Entanglement Entropy and the Arrow of Time in Wheeler-DeWitt Quantum Cosmology

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Presentation Overview

Motivation

- Explain the time asymmetry from fundamental QG theory.
- Gravitational arrow of time as master arrow of time.
- To explain Quantum Weyl Hypothesis from full quantum theory of gravity: Quantum Geometrodynamics.

Quantum Geometrodynamics

- Background independence $\rightarrow$ Reparameterization invariance $\rightarrow$ Existence of Constraints.
  Classical $H = 0 \implies \hat{H}\Psi = 0$ in quantum theory
- In GR, Hamiltonian constraint $\rightarrow$ *Timeless* Wheeler-DeWitt equation:
  $\hat{\mathcal{H}}_{\perp}\Psi := \left(-16\pi G\hbar^2 G_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - \frac{\sqrt{h}}{16\pi G}\left({}^{(3)}R - 2\Lambda\right)\right)\Psi = 0.$
- *Superspace*: Configuration space for the wave function $(h_{ij}(\mathbf{x}), \Phi(\mathbf{x}))$
- *Minisuperspace*: Superspace with finite number of d.o.f:
- Time can be introduced in semiclassical approximation.

Decoherence in quantum mechanics

- Considering the role of Environment:

$$\left(\sum_n c_n |n\rangle\right) |\Phi_o\rangle |E_o\rangle \xrightarrow{t} \sum_n c_n |n\rangle |\Phi_n\rangle |E_n\rangle \quad (1)$$

$$\implies \rho_{SA} \approx \sum_n |c_n|^2 |n\rangle \langle n| \otimes |\Phi_n\rangle \langle \Phi_n| \qquad \text{if} \qquad \langle E_n|E_m\rangle \approx \delta_{nm} \quad (2)$$

- Pure State: $\rho = |\alpha\rangle \langle\alpha| \implies \rho^2 = \rho \implies tr\rho^2 = 1.$
- Mixed State: $\rho = \sum_i p_i |\alpha_i\rangle \langle\alpha_i| \implies \rho^2 \neq \rho \implies tr\rho^2 \neq 1$
- Entanglement Entropy ($k_B = 1$):

$$S = -tr\rho \log \rho \rightarrow \text{von Neumann entropy} \quad (3)$$

$$S_{\text{lin}} = tr\rho - tr\rho^2 \rightarrow \text{Linear Entropy} \quad (4)$$

Wheeler-DeWitt evolution for the toy model

$$\left[\frac{1}{m_p^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi_1^2} - \frac{\partial^2}{\partial\phi_2^2} + a_o^6 e^{6\alpha}\left(\mathcal{V}(\phi_1,\phi_2) + m_p^2\frac{\Lambda}{3}\right)\right]\Psi(\alpha,\phi_1,\phi_2) = 0 \quad (5)$$

with

$$\mathcal{V}(\phi_1,\phi_2) = m^2(\phi_1^2 + \phi_2^2) \quad (6)$$

To simplify calculations

$$\alpha = \alpha_o + s \implies s = \alpha - \alpha_o \quad (7)$$

where s is shifted intrinsic time with

$$\alpha_o := \lim_{\alpha\to-\infty}\alpha \qquad s.t. \qquad \xi := e^{6\alpha_o} << 1, \tilde{m}^2 := \xi m^2 << m^2 \quad (8)$$

$$\left[\frac{1}{m_p^2}\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial\phi_1^2} - \frac{\partial^2}{\partial\phi_2^2} + \underbrace{a_o^6 e^{6s}\left(\tilde{m}^2(\phi_1^2+\phi_2^2) + \xi m_p^2\frac{\Lambda}{3}\right)}_{=:V(s,\phi_1,\phi_2)\propto\xi}\right]\Psi(s,\phi_1,\phi_2) = 0 \quad (9)$$

Wheeler-DeWitt evolution for the toy model

In the limit $\alpha \to -\infty$ i.e. $s \to 0$:

$$\left[\frac{1}{m_p^2}\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial \phi_1^2} - \frac{\partial^2}{\partial \phi_2^2}\right]\Psi_h(s, \phi_1, \phi_2) = 0 \tag{10}$$

