Prescriptions and analytic control over quantum dynamics in loop quantum cosmology

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Outline

Classical theory

Phase Space

We are going to consider simplest setting: a model of a flat FLRW universe admitting the massless scalar field. Ashtekar-Barbero variables takes form:

$$A_i^a = cV_0^{-\frac{1}{3}}\delta_i^a \qquad E_a^i = pV_0^{-\frac{2}{3}}\delta_a^i$$

with Poisson Bracket:

$$\{c, p\} = \frac{8\pi}{3}G\gamma$$

Constraint

Due to symmetries and partial gauge fixing the Gauss and Diffeonorphism one are already satisfied. The only remaining is Hamiltonian one: one:

$$C = C_{gr}^{(E)} - 2(1+\gamma^2)C_{gr}^{(L)} + 8\pi G p^{-\frac{3}{2}} p_\phi^2$$

$$C_{gr}^{(E)} = \int_{\mathcal{V}} d^3x \epsilon_{ijk} \frac{1}{\sqrt{det(E)}} E^{ai}E^{bj}F_{ab}^k \qquad C_{gr}^{(L)} = \int_{\mathcal{V}} d^3x \frac{1}{\sqrt{det(E)}} E^{ai}E^{bj}K_{[a}^j K_{b]}^i$$

Quantization

On the kinematical level

Kinematical Hilbert space is a tensor product of $\mathcal{H}^{\phi} = L^2(\phi, d\phi)$ (the basic operators $\hat{\phi}$ and $\hat{p}_{\phi}$) and $\mathcal{H}^{gr} = L^2(\bar{\mathbb{R}}, d\mu)$ with elementary operators:

$$\hat{V}\,|v\rangle = \alpha|v|\,|v\rangle \qquad \hat{N}\,|v\rangle = |v+1\rangle$$

$$\alpha = 2\pi G\hbar\gamma\sqrt{\Delta} \qquad \hat{N} := e^{i\frac{\bar{\mu}c}{2}}$$

Of the constraint

Due to ambiguities of the regularization procedure, quantization of the gravity part of the constraint is not unique. So far three distinct examples of such have been proposed in the literature.

LQC and Strict Thiemann regularization

Constraint in LQC

- on classical level:

$$E^{ai}E^{bj}K^j_{[a}K^i_{b]} = \frac{1}{2\gamma^2}\epsilon_{ijk}E^{ai}E^{bj}(F^k_{ab} - \Omega^k_{ab})$$

  for flat model $\Omega^k_{ab} = 0$, then the Lorentzian part can be subsumed into the Euclidean part.
- $\hat{\Theta} = \sqrt{\hat{V}}\hat{C}_{gr}\sqrt{\hat{V}}$ acts on the state in volume representation as a second order difference operator and
- is a self-adjoint operator

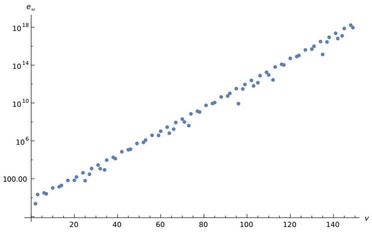
Constraint in Strict Thiemann regularization

- by using Thiemann's algorithm K^i_a can be obtained by a Poisson bracket which in cosmological settings (in improved dynamics scheme) reduce to:

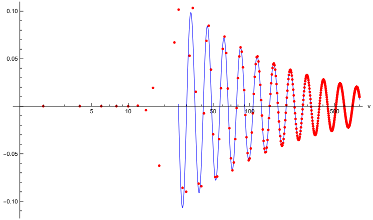
$$K^i_a = -\frac{2}{3\kappa\gamma^3\bar{\mu}} h_i^{(\bar{\mu})}\{(h_i^{(\bar{\mu})})^{-1}, \{C^E, V\}\}$$

- $\hat{\Theta}$ acts on the state in volume representation as a fourth order difference operator and
- admits multiple self-adjoint extensions

Yang-Ding-Ma regularization $K_a^i = \frac{1}{\gamma} A_a^i$



(a)



(b)

Figure: Plot of an example of a (generic) solution to an initial value problem corresponding to eigenvalue equation in volume representation (1a, here corresponding to $\omega = 10$) is compared against the actual element of the energy eigenbasis (1b, an example of $e_k(v)$ for $k = 20$). In the latter we observe the exponential suppression in classically forbidden (sub-bounce) region and a convergence to a WDW standing wave in large volume limit.

