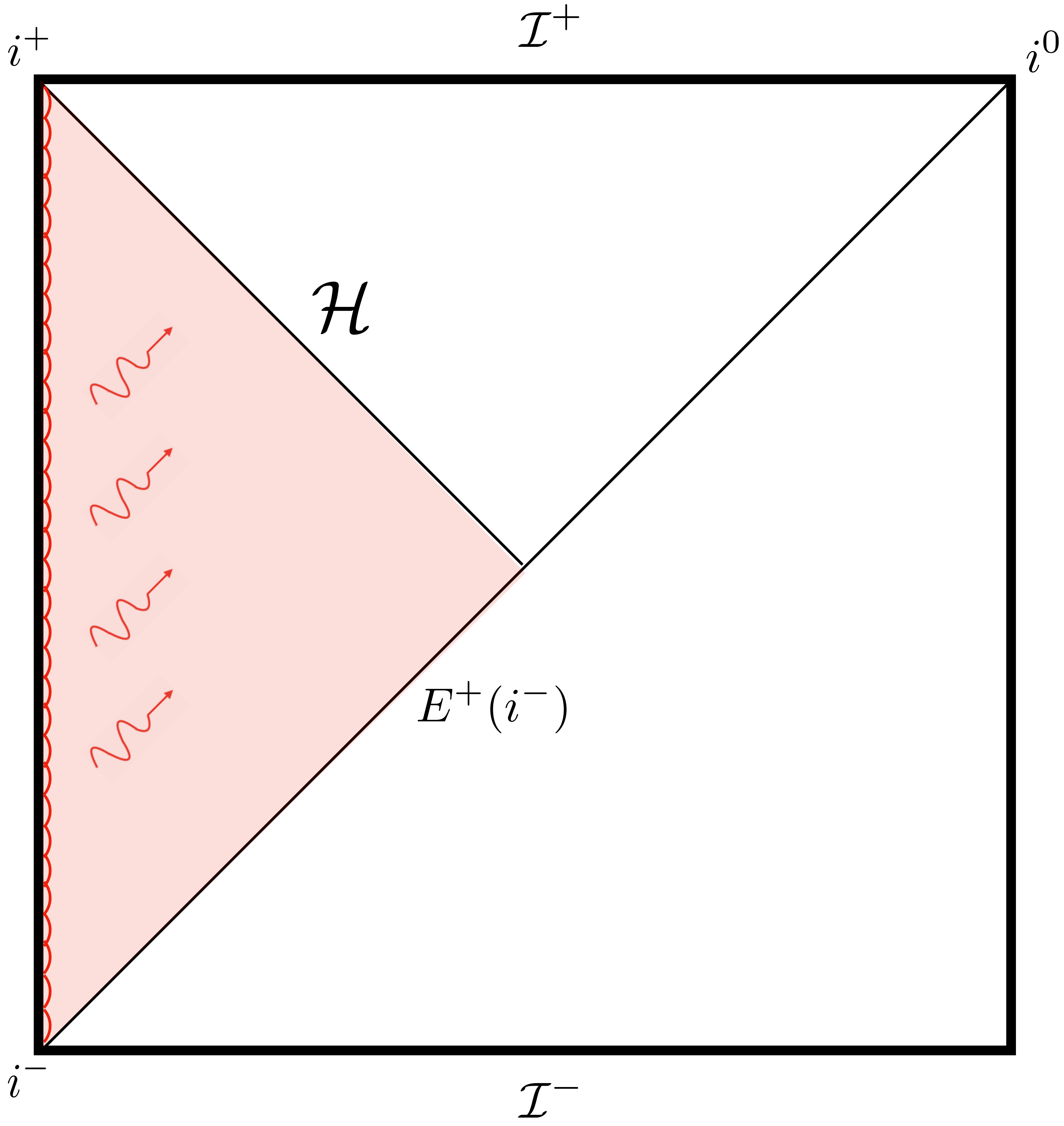