$\implies$

$$\Psi_h(s, \phi_1, \phi_2) = \exp\{-i\,(m_p K s + k_1\phi_1 + k_2\phi_2)\} \to \text{Plane waves} \tag{11}$$

$$K = \sqrt{k_1^2 + k_2^2} \tag{12}$$

As the universe expands:

$$\Psi_{k_1,k_2}(s, \phi_1, \phi_2) = \Psi_h(s, \phi_1, \phi_2) + \Psi_1(s, \phi_1, \phi_2) \tag{13}$$

Ansatz for Ψ_1:

$$\Psi_1(s, \phi_1, \phi_2) = \xi e^{6s} C_1(\phi_1, \phi_2) e^{-i m_p K s} \tag{14}$$

Wheeler-DeWitt evolution for the toy model

$$\left(\frac{1}{m_p^2}\frac{\partial}{\partial s^2}-\frac{\partial^2}{\partial \phi_1^2}-\frac{\partial^2}{\partial \phi_2^2}\right)(\Psi_h+\Psi_1)=-\underbrace{V(s,\phi_1,\phi_2)\Psi_h}_{\propto \xi}-\underbrace{V(s,\phi_1,\phi_2)\Psi_1}_{\propto \xi^2} \quad (15)$$

The solution for C_1 is obtained as:

$$C_1(\phi_1,\phi_2)=-\frac{a_o^6 m^2 e^{-i(k_1\phi_1+k_2\phi_2)}}{\underbrace{K^2+\Omega^2}_{=:R^2\approx-\frac{12iK}{m_p}}}\Bigg\{\phi_1^2+\phi_2^2+m_p^2\frac{\bar{\Lambda}}{3}+\frac{4}{K^2+\Omega^2}(1- i(k_1\phi_1+k_2\phi_2))-\frac{8}{(K^2+\Omega^2)^2}\left(k_1^2+k_2^2\right)\Bigg\} \quad (16)$$

with $\bar{\Lambda}:=\frac{\Lambda}{m^2}$.

Wheeler-DeWitt evolution for the toy model

The expression for general solution is:

$$\Psi(s,\phi_1,\phi_2) = N \exp\left\{ -\sum_{n=1}^{2}\left(\frac{b^2}{2}\left(\frac{\bar{k}_n}{\bar{K}}m_p s+\phi_n\right)^2 + i\left(\frac{\bar{k}_n}{\bar{K}}m_p s+\phi_n\right)\bar{k}_n\right)\right\}$$
$$\left[1-A(s)-\sum_{n=1}^{2}\left(B_n(s)\phi_n+C(s)\phi_n^2\right)\right] \tag{17}$$

Similar to the **squeezed** state!

Linear entropy:

$$S_{\text{lin}} = tr\rho - tr\rho_{\text{red}}^2$$

Wheeler-DeWitt evolution for the toy model

$$S_{\text{lin}} \approx \frac{a_o^{12}\tilde{m}^4 \bar{\bar{C}} \left\{ \frac{e^{12\bar{s}}\bar{k}_2^2 m_p^4}{864\bar{K}^4 b^4}(s+\bar{s}) + ... \right\} + (\text{higher order terms in } a_o^{12}\tilde{m}^4)}{Q} > 0 \tag{18}$$

with $Q(s) = 1 + \xi(\text{...s dependent terms...})$ such that $Q'(s) \propto \xi$.

$$S'_{\text{lin}} \approx \frac{a_o^{12}\tilde{m}^4 \bar{\bar{C}} \left\{ \frac{e^{12\bar{s}}\bar{k}_2^2 m_p^4}{864\bar{K}^4 b^4} + ... \right\} + (\text{higher order terms in } a_o^{12}\tilde{m}^4)}{Q^2} > 0 \tag{19}$$

From semiclassical approximation:

$$\frac{\partial}{\partial t} = 2a_o^3\xi^{\frac{1}{2}}\sqrt{\frac{\Lambda}{3}}e^{3s}\frac{\partial}{\partial s} \implies \frac{\partial S_{\text{lin}}}{\partial t} = 2a_o^3\xi^{\frac{1}{2}}\sqrt{\frac{\Lambda}{3}}e^{3s}\frac{\partial S_{\text{lin}}}{\partial s} > 0 \tag{20}$$

Thus linear entropy increases as the universe expands and time moves forward, thus providing appropriate arrow of time.

