Rainbow Oppenheimer-Snyder collapse and the entanglement entropy production.

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59. Winter School of Theoretical Physics and third COST Action CA18108 Training School "Gravity – Classical, Quantum and Phenomenology"

14th of February 2023

Motivation

- Classical singularities resulting from gravitational collapse are not so *satisfactionary*.
- Quantum fields incorporated into black hole backgrounds induces Hawking radiation. It leads to *the black hole information paradox*, which still is the subject of ongoing debates.
- Black-to-white hole transition paradigm seem to be a more *satisfactionary* description in the above context. In principle, an observer located at $\mathscr{I}^+$ can reconstruct the entire space-time (no information paradox).
- *Quantum* treatment (LQC) of the Oppenheimer-Snyder collapse scenario may model the black-to-white hole transition. What is the global picture? What about the time dynamics of the entanglement entropy?

Plan

1. Effective dust ball interior geometry.
   - Quantized FRW interior.
   - Effective dynamics.
   - *Rainbow* metric for the interior.
2. Extracted exterior geometry.
   - Imposed general form of exterior metric.
   - Junction conditions.
   - Resulting space-time and its conformal diagram.
3. Calculation of so-called entanglement entropy production.

Quantized interior of the collapsing dust ball

The framework was developed by Parvizi et al., 2021:

- **Classical action**: gravity coupled to the dust and the scalar field $S = \int \mathrm{d}x\sqrt{-g}\left[\frac{\mathcal{R}}{16\pi G} + \mathcal{L}_T\right] + S_\phi$, $\mathrm{d}s^2 = -N(x_0)^2\mathrm{d}x_0^2 + a^2(x_0)\,\mathrm{d}\mathbf{x}^2$
- **Canonical quantization**: gravitational sector reduced phase space quantization according to LQC. That procedure mimicks the constructions used in the full LQG. Polymer representation for gravitational degrees of freedom, Schrödinger represantation for the dust and the scalar field. Dust field is the clock.
- **Dynamics**: The evolution is governed by Schrödinger-like equation

$$i\hbar\partial_T\Psi = \left(\hat{H}_{\mathrm{grav}} + \hat{H}_\phi\right)\Psi \quad (1)$$

For gravity, basic operators are volume and associated momentum operator

$$\hat{H}_{\mathrm{grav}} = -\frac{3\pi G\hbar^2}{2\alpha_o}\sqrt{|\hat{\mathfrak{v}}|}\sin^2(\hat{\mathfrak{b}})\sqrt{|\hat{\mathfrak{v}}|}\,. \quad (2)$$

- **Scalar field**: the assembly of harmonic oscillators (I pick one of them)

$$\hat{H}_\phi = \sum_{\mathbf{k}}\hat{H}_{\mathbf{k}} \to \frac{1}{2}\alpha_o^{-1}\hat{\mathfrak{v}}^{-1}\otimes\hat{P}_{\mathbf{k}}^2 + \frac{1}{2}k^2\alpha_o^{1/3}\hat{\mathfrak{v}}^{1/3}\otimes\hat{Q}_{\mathbf{k}}^2\,, \quad (3)$$

with $[\hat{Q}_{\mathbf{k}}, \hat{P}_{\mathbf{k}}] = i\hbar$. The field is massless.

Effective dynamics

I do not exactly solve Schrödinger-like equation. I move to so called *effective dynamics* (Bojowald, et al. 2005), here performed as follows. I assume there exist semi-classical coherent state $\Psi_{(v,b,Q_{\mathbf{k}},P_{\mathbf{k}})}$ peaked on the phase-space variables. I define effective Hamiltonian as
$H_{\text{eff}}(v,b,Q_{\mathbf{k}},P_{\mathbf{k}}) := \langle\Psi_{(v,b,Q_{\mathbf{k}},P_{\mathbf{k}})}|\left(\hat{H}_{\text{grav}} + \hat{H}_{\phi}\right)|\Psi_{(v,b,Q_{\mathbf{k}},P_{\mathbf{k}})}\rangle$.

I calculate expectation values of the operators, for example $v := \langle\hat{v}\rangle$. **The time dynamics of those correspond to quantum-corrected trajectories living on classical phase-space**. For example, $\dot{v} = \{v, H_{\text{eff}}\}$. Non-vanishing Poisson brackets are $\{b, v\} = \frac{2}{\hbar}$ and $\{Q_{\mathbf{k}}, P_{\mathbf{k}}\} = 1$.

