Energy and entropy in the Geometrical Trinity of gravity

In collaboration with J. B. Jiménez, T. S. Koivisto (Phys. Rev. D 107, 024044, 2023)

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Motivation

Noether Theorem

Symmetry $\rightarrow$ Conserved Charges

What are the conserved charges for gravity?

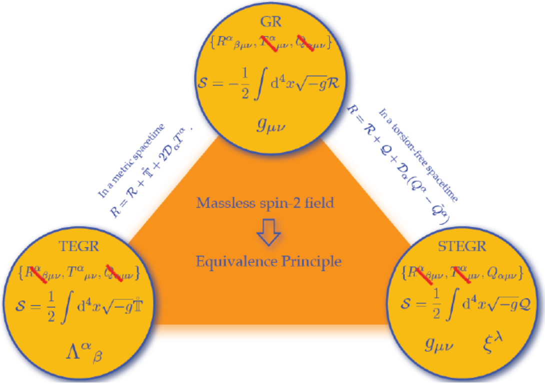

Figure: J. B. Jimenez, L. Heisenberg, T. S. Koivisto, Universe 2019, 5(7), 173.

Notation and conventions

A general connection can be decomposed as

$$\Gamma^{\alpha}{}_{\mu\nu} = \mathring{\Gamma}^{\alpha}{}_{\mu\nu} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}\,, \tag{1}$$

where the contortion and disformation are defined as

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2} T^{\alpha}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\alpha}\,, \tag{2a}$$

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2} Q^{\alpha}{}_{\mu\nu} - Q_{(\mu\nu)}{}^{\alpha}\,. \tag{2b}$$

The general connection is associated to a general covariant derivative ∇_{α}, which defines the non-metricity $Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu}$. From the general connection, the curvature tensor and torsion can be defined

$$R^{\alpha}{}_{\beta\mu\nu} = 2\partial_{[\mu}\Gamma^{\alpha}{}_{\nu]\beta} + 2\Gamma^{\alpha}{}_{[\mu|\lambda|}\Gamma^{\lambda}{}_{\nu]\beta}\,, \tag{3a}$$

$$T^{\alpha}{}_{\mu\nu} = 2\Gamma^{\alpha}{}_{[\mu\nu]}\,. \tag{3b}$$

From the torsion, we can define the torsion scalar

$$T = \frac{1}{4} T_{\alpha\mu\nu} T^{\alpha\mu\nu} + \frac{1}{2} T_{\mu\alpha\nu} T^{\alpha\mu\nu} - T_{\alpha} T^{\alpha}, \tag{4}$$

where $T_{\alpha} = T^{\beta}{}_{\alpha\beta}$ is the trace of the torsion. The Levi-Civita connection and the torsion scalar are related by

$$\mathring{R} = -T - 2\mathring{\nabla}_{\alpha} T^{\alpha}. \tag{5}$$

From the non-metricity, we can define the non-metricity scalar

$$Q = \frac{1}{4} Q^{\alpha\beta\gamma} Q_{\alpha\beta\gamma} - \frac{1}{2} Q^{\alpha\beta\gamma} Q_{\beta\alpha\gamma} + \frac{1}{2} Q_{\alpha} \tilde{Q}^{\alpha} - \frac{1}{4} Q_{\alpha} Q^{\alpha}, \tag{6}$$

where $Q_\alpha = Q_{\alpha\mu}{}^{\mu}$ and $\tilde{Q}^{\mu} = Q_{\alpha}{}^{\mu\alpha}$ are the non-metricity traces. The Levi-Civita connection and the non-metricity scalar are related by como

$$\mathring{R} = -Q - \mathring{\nabla}_{\alpha}(Q^{\alpha} - \tilde{Q}^{\alpha}). \tag{7}$$

Equations of Motion

We consider a Langrangian $L_G = L_G(g^{\mu\nu}, Q_\lambda{}^{\mu\nu}, T^\alpha{}_{\mu\nu}, R^\alpha{}_{\mu\lambda\nu})$. The variation of the action

$$I = \int \mathrm{d}^n x \sqrt{-\mathfrak{g}} L \,. \tag{8}$$

gives

$$\begin{aligned} E_{\mu\nu} &= -\frac{1}{2} T_{\mu\nu} + \hat{\nabla}_\alpha q^\alpha{}_{\mu\nu} + \frac{\partial L_G}{\partial g^{\mu\nu}} - \frac{1}{2} L_G g_{\mu\nu} \,, & \text{(9a)} \\ \frac{1}{2} E_\alpha{}^{\mu\nu} &= -\frac{1}{2} Z_\alpha{}^{\mu\nu} + \hat{\nabla}_\beta r_\alpha{}^{\nu\mu\beta} + \frac{1}{2} T^\mu{}_{\beta\gamma} r_\alpha{}^{\nu\beta\gamma} \\ &+ t_\alpha{}^{\mu\nu} - q^{\mu\nu}{}_\alpha \,, & \text{(9b)} \end{aligned}$$