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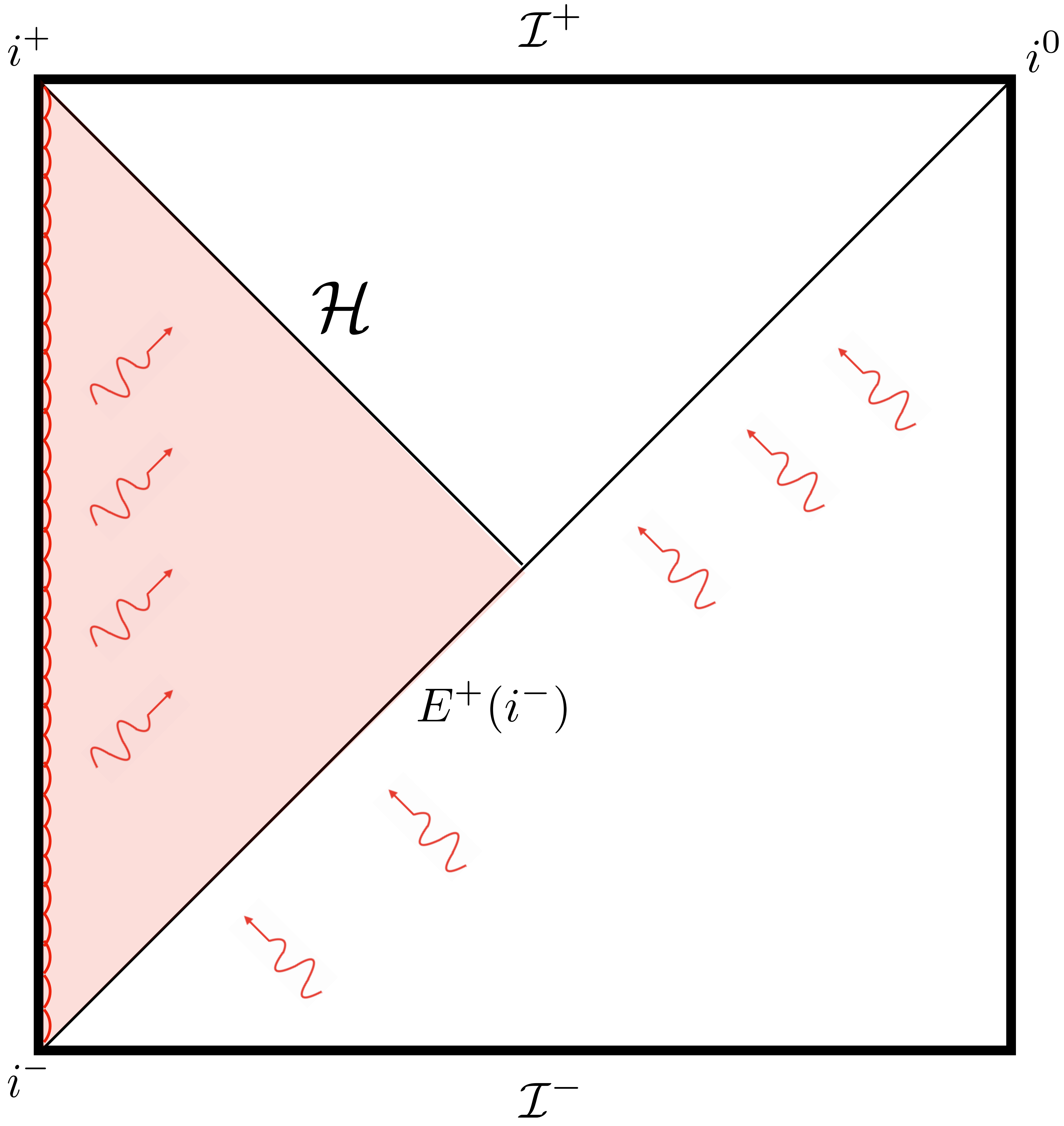

