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Fundamental decoherence from quantum spacetime

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Contents

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Decoherence in Quantum Gravity

Gravitational field (or spacetime) as an omnipresent environment [*A. Bassi et al, Class. Quant. Grav. 34, 193002 (2017).*].

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Gravitational field (or spacetime) as an omnipresent environment [*A. Bassi et al, Class. Quant. Grav. 34, 193002 (2017).*].

Several frameworks: Quantum clocks [*R. Gambini et al, Phys.Rev.Lett. 93 (2004)*], metric fluctuations [*H.P. Breuer et al,Class. Quant. Grav. 26, 105012 (2009)*], fluctuating minimal length and GUP models [*L. Petruzziello and F. Illuminati, Nat.Comm. 12, 4449 (2021)*], . . .

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Deformed symmetries are described with Hopf algebras

Non-linear algebraic sector

$$[X_i, X_j] = f(\boldsymbol{X})$$

Coalgebra becomes relevant

$$\Delta : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H} \text{ (coproduct)}$$

$$S : \mathbb{H} \to \mathbb{H} \text{ (antipode)}$$

Deformed symmetries in QM

κ-Galilei algebra in classical basis: undeformed algebra and deformed coalgebra.

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Undeformed coalgebra	Deformed coalgebra
$$\Delta P_\mu \Big(\|p\rangle \otimes \|q\rangle \Big) = (p_\mu + q_\mu) \|p\rangle \otimes \|q\rangle \ , \quad (1)$$	$$\Delta P_\mu \Big(\|p\rangle \otimes \|q\rangle \Big) = (p_\mu \oplus q_\mu) \|p\rangle \otimes \|q\rangle \ , \quad (3)$$
$$P_\mu \langle q\| := \langle q\| S(P_\mu) = -q_\mu \langle q\| \ . \quad (2)$$	$$P_\mu \langle q\| := \langle q\| S(P_\mu) \neq -q_\mu \langle q\| \ . \quad (4)$$

Deformed symmetries in QM

κ-Galilei algebra in classical basis: undeformed algebra and deformed coalgebra.

Deforming the coalgebra means deforming the action of generators on the Hilbert space.

Undeformed coalgebra

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Deformed coalgebra

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Non trivial structures on the coalgebra are

$$\Delta P_0 = P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0 + \frac{1}{\kappa} P_n \otimes P^n \ , \ S(P_0) = -P_0 + \frac{1}{\kappa} \boldsymbol{P}^2 \ . \quad (5)$$

Adjoint Action

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Adjoint action defined as [*H. Ruegg & V. N. Tolstoy, Lett.Math.Phys. 32, 85–101 (1994)*]

$$\boxed{\mathrm{ad}_A(B) := (id \otimes S)\Delta A \diamond B} \ , \quad (a \otimes b) \diamond O := a\, O\, b\, . \tag{6}$$

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If coalgebra structures are trivial

$$\mathrm{ad}_A(B) = (id\otimes S)\underbrace{(A\otimes\mathbb{1}+\mathbb{1}\otimes A)}_{\Delta A}\diamond B = (A\otimes\mathbb{1} - \underbrace{\mathbb{1}\otimes A}_{S(A)=-A})\diamond B = [A,B]\,. \tag{7}$$

Density operator time evolution

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- Time evolution preserves the hermiticity of ρ (required to interpret the eigenvalues of ρ as probabilities).

These requirements are met with

$$\boxed{i\,\partial_t\rho = \frac{1}{2}\left\{\mathrm{ad}_{P_0}(\rho) - \left[\mathrm{ad}_{P_0}(\rho)\right]^\dagger\right\}}\,. \tag{8}$$

leading to

$$\partial_t\rho = -i\,[P_0,\rho] - \frac{1}{2\,\kappa}\left(\rho\,\boldsymbol{P}^2 + \boldsymbol{P}^2\,\rho - 2\,P_n\rho\,P^n\right)\,. \tag{9}$$