where $\hat{\nabla}_\mu = \nabla_\mu + T_\mu + \frac{1}{2} Q_\alpha$ and

$$q^\alpha{}_{\mu\nu} = \frac{\partial L_G}{\partial Q_\alpha{}^{\mu\nu}}, \quad t_\alpha{}^{\mu\nu} = \frac{\partial L_G}{\partial T^\alpha{}_{\mu\nu}}, \quad r_\alpha{}^{\beta\mu\nu} = \frac{\partial}{\partial R^\alpha{}_{\beta\mu\nu}} \,.$$

Noether Current

For a diffeomorphism, the Noether current is given by:

$$\begin{aligned} J^\mu &= -2q^\mu{}_{\alpha\beta}\mathring{\nabla}^\alpha v^\beta + 2E^\mu{}_\nu v^\nu \\ &- 2r_\alpha{}^{\nu\mu\beta}\left[\nabla_\beta\nabla_\nu v^\alpha + \nabla_\beta T^\alpha{}_{\gamma\nu}v^\gamma + R^\alpha{}_{\nu\gamma\beta}v^\gamma\right] \\ &- E_\alpha{}^{\mu\nu}\nabla_\nu v^\alpha + \hat{\nabla}_\nu E_\alpha{}^{\nu\mu}v^\alpha + T^\alpha{}_{\nu\beta}E_\alpha{}^{\mu\nu}v^\beta \\ &+ Lv^\mu + \frac{\partial L_M}{\partial\nabla_\mu\psi}\delta_v\psi\,. \end{aligned} \tag{10}$$

There exists an antisymmetric 2nd rank tensor $J^{\mu\nu} = J^{[\mu\nu]}$ such that $J^\mu = \mathring{\nabla}_\nu J^{\mu\nu}$. The Noether potential for Palatini-Einstein Lagrangian leads to the Komar superpotential:

$$J^{\mu\nu}_G = m_P^2\mathring{\nabla}^{[\mu}v^{\nu]}\,. \tag{11}$$

It has been proposed that the Komar superpotential should be modified as:

$$J_G^{\mu\nu} = m_P^2 \mathring{\nabla}^{[\mu} v^{\nu]} + m_P^2 A^{[\mu} v^{\nu]}. \tag{12}$$

When we consider the symmetric teleparallel equivalent to GR, we arrive at

$$J^{\mu\nu} = m_P^2 \mathring{\nabla}^{[\mu} v^{\nu]} - Q^{[\mu} v^{\nu]} + \tilde{Q}^{[\mu} v^{\nu]}, \tag{13}$$

with $A^\mu = -Q^\mu + \tilde{Q}^\mu$.

space of connections	Noether potential $J^{\mu\nu}$
all	$\nabla^{[\mu} v^{\nu]} - h^{\mu\nu}{}_{\alpha} v^{\alpha}$
flat	$-h^{\mu\nu}{}_{\alpha} v^{\alpha}$
symmetric	inequivalent to GR
flat & symmetric	$-2q^{[\mu\nu]}{}_{\alpha} v^{\alpha}$

In electromagnetism, we have the charge

$$q = \frac{1}{2} \oint_{\partial \mathcal{V}} \mathrm{d}^2 \sigma_{\mu\nu} \sqrt{-\mathfrak{g}} A^{\mu\nu} \,, \tag{14}$$

where $\nabla_\mu(\sqrt{g} F^{\mu\nu}) = \sqrt{g} J^\mu$.
The energy-momentum will be defined as:

$$C_\alpha = \frac{1}{2} \oint_{\partial \mathcal{V}} \mathrm{d}^2 \sigma_{\mu\nu} \mathfrak{h}^{\mu\nu}{}_\alpha \,, \tag{15}$$

where the superpotential obeys

$$\nabla_\alpha \mathfrak{h}^{\alpha\mu}{}_\nu = \mathfrak{T}^\mu{}_\nu + \mathfrak{G}^\mu{}_\nu \,. \tag{16}$$

Excitation Tensors

Examples of excitation tensors that have been proposed:

- The von Freud superpotential [1]

$$\underline{\mathfrak{h}}_{vF}{}^{\mu\nu}{}_{\alpha} = -\frac{1}{2}\sqrt{-\mathfrak{g}}\,\delta^{\mu\nu\gamma}_{\lambda\sigma\alpha}\, g^{\beta\lambda}\left(\mathring{\Gamma}^{\lambda}{}_{\beta\gamma} - \underline{\mathring{\Gamma}}^{\lambda}{}_{\beta\gamma}\right); \tag{17}$$

- Landau and Lifshitz superpotential[2]