Expectation value of volume observable

With observation, that equation of the constraint can be formulated in form of Klein Gordon equation:

$$\partial_\phi^2 \Psi(x,\phi) = \beta^2 \partial_x^2 \Psi(x,\phi)$$

and by knowing the form of normalization constant we are able to write down wave function as:

$$[\mathcal{P}^\pm \psi](x) = \frac{1}{2\sqrt{2}} \int_0^{+\infty} dk \, \frac{\psi(k)}{\sqrt{k}} e^{\pm ikx} e^{i\beta k\phi}$$

and $\hat{V}$ could be rewritten:

$$\hat{V} = -iAf(x)\partial_x(P^+ - P^-)$$

we obtained general formula to probe Loop Quantum Cosmology system where constant A and β alongside with function f is define via model we considering.

In th energy k representation expectation value becomes:

$$\langle \hat{V}_\phi \rangle = A \sum_{l=0}^{\infty} \frac{1}{l!} f^{(l)}(\beta\phi) \langle \sqrt{k}(i\partial_k)^l \sqrt{k} \rangle$$

Expectation value of the product of operators

The expectation value of the product of operators, in general, is not equal to products of expectation values, to describe the system completely after quantization we need moments defined as:

$$\tilde{G}^{a,b} := \sum_{k=0}^{a} \sum_{l=0}^{b} (-1)^{(a+b)-(k+l)} \binom{a}{k} \binom{b}{l} \langle \hat{k} \rangle^{a-k} \langle i\partial_k \rangle^{b-l} \tilde{F}^{k,l}$$

$$\tilde{F}^{a,b} := \langle \hat{k}^a (i\partial_k)^b \rangle_{Weyl}$$

and to be allowed to put it to good use we have to strengthen our restrictions on the studied semiclassical state, further requiring that:

- All the expectation values $\langle \hat{k} \rangle$, $\langle i\partial_k \rangle$, $\langle \hat{k}^a (i\partial_k)^b \rangle_{Weyl}$ are finite.
- The state is peaked at large "energy" $\langle \hat{k} \rangle \gg 1$.

Remark

After quantization of product of classical variables ordering of corresponding operators is not unique. Because of that, relation of general arbitrary ordering x and Weyl ordering is required.

Using Central Moments

Rewriting product of the operator in expectation value of Volume as a Weyl ordering

$$\sqrt{k}(i\partial_k)^l\sqrt{k} =: k(i\partial_k)^l : + \frac{1}{2}[\sqrt{k},[(i\partial_k)^l,\sqrt{k}]]$$

In the next step, we need to deal with the double commutator appearing above. In order to do so we recall that since we are considering semiclassical states peaked at large k we can estimate the expression by its leading term:

$$[\sqrt{k},[(i\partial_k)^l,\sqrt{k}]] = \frac{l(l-1)}{8}\left((i\partial_k)^{l-2}\frac{1}{k} + \frac{1}{k}(i\partial_k)^{l-2}\right) + O(\langle k\rangle^{-2}) \ .$$

Treating $\widehat{1/k}$ as a composite operator we add one more additional restriction:

$$\forall_{n\in\mathbb{N}} \ \langle\widehat{k^{-n}}\rangle < \infty$$

to be able to convert it to a series involving constant k_0 which will lead to the expectation value of a double commutator in simple symmetry ordering

$$\frac{1}{2}\langle[\sqrt{k},[(i\partial_k)^l,\sqrt{k}]]\rangle = \frac{l(l-1)}{16}\frac{1}{k_0}\sum_{n=0}^{\infty}\sum_{a=0}^{n}\frac{(-1)^{n+a}}{k_0^{n-a}}\binom{n}{a}\langle k^{n-a}(i\partial_k)^{l-2}\rangle_s$$