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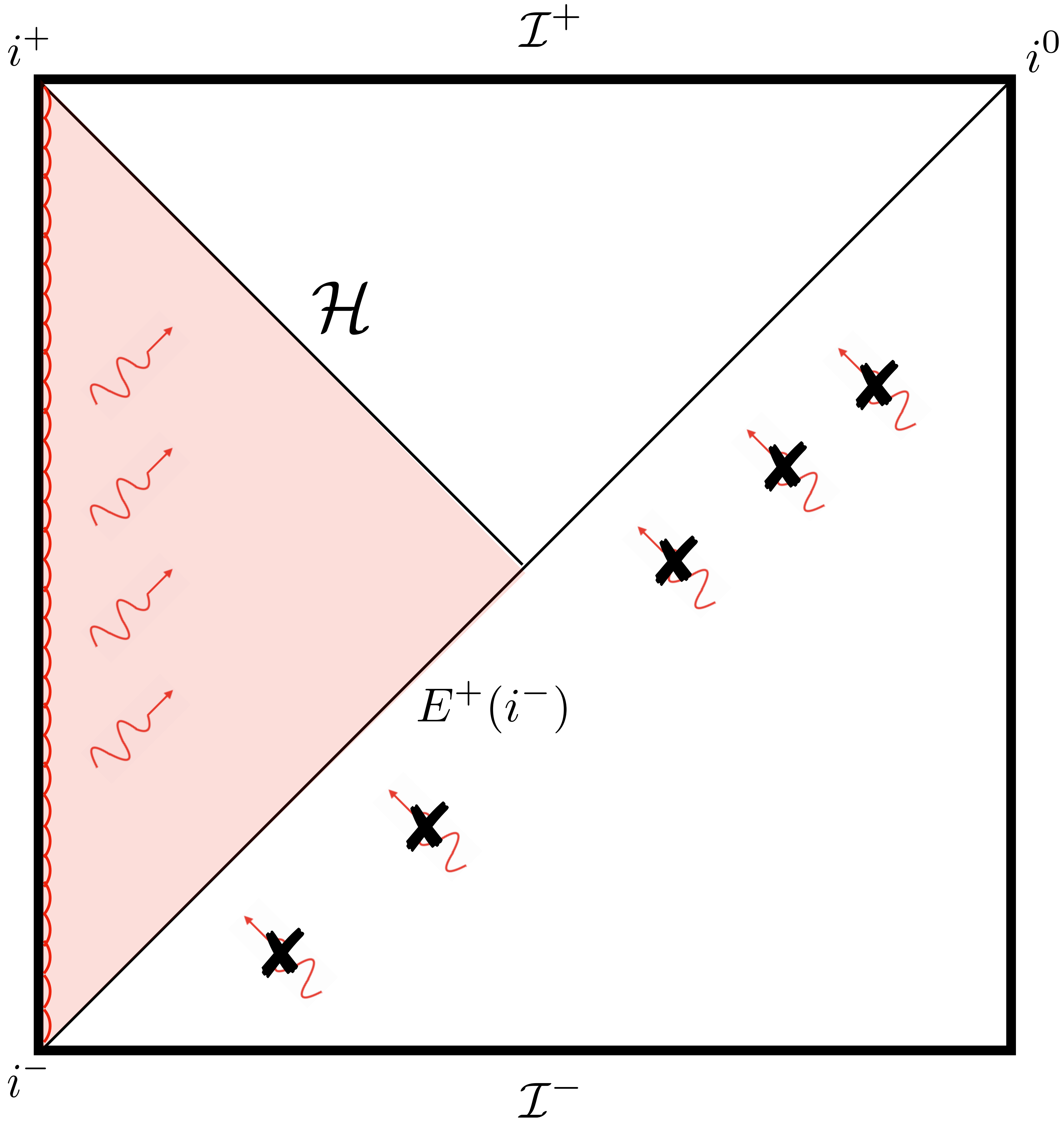