I end up with the set of 4 coupled differential equations. Initial conditions are $(v_{\text{in}}, b_{\text{in}}, Q_{\mathbf{k}\,(\text{in})}, P_{\mathbf{k}\,(\text{in})})$. I set $b_{\text{in}}(0) = \pi/2$ so that the *bounce* will occur at $T = 0$.

Effective interior geometry

I write the effective metric **for the interior** as

$$\mathrm{d}s_-^2 = -\mathrm{d}T^2 + a(T)^2\mathrm{d}r^2 + r^2 a(T)^2\mathrm{d}\Omega^2\,, \tag{4}$$

where $a(T) = (\alpha_o \mathfrak{v})^{1/3}$.

I consider two cases

- No scalar field ($Q_\mathbf{k} = P_\mathbf{k} = 0$)

$$a(T) = (\alpha_o \mathfrak{v}(T))^{1/3} \to \left(\frac{9\pi^2 G^2 \hbar^2 \mathfrak{v}_{\text{in}}}{\alpha_o} T^2 + \mathfrak{v}_{\text{in}}\alpha_o\right)^{1/3} \tag{5}$$

- Single mode of the scalar field is included. Only numerical form of $a(T)$ is available. Different **k** induce different geometries (hence the name *rainbow*).

Now, the interior geometry is determined.

Extraction of the Exterior Geometry

Ingoing and outgoing Eddington-Finkelstein-like metric in the exterior region

I assume that the exterior is described by $x^\alpha = (v, X, \theta, \phi)$

$$\mathrm{d}s_+^2 = -F(X)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}X + X^2\mathrm{d}\Omega^2 \tag{6}$$

or $x^\alpha = (u, X, \theta, \phi)$

$$\mathrm{d}s_+^2 = -F(X)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}X + X^2\mathrm{d}\Omega^2 \tag{7}$$

with yet-to-be-extracted $F(X)$. **Special case of generalized Vaidya metric:** $F(v, X) = F(X)$, $F(u, X) = F(X)$

Exterior geometries need to be matched with

$$\mathrm{d}s_-^2 = -\mathrm{d}T^2 + a(T)^2\mathrm{d}r^2 + r^2a(T)^2\mathrm{d}\Omega^2\,, \tag{8}$$

Boundary Σ: the surface of the dust ball parametrized by $(T, r = r_b = \text{const.})$ and *from outside* by (v = V(T), X = R(T)).

Four-velocity of an observer comoving with the surface: $l^\alpha = \partial x^\alpha/\partial T$. Normal n^α to Σ: $n^\alpha n_\alpha = 1$ and $n^\alpha l_\alpha = 0$, chosen so that it points outwards.

Conditions for a smooth joining interior and exterior regions at Σ

- continuity of induced metrics: $h^-_{ab} = h^+_{ab}$
- continuity of extrinsic curvatures $K^-_{ab} = K^+_{ab}$, where $K_{ab} := n_{\alpha;\beta} e^\alpha_a e^\beta_b = n_{\alpha;\beta} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b}$

F(X)

From the the junction conditions, I have $F(T) = 1 - \dot{R}^2(T)$ and $X\Big|_\Sigma = R(T) = r_b a(T)$. From the latter it is possible to extract $T^2(X)$. Due to assumed symmetries I can write

$$F(X) = 1 - \dot{R}(T^2(X))^2 = 1 - \frac{2GM}{X} + \frac{2GM\alpha_0 r_b^3 \vartheta_{\text{in}}}{X^4}\,. \tag{9}$$

where the mass is $M = (4\pi/3)\rho R^3 = 2\pi^2 G\hbar^2 r_b^3 \vartheta_{\text{in}}/\alpha_o$.

Two roots X_- and X_+ for masses $M > M_{\text{ext}} := 16\alpha_o r_b^3/(9\sqrt{3}G^2\hbar) \approx 0.83\, m_{\text{Pl}}$, exactly one root for $M = M_{\text{ext}}$, and no roots for $M < M_{\text{ext}}$.

Restriction

I restrict the discussion to $M \gg M_{\text{ext}}$, because otherwise the dispersions of quantum observables are too high to maintain the reliability of the employed semi-classical framework.