Decoherence time and fundamental constraint on the mass

Free particle solution in momentum basis is

$$\langle \boldsymbol{p}|\rho|\boldsymbol{q}\rangle := \rho_{pq}(t) = \rho_{pq}(0)\exp\left\{-i\,t[E(p)-E(q)] - \frac{t}{2\,\kappa}(\boldsymbol{p}-\boldsymbol{q})^2\right\}. \tag{10}$$

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Off-diagonal coherences vanish after a decoherence time

$$\boxed{\tau_D = \frac{2\,\kappa}{(\delta \boldsymbol{p})^2}}. \quad (11)$$

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"Lifetime" of a state of a quantum system

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With $E = (2\,m)^{-1}\boldsymbol{p}^2$, from (13) and for momentum localized states, $\frac{\delta p}{p} \ll 1$

$$\boxed{m \lesssim \kappa}. \quad (14)$$

Phenomenological opportunities?

Corrections to the oscillation probability formula

$$P(\beta \to \alpha) = \sum_{i,j} U^*_{\beta i} U_{\beta j} U^*_{\alpha j} U_{\alpha i} \, e^{-i\left[E(\boldsymbol{p}_i)t - \boldsymbol{p}_i \cdot \boldsymbol{x}\right] + i\left[E(\boldsymbol{p}_j)t - \boldsymbol{p}_j \cdot \boldsymbol{x}\right]} \, e^{-\frac{t}{2\kappa}\left(\boldsymbol{p}_i - \boldsymbol{p}_j\right)^2} \, . \tag{15}$$

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Neutrino fluxes at the detector depends on oscillation probability

$$\Phi^D_{\nu_\alpha} = \sum_\beta P(\beta \to \alpha)\Phi^S_{\nu_\beta} . \tag{16}$$

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Corrections to the amount of decoherence of a system with respect to an environment. For optomechanical cavities

$$\mathrm{Dec}(A \mid E) = \frac{1}{4}\left\{1 + \sqrt{1 - \exp\left[-4(2\bar{n}+1)\frac{g_0^2}{\omega_m^2}\sin\left(\frac{\omega_m t}{2}\right)\right]}\right\} , \tag{17}$$

with

$$\bar{n} = \frac{k_B T}{\hbar\, \omega_m} + \frac{2Q}{\omega_m \tau_D} . \tag{18}$$

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- From the decoherence time, a fundamental constraint on the mass of quantum systems arises.
- Phenomenology: neutrino oscillation, table-top experiments with optomechanical cavities.
- Can this framework be extended to field theories? Primordial perturbations decoherence...

Thank you!

Interaction with the Environment

von Neumann measurement:

$$|o_n\rangle \otimes |R\rangle \mapsto |o_n\rangle \otimes |a_n\rangle \Rightarrow |\psi\rangle \otimes |R\rangle \mapsto \sum_n c_n |o_n\rangle \otimes |a_n\rangle \ , \tag{19}$$

$|a_n\rangle \in \mathscr{H}_A$, $|o_n\rangle \in \mathscr{H}_S$.
Two-level system: $|\psi_i\rangle \otimes |E_0\rangle \mapsto |\psi_i\rangle \otimes |E_i\rangle$, $i = 1,2$. Environment-system entanglement emerges dynamically

$$|\psi\rangle \otimes |E_0\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle + |\psi_2\rangle\Big) \otimes |E_0\rangle \mapsto \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle \otimes |E_1\rangle + |\psi_2\rangle \otimes |E_2\rangle\Big) \ . \tag{20}$$

The density operator of the system is

$$\rho_S = \mathrm{tr}_E\{\rho\} = \frac{1}{2}\Big(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_2|\langle E_1|E_2\rangle + |\psi_2\rangle\langle\psi_1|\langle E_2|E_1\rangle\Big) \ . \tag{21}$$

For macroscopic systems $|E_i\rangle = \prod_{\alpha=1}^N \left|e_\alpha^{(i)}\right\rangle$, $\left\langle e_\alpha^{(1)} \middle| e_\beta^{(2)} \right\rangle = \varepsilon \lesssim 1$, thus $\boxed{\langle E_1|E_2\rangle \sim \varepsilon^N \sim 0}$