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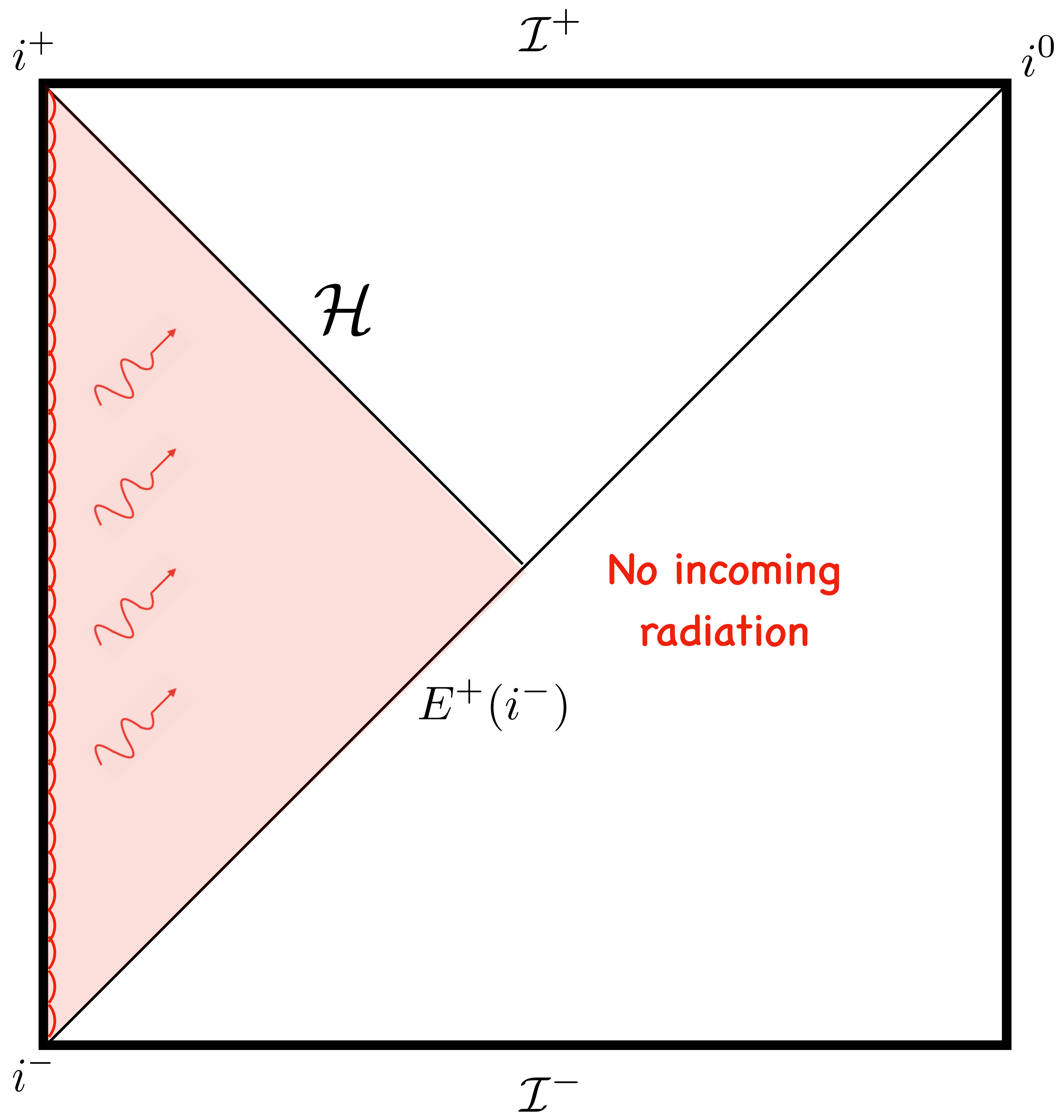