I need to solve for $V(T)$ and $U(T)$ to fully determine the trajectory of the collapsing ball.

$$\dot{V}(T) = \frac{\mathrm{sgn}(T)\sqrt{1 - F(T)} + 1}{F(T)} \qquad \dot{U}(T) = \frac{-\mathrm{sgn}(T)\sqrt{1 - F(T)} + 1}{F(T)} \tag{10}$$

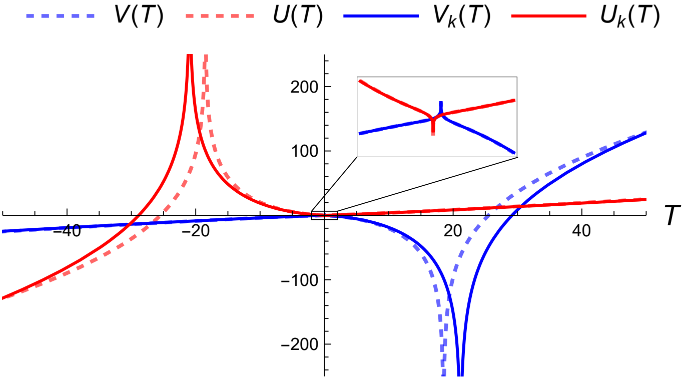
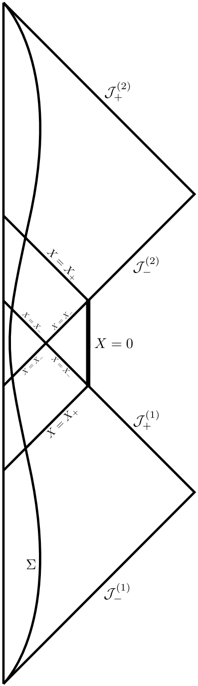


Figure: Initial conditions are $v_{\text{in}} = 10$, $b_{\text{in}} = \pi/2$, $Q_{\mathbf{k},(\text{in})} = P_{\mathbf{k},(\text{in})} = 0.3$, $V(0) = V_k(0) = 0$, $U(0) = U_k(0) = 0$ and also $k = G = c = \hbar = 1$, $\gamma = 0.23\ldots$.



Metric in double null coordinates

Comparing ingoing and outgoing exterior metric suggest a new coordinate $\mathrm{d}u := \mathrm{d}v - \frac{2}{F(X)}\mathrm{d}X$. Then the space-time is decribed with (u, v, Ω)

$$\mathrm{d}s_+^2 = -F(X)\mathrm{d}u\mathrm{d}v + X^2\mathrm{d}\Omega^2\,. \qquad (11)$$

Time-like singularity

Coordinate X is an affine parameter on null geodesics. Outgoing/ingoing null geodesics are reaching $X = 0$, where null condition and Kretschmann scalar diverge.

Entanglement entropy production

Intuition

Consider a test massless scalar field propagating in the space-time. An asymptotic observer at $\mathcal{I}^+$ is measuring the field in a state $|s\rangle$ for a finite time $A = [u_0, u]$. The results of all his measurements are coded in the reduced density matrix $\rho_A = \mathrm{Tr}_{\bar{A}} |s\rangle\langle s|$. The entanglement entropy is $S(A) = -\mathrm{Tr}_{\bar{A}} \left(\rho_A \log \rho_A\right)$ – **rather a formal writing**.

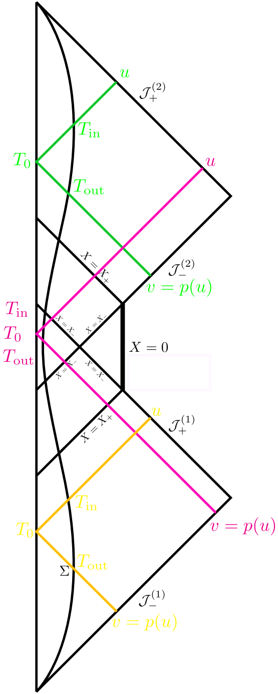
I calculate so-called *covariantly regularized entanglement entropy production*[1] (Bianchi, et al. 2014). It measures the entanglement of Hawking quanta produced in the collapse ("radiation entropy"). Specifically, characteristic up and down behaviour of the entropy corresponds to the Page curve.

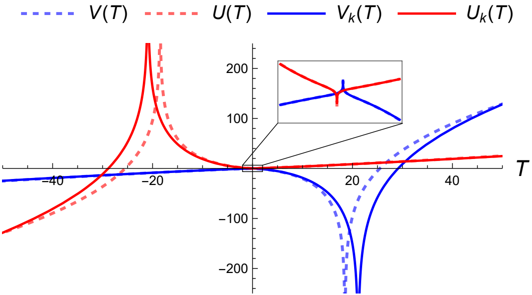
The workable formula for the entropy is

$$S(u) = \lim_{u_0 \to -\infty} \frac{1}{12} \log \frac{(p(u) - p(u_0))^2}{\dot{p}(u)\dot{p}(u_0)(u - u_0)^2} = -\frac{1}{12} \log \dot{p}(u) \,. \qquad (12)$$

The last equality is satisfied if $\lim_{u \to -\infty} \dot{p}(u) = 1$ holds. $p(u)$ is so-called ray-tracing map.