- Obtain squeezed states near the big bang singularity $\implies$ can develop interaction term as the universe expands.
- Get entropy generation and arrow of time for fields not interacting explicitly with each other in the full Wheeler-DeWitt picture.
- Entanglement entropy comes out to be very **small!** $\rightarrow$ justifying Quantum Weyl Curvature hypothesis as anticipated.

- The treatment for toy model in full Wheeler-DeWitt picture can be used to investigate the case for primordial fluctuations [Brizuela et. al.,2016]:

$$\left\{ e^{-2\alpha}\left[\frac{1}{m_p^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + 2e^{6\alpha}\mathcal{V}(\phi)\right] - \frac{\partial^2}{\partial v_{k_1}^2} + \omega_{k_1}^2(\alpha,\phi)v_{k_1}^2 - \frac{\partial^2}{\partial v_{k_2}^2} + \omega_{k_2}^2(\alpha,\phi)v_{k_2}^2 \right\}\Psi(\alpha,\phi,v_{k_1},v_{k_2}) = 0 \quad (21)$$

$$\Longrightarrow \left\{ \frac{1}{m_p^2}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial \tilde{v}_{k_1}^2} - \frac{\partial^2}{\partial \tilde{v}_{k_2}^2} + e^{4\alpha}\left(k_1^2\tilde{v}_{k_1}^2 + k_2^2\tilde{v}_{k_2}^2\right)\right\}\Psi(\alpha,\phi,\tilde{v}_{k_1},\tilde{v}_{k_2}) = 0 \quad (22)$$

with $\tilde{v}_k = e^{-\alpha}v_k$.

Conclusions and future directions

Future directions

- In the presence of interaction: $\tilde{S}_{\text{lin}} = S_{\text{lin}} + g^n(...)$.
- Knowledge of unitarity for Wheeler-DeWitt equation can help understand entropy generation.
- Get correct behaviour for arrow of time due to asymmetric term. source: invariant volume measure $\int d^4x\sqrt{-g}$ for Friedmann universe.
- Relationship between entanglement entropy and thermodynamical entropy in context of Quantum Geometrodynamics.

References

Claus Kiefer, Ingo Lohmar, David Polarski, and Alexei A Starobinsky
Pointer states for primordial fluctuations in inflationary cosmology.
Classical and Quantum Gravity, 24(7):1699, March 2007.

ROBERT BRANDENBERGER, RAYMOND LAFLAMME, and MILAN MIJIĆ.
Classical perturbations from decoherence of quantum fluctuations in the inflationary universe.
Modern Physics Letters A, 05(28):2311–2317, 1990.

H.D. Conradi and H.D. Zeh.
Quantum cosmology as an initial value problem.
Physics Letters A, 154(7):321–326, 1991

Claus Kiefer and David Polarski.
Emergence of classicality for primordial fluctuations: Concepts and analogies.
Annalen der Physik, 510(3):137–158, 1998

References

David Brizuela, Claus Kiefer, and Manuel Krämer.
Quantum-gravitational effects on gauge-invariant scalar and tensor perturbations during inflation: The de sitter case.
Phys. Rev. D, 93:104035, May 2016

The End

Questions? Comments?

Backup Slides

Wheeler-DeWitt evolution for the toy model

General solution:

$$\Psi = \int_{-\infty}^{+\infty} dk_1 dk_2 C(k_1, k_2) \Psi_{k_1, k_2} \tag{23}$$

We choose a narrow Gaussian for $C(k_1, k_2)$:

$$C(k_1, k_2) = \frac{1}{\sqrt{\pi} b} \exp\left\{ -\left(\frac{\left(k_1 - \bar{k}_1\right)^2 + (k_2 - \bar{k}_2)^2}{2b^2} \right) \right\} \tag{24}$$

$\implies$

$$\Psi(s, \phi_1, \phi_2) = N \exp\left\{ -\sum_{n=1}^{2} \left(\frac{b^2}{2} \left(\frac{\bar{k}_n}{\bar{K}} m_p s + \phi_n \right)^2 + i \left(\frac{\bar{k}_n}{\bar{K}} m_p s + \phi_n \right) \bar{k}_n \right) \right\}$$
$$\left[1 - A(s) - \sum_{n=1}^{2} \left(B_n(s) \phi_n + C(s) \phi_n^2 \right) \right] \tag{25}$$