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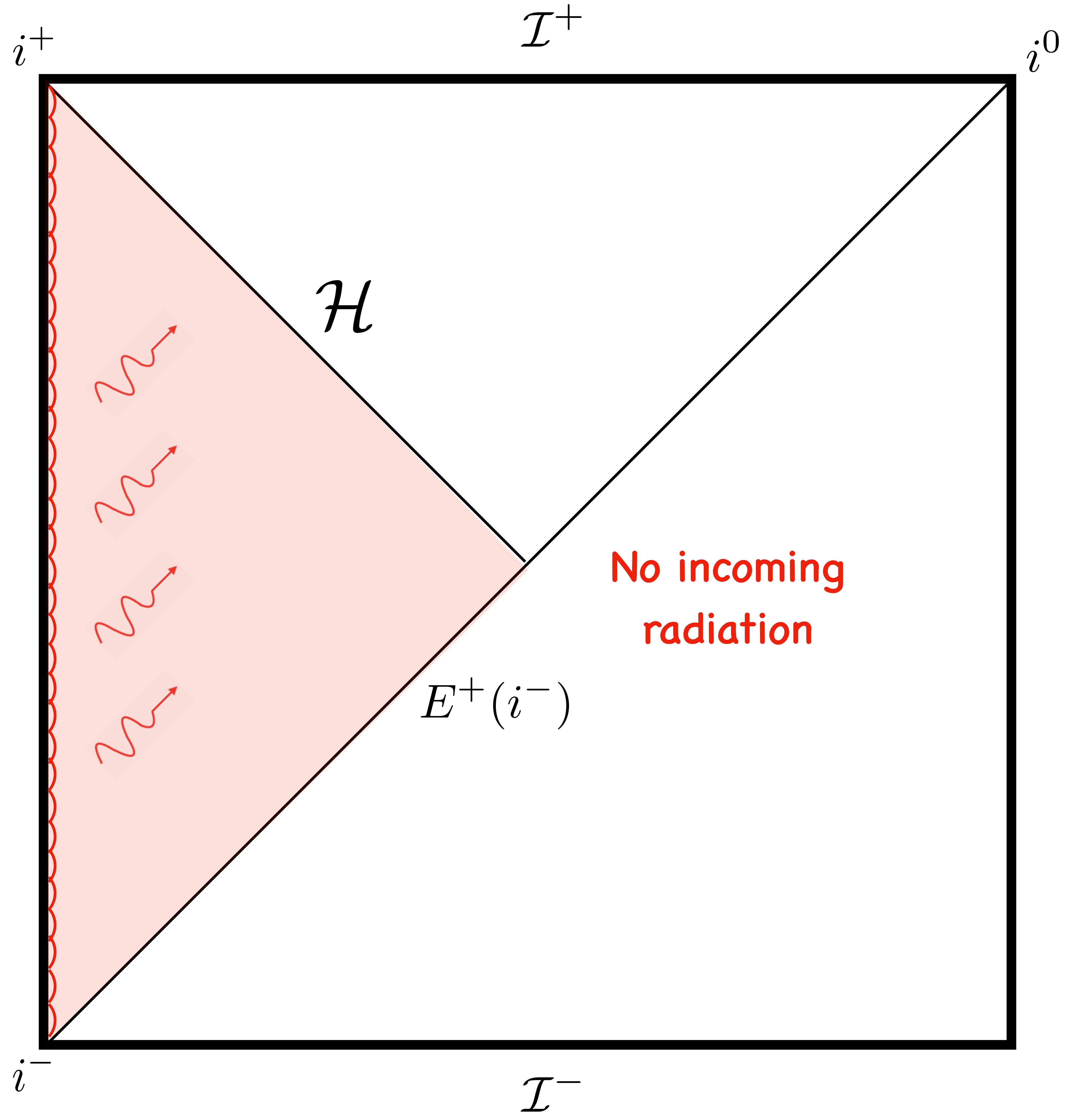