[1] This is a generalization of Holzhey-Larsen-Wilczek *geometric entropy* (Holzhey, et al., 1994)





3 ray-tracing maps $p(u)$: black hole region, transition region, white hole region.

$$S(u) = -\frac{1}{12}\log \dot{p}(u)\,. \qquad (13)$$

Information Paradox

Finitness of $S(u)$ for all times u is a necessary condition for unitarity of quantum evolution of a massless field (no paradox).

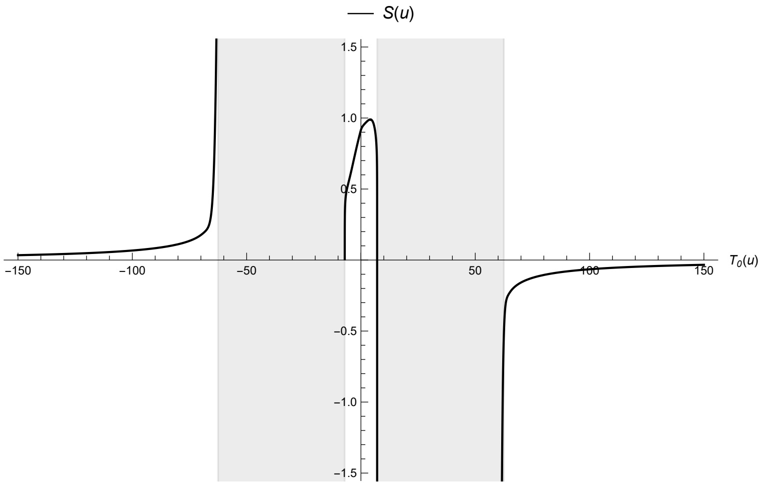


Figure: Parametric plot of $(T_0(u), S(u))$ – pure dustball collapse. $v_{\text{in}} = 10$.

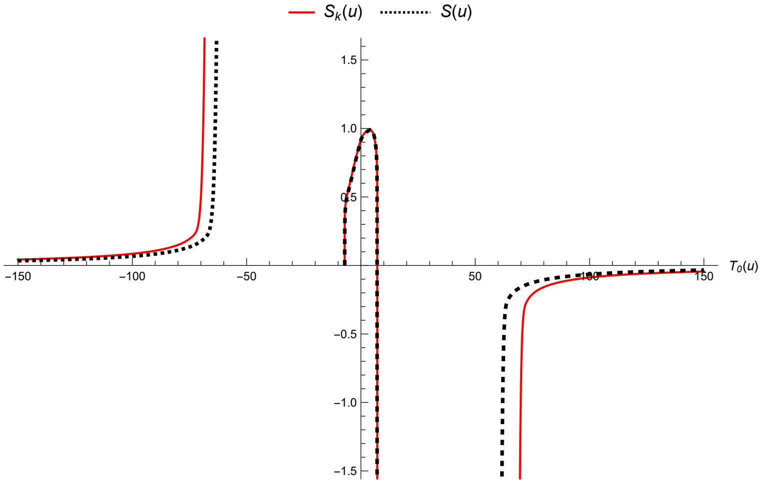


Figure: Parametric plot of $(T_0(u), S(u))$ and $(T_0(u), S_k(u))$ – pure dustball collapse and dustball + scalar field. $v_{\text{in}} = 10$, $k = 1$.

Conclusions

- I derived quantum-corrected exterior geometry in quantum gravitational collapse. The global picture of the collapse resembles the Reissner–Nordström space-time. The presented quantum dynamics of gravity mimics the repulsive character of collapsing charged matter in a classical theory.
- The black hole information paradox still exists in this model. This is manifestly visible from the time dynamics of the *entanglement entropy production*. For black hole and white regions, the time dynamics of the entropy looks similar to one obtained in *black hole fireworks* model (Haggard et. al. 2014).
- Incorporated perturbations from a single mode of the scalar field do not change the global picture. They effectively increase the mass of collapsing ball. Resulting singularity is problematic (especially when one includes a scalar field). The form of the exterior metric is determined by assumed staticity – this assumption may have been inappropriate.

This talk was supported in part by NCN grant OPUS 2020/37/B/ST2/03604.

Thank you for your attention!