Example of system-environment interaction

Two-level system $\{ |0\rangle , |1\rangle \}$; environment composed of N two-level systems $\{ |\uparrow\rangle_i , |\downarrow\rangle_i \}$.

$$\hat{H}_{\text{int}} = \frac{1}{2} \hat{\sigma}_z \otimes \hat{E} \ , \ \ \hat{E} := \sum_{i=1}^{N} g_i \, \hat{\sigma}_z^{(i)} \ . \tag{22}$$

$2^N - 1$ energy levels for E given by $|n\rangle = |\uparrow\rangle_i |\downarrow\rangle_i \dots |\uparrow\rangle_i$ with $\varepsilon_n = \sum_{i=1}^{N} (-1)^{n_i} g_i$.

$$e^{-it\hat{H}_{\text{int}}} |\psi\rangle = e^{-it\hat{H}_{\text{int}}} \Big(a|0\rangle + b|1\rangle \Big) \otimes \sum_{i=1}^{2^N-1} c_n |n\rangle = a|0\rangle |\varepsilon_0(t)\rangle + b|1\rangle |\varepsilon_1(t)\rangle \ , \tag{23}$$

with $|\varepsilon_0(t)\rangle = |\varepsilon_1(-t)\rangle = \sum_{i=1}^{2^N-1} c_n e^{-it\frac{\varepsilon_n}{2}}$.
The decoherence parameter

$$r(t) = \langle \varepsilon_1(t) | \varepsilon_0(t) \rangle = \sum_n |c_n|^2 e^{-i\varepsilon_n t} \ \Rightarrow \ \boxed{\langle |r(t)| \rangle \sim 2^{-N}} \ . \tag{24}$$

Recurrence time τ_{rec} exists since N is always finite. $g_i = g \ \forall \, i$ and $\sum_{i=1}^{2^N-1} c_n |n\rangle = \bigotimes_{i=1}^{N} \frac{1}{\sqrt{2}} \Big(|\downarrow\rangle_i + |\uparrow\rangle_i \Big)$ give $r(t) = \big[\cos(gt) \big]^N$ with $\tau_{\text{rec}} = \frac{\pi}{g}$. Highly improbable, typically $\tau_{\text{rec}} \propto N!$

Environmental superselection

Interaction with the environment selects the preferred basis.

$$|\psi_\pm\rangle \otimes |E_0\rangle \mapsto \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle \otimes |E_1\rangle \pm |\psi_2\rangle \otimes |E_2\rangle \Big) . \qquad (25)$$

$|\psi_\pm\rangle$ get entangled with the environment.
Superselected states are the ones that get least entangled with the environment ($|\psi_i\rangle$).
Typically $\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{\text{int}}$.
Quantum measurement limit: $\hat{H} \approx \hat{H}_{\text{int}}$. Typically $\hat{H}_{\text{int}} = \hat{S} \otimes \hat{E}$, thus eigenstates of $\hat{S}$ get selected

$$e^{-it\hat{S}\otimes\hat{E}} |s_i\rangle \otimes |E_0\rangle = |s_i\rangle \otimes e^{-it\lambda_i\hat{E}} |E_0\rangle := |s_i\rangle \otimes |E_i(t)\rangle \ . \qquad (26)$$

Quantum limit: $\hat{H} \approx \hat{H}_S$. Constants of motion of S (energy) get selected.