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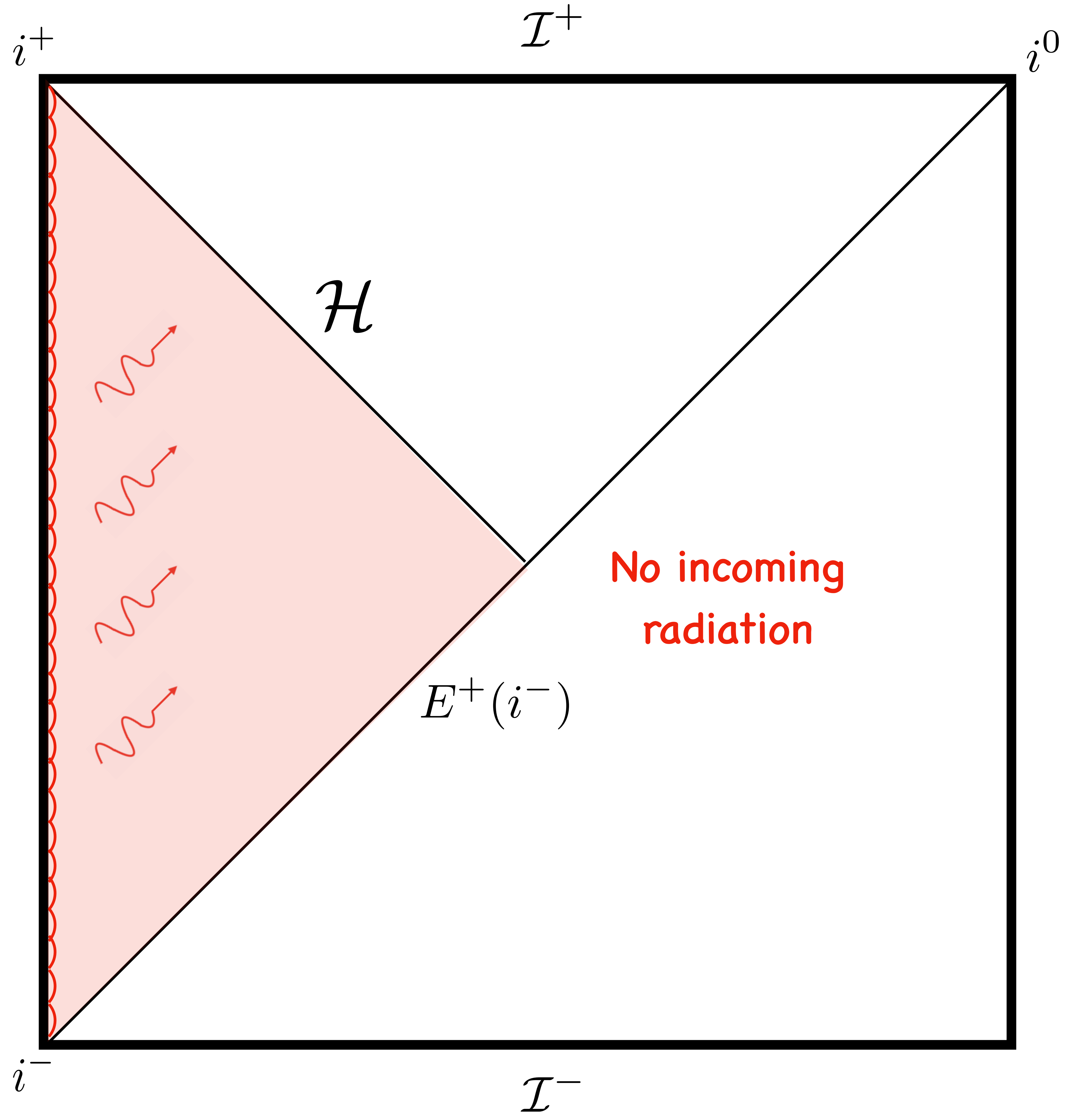