Wheeler-DeWitt evolution for the toy model

$$
\begin{aligned}
\text{diff} \approx a_o^{12}\tilde{m}^4\bar{\bar{C}} \Bigg\{ & \frac{ie^{12\bar{s}}\bar{k}_1^2\bar{k}_2^2 m_p^7}{2592\bar{K}^7}\left(27s^3\bar{s} + 9s^2\bar{s}^2 + 9s\bar{s}^3 + 27\bar{s}^4 + s^3 + s^2\bar{s} - s\bar{s}^2\right. \\
& \left. -\bar{s}^3\right) - \frac{e^{12\bar{s}}\bar{k}_2 m_p^6}{2592\bar{K}^6}\left(\bar{s}^2 + s\bar{s}\right) + \frac{e^{12\bar{s}}\bar{k}_2^2 m_p^4}{864 b^4\bar{K}^4}\left(s + \bar{s}\right) \\
& + \frac{1}{b^2}\Bigg(\frac{ie^{12\bar{s}}\bar{k}_2^2 m_p^5}{2592\bar{K}^5}\left(s + \bar{s} + 3s\bar{s} + 3\bar{s}^2\right) \\
& - \frac{e^{12\bar{s}}\bar{k}_1^2\bar{k}_2 m_p^6}{2592\bar{K}^6}\left(3\left(s+\bar{s}\right)^3 - \left(s+\bar{s}\right)^2\right)\Bigg)\Bigg\}
\end{aligned} \tag{26}
$$

and some complicated expression for $Q(s)$ of the form $Q(s) = 1 + \xi(\ldots\text{s dependent terms})$ such that $Q'(s) \propto \xi$.

Wheeler-DeWitt evolution for the toy model

$$\Psi(s,\phi_1,\phi_2) = N\exp\left\{-\sum_{n=1}^{2}\left(\frac{b^2}{2}\left(\frac{\bar{k}_n}{\bar{K}}m_p s+\phi_n\right)^2 + i\left(\frac{\bar{k}_n}{\bar{K}}m_p s+\phi_n\right)\bar{k}_n\right)\right\}$$
$$\left[1-A-\sum_{n=1}^{2}\left(B_n\phi_n + C\phi_n^2\right)\right] \tag{27}$$

with

$$A(s) = e^{6s}a_o^6\tilde{m}^2\left(-\frac{4m_p^2}{144\bar{K}^2}+\frac{im_p^3}{12\bar{K}}\left(\frac{\bar{\Lambda}}{3}+\frac{1}{18}\right)\right) \tag{28a}$$

$$B_n(s) = e^{6s}a_o^6\tilde{m}^2\frac{4i\bar{k}_n m_p^2}{144\bar{K}^2}\left(1-\frac{ib^2 m_p s}{\bar{K}}\right) \tag{28b}$$

$$C(s) = e^{6s}a_o^6\tilde{m}^2\frac{m_p}{12\bar{K}}\left(i+\frac{4m_p b^2}{12\bar{K}}\right) \tag{28c}$$

Wheeler-DeWitt evolution for the toy model

$$S'_{\text{lin}} \approx \frac{\text{diff}'}{Q^2} \tag{29}$$

In the above expression, dominating terms is diff′

$$\begin{aligned}(\text{diff})' \approx a_o^{12}\tilde{m}^4\bar{\bar{C}}\Bigg\{ &\frac{ie^{12\bar{s}}\bar{k}_1^2\bar{k}_2^2 m_p^7}{2592\bar{K}^7}\left(81s^2\bar{s}+18s\bar{s}^2+9\bar{s}^3+3s^2+2s\bar{s}-\bar{s}^2\right)\\ &-\frac{e^{12\bar{s}}\bar{k}_2 m_p^6}{2592\bar{K}^6}\bar{s}+\frac{e^{12\bar{s}}\bar{k}_2^2 m_p^4}{864b^4\bar{K}^4}+\frac{1}{b^2}\Bigg(\frac{ie^{12\bar{s}}\bar{k}_2^2 m_p^5}{2592\bar{K}^5}(1+3\bar{s})\\ &-\frac{e^{12\bar{s}}\bar{k}_1^2\bar{k}_2 m_p^6}{2592\bar{K}^6}\left(9(s+\bar{s})^2-2(s+\bar{s})\right)\Bigg)\Bigg\}\end{aligned} \tag{30}$$

$(\text{diff})' > 0 \implies S'_{\text{lin}} > 0$