Lindblad equation

In general $\hat{\rho}_S(t) = \mathrm{tr}_E\left\{\hat{U}(t)\hat{\rho}_{S,E}(0)\hat{U}^\dagger(t)\right\}$. Master equations give approximations

$$\partial_t\hat{\rho}_S(t) = -i\left[\hat{H}_S,\hat{\rho}_S\right] + \hat{D}\left[\hat{\rho}_S(t)\right]\,. \tag{27}$$

Lindblad equation: most general master equation preserving the positivity of $\hat{\rho}_S$.

$$\boxed{\partial_t\hat{\rho} = -i\left[\hat{H}_S,\hat{\rho}_S\right] - \frac{1}{2}g^{mn}\left(\hat{Q}_m\hat{Q}_n\,\hat{\rho} + \hat{\rho}\,\hat{Q}_m\hat{Q}_n - 2\,\hat{Q}_m\,\hat{\rho}\,\hat{Q}_n\right)}\,, \tag{28}$$

with $g^{mn}\in\mathbb{R}$ and $u_m\,u_n\,g^{mn}\geq 0\,,\ \forall u$.

κ-Galilei as deformed symmetry group

κ-Poincaré algebra in classical basis is undeformed while the relevant coalgebra structures are

$$\begin{aligned} \Delta P_0 &= P_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{\kappa} P_n \Pi_0^{-1} \otimes P^n \ , \ \Delta P_n = P_n \otimes \mathbb{1} + \mathbb{1} \otimes P_n \\ S(P_0) &= -P_0 + \frac{1}{\kappa} \boldsymbol{P}^2 \Pi_0^{-1} \ , \ S(P_i) = -P_i \Pi_0^{-1} \\ \Pi_0 &= \frac{1}{\kappa} P_0 + \sqrt{\mathbb{1} - \frac{1}{\kappa^2} P^\mu P_\mu} \ , \ \Pi_0^{-1} = \frac{-\frac{1}{\kappa} P_0 + \sqrt{\mathbb{1} - \frac{1}{\kappa^2} P^\mu P_\mu}}{\mathbb{1} - \frac{1}{\kappa^2} P^\mu P_\mu} \end{aligned} \tag{29}$$

The contraction [*A. Ballesteros et al, Phys.Lett.B 805, 135461 (2020)*] is carried out with

$$\boldsymbol{N} \mapsto c^{-1} \boldsymbol{N} \ , \ \boldsymbol{P} \mapsto c^{-1} \boldsymbol{P} \ , \ \kappa \mapsto c^{-2} \kappa \tag{30}$$

and $c \to \infty$.

The algebra becomes the standard Galilei algebra. Non trivial structures on the coalgebra are

$$\Delta P_0 = P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0 + \frac{1}{\kappa} P_n \otimes P^n \ , \ S(P_0) = -P_0 + \frac{1}{\kappa} \boldsymbol{P}^2 \ . \tag{31}$$

Details of the contraction

The contraction

$$\boldsymbol{N} \mapsto c^{-1}\boldsymbol{N} \ , \ \boldsymbol{P} \mapsto c^{-1}\boldsymbol{P} \ , \ \kappa \mapsto c^{-2}\kappa \tag{32}$$

gives

$$\Pi_0 = \frac{1}{c^2\kappa}P_0 + \sqrt{\mathbb{1} - \frac{1}{c^4\kappa^2}\left(P_0^2 - c^2\boldsymbol{P}^2\right)} \to \mathbb{1} \ , \ \Pi_0^{-1} = \frac{-\frac{1}{c^2\kappa}P_0 + \sqrt{\mathbb{1} - \frac{1}{c^4\kappa^2}\left(P_0^2 - c^2\boldsymbol{P}^2\right)}}{\mathbb{1} - \frac{1}{c^4\kappa^2}\left(P_0^2 - c^2\boldsymbol{P}^2\right)} \to \mathbb{1} \ , \tag{33}$$

thus

$$\begin{aligned} \Delta P_0 &= P_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{c^2\kappa} cP_n \Pi_0^{-1} \otimes cP^n \to P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0 + \frac{1}{\kappa} P_n \otimes P^n \ , \\ S(P_0) &= -P_0 + \frac{1}{c^2\kappa} c^2 \boldsymbol{P}^2 \Pi_0^{-1} \to -P_0 + \frac{1}{\kappa}\boldsymbol{P}^2 \ , \quad cS(P_i) = -cP_i\Pi_0^{-1} \Rightarrow S(P_i) = -P_i \ . \end{aligned} \tag{34}$$