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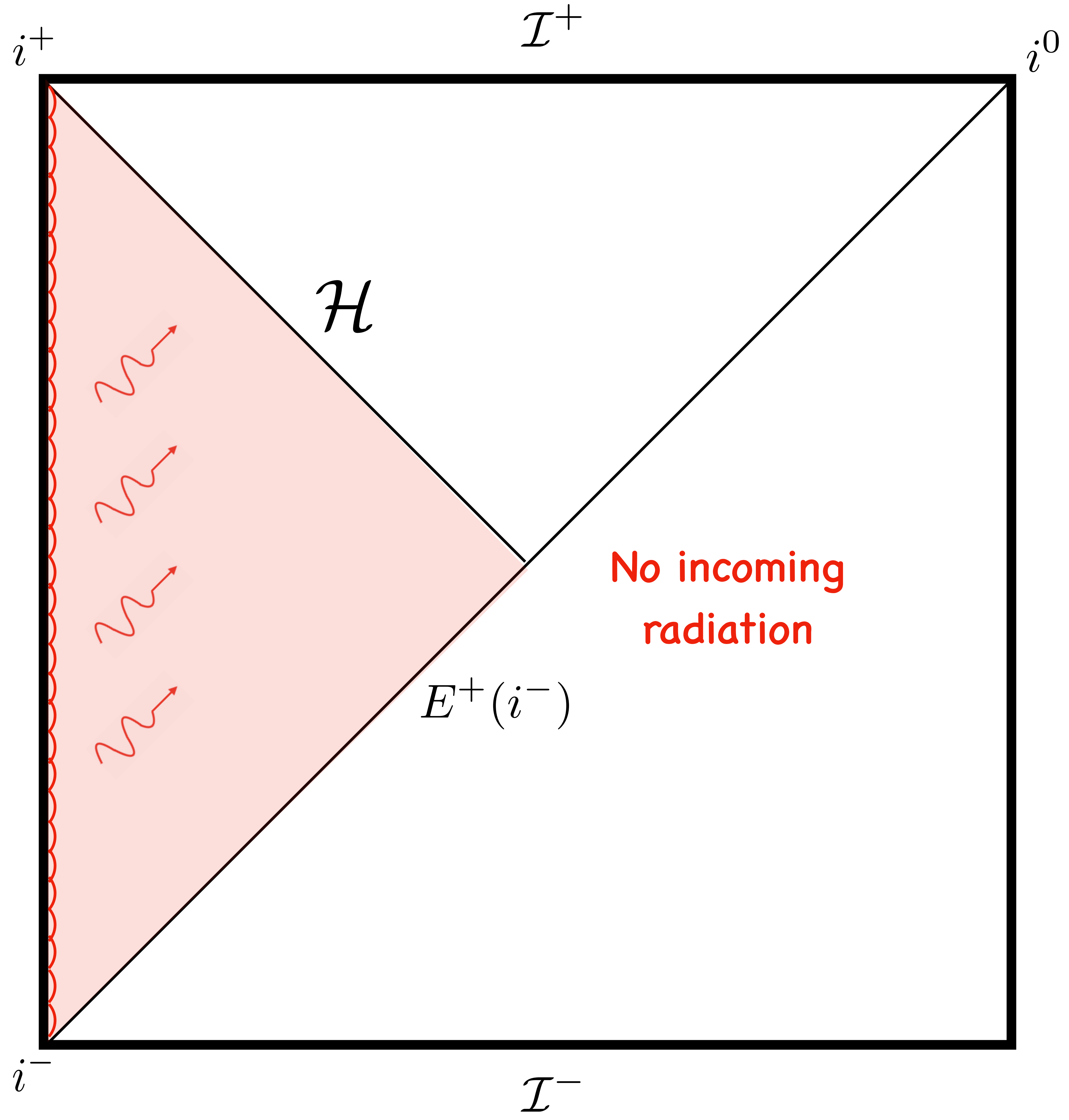