Result

In order to obtain a final result we make the following observation:

- to be able to use expansion in terms of central moments simple symmetry ordering has been rewritten into Weyl symmetry ordering,
- because we operate on the constant of motion we can set $k_0 = \langle k \rangle$ which simplifies our considerations,
- by thus we have at our disposal the trajectory of volume at given ϕ expressed in terms of the explicit functions of ϕ, expectation values $\langle k \rangle$, $\langle i\partial_k \rangle$ and the auxiliary central moments $\tilde{G}^{a,b}$,
- what is important here k is in fact dimensionless and the physical interpretation of the operators is not immediately obvious

The last issue is however easy to fix:

$$\hat{p}_\phi = \hbar\beta\hat{k} \qquad \hat{\phi}_0 := i\beta^{-1}\partial_k \qquad G^{ab} = \hbar^a\beta^{a-b}\tilde{G}^{a,b}$$

By thus, the trajectory of the volume in physical variables reads

$$\langle \hat{V}_\phi \rangle = \frac{A}{\hbar}\sum_{i=0}^{\infty}\frac{\beta^{i-1}}{i!}f^{(i)}(\beta(\phi-\phi_0))\left(p_\phi G^{0,i} + G^{1,i}\right) + O({p_\phi}^{-1})$$

where again only information that we need to know about the considered model is the form of the function f and value of constant A and β. By similar reasoning, we obtained a variance given by the standard deviation.

Application

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To check the validity of these results in the lowest order of central moments we should obtain classical trajectories which should be consistent with predictions of effective dynamics. From what we checked for Yang-Ding-Ma regularization:

$$\begin{aligned}\langle \hat{V}_\phi \rangle = & \frac{\gamma\sqrt{\pi G \Delta}}{\sqrt{3(\gamma^2+1)}} p_\phi \left(\gamma^2 \tanh^2(\beta(\phi-\phi_0))+1\right)\cosh(\beta(\phi-\phi_0)) \\ & + \frac{2\pi\gamma G\sqrt{\Delta}}{\sqrt{\gamma^2+1}}\left((\gamma^2+1)\sinh(\beta(\phi-\phi_0))+\gamma^2\frac{\sinh(\beta(\phi-\phi_0))}{\cosh^2(\beta(\phi-\phi_0))}\right) G^{1,1} \\ & + \frac{\gamma(\pi G)^{3/2}\sqrt{\Delta}}{\sqrt{\gamma^2+1}}\left((\gamma^2+1)\cosh(\beta(\phi-\phi_0))-\gamma^2\frac{\cos^2(\beta(\phi-\phi_0))-2}{\cos^3(\beta(\phi-\phi_0))}\right)\left(p_\phi G^{0,2}+G^{1,2}\right) \\ & + \frac{2\pi\gamma G\sqrt{\Delta}}{\sqrt{\gamma^2+1}}\sum_{i=3}^{\infty}\frac{(12\pi G)^{(i-1)/2}}{i!} f^{(i)}(\beta(\phi-\phi_0))\left(p_g G^{0,i}+G^{1,i}\right)+O(p_\phi^{-1})\end{aligned}$$

the first line recreates effective dynamics and the rest are higher-order quantum corrections. The same will be true as well for mainstream LQC in which in leading terms we will recreate cosh dependence on a scalar field and even in Geometrodynamics when we recreate exponential behaviour

Summary

In our work:

- we investigated properties of operator Θ in for Yang-Ding-Ma regularization,
- to probe its dynamics we propose an analytical method based on evaluating the desired expectation values on the variable classically corresponding to the canonical momentum of the volume and transforming the result to a form of an explicit function of the scalar field "time" and a series of the so-called central moments built off constants of motions
- such a method allows us to have good control over quantum corrections to an arbitrary order
- it is easily applied for a wide variety of models, either in LQC or geometrodynamics and allows to control of dynamics of the quantum state without resorting to numerical methods

Thank you for your attention