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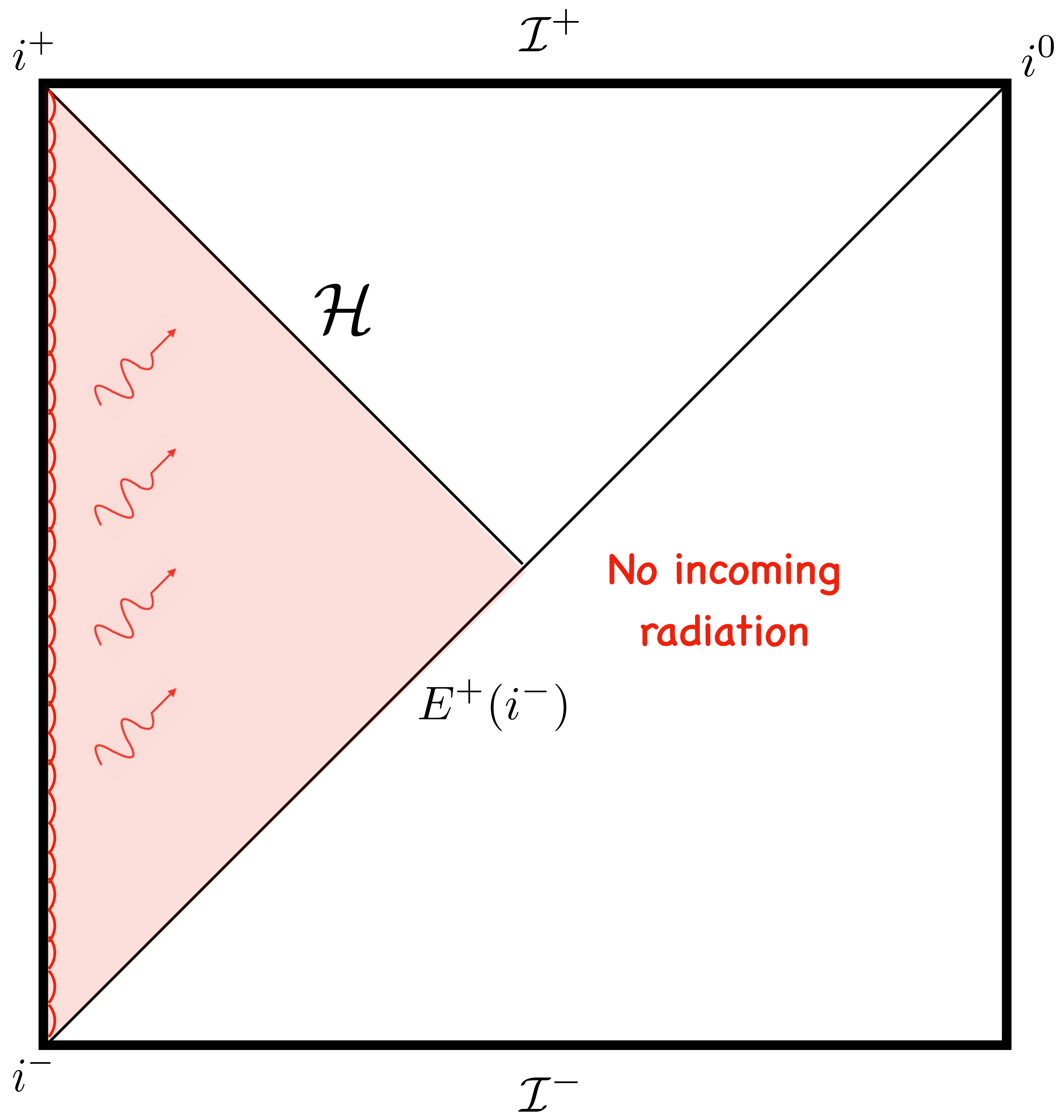

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Ashtekar, Bonga, Kesavan
arXiv:1510.05593

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null surface → cosmological horizon $r = \frac{1}{H}e^{-Ht}$

shear of ℓ → shear of T

expansion of ℓ → expansion of T

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To sustain its null character with respect to the perturbed geometry a suitable gauge is applied:

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It may be interpreted as a deformation procedure for $\mathscr{H}$, that is performed in such a way that, given the original perturbation of spacetime, $\mathscr{H}$ remains null.

A generalization of the Einstein's quadrupole formula for positive cosmological constant

◦

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where $$q_{ab}^{(i)} := Q_{ab}^{(i)} - \frac{1}{3}\mathring{q}_{ij} Q^{(i)}$$

The limit for $\Lambda \to 0$, or equivalently $H \to 0$, recovers the famous Einstein quadruple formula:

$$E_T = \frac{1}{45}\int dt \sum_{i,j=1}^{3}\left[\left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3}\right)^2\right](t_{ret})$$

Wald, 1984

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In such decomposition, only the transverse-traceless part, A_{ij}^{TT}, is gauge invariant, and therefore is regarded as physical component of the field A_{ij}. Generally, it is highly non-trivial to extract the transverse-traceless part of the field A_{ij} (see Bonga & Hazboun for explicit example).

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There also exists a distinct notion of transverse-traceless tensors often used in the context of gravitational waves. It is easier for calculations, however generically inequivalent to the other notion. To extract the transverse-traceless part of a rank-2 tensor one simply uses an algebraic projection operator:

$$P_i{}^j = \delta_i^j - \tilde{x}_i \tilde{x}^j \qquad\qquad \Lambda_{ij}{}^{kl} = \frac{1}{2}\left(P_i{}^k P_j{}^l + P_i{}^l P_j{}^k - P_{ij} P^{kl}\right)$$

where $\tilde{x}^i = x^i / r$. In this notion the transverse traceless part of the field is often written with a tt in a superscript, namely:

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However, if we restrict ourselves to large radial distances away from the source it often happens that TT coincides with tt (Example: power radiated by a spatially compact circular binary system, Bonga & Hazboun 2017 and Hoque & Aggarwal 2017).

Thank you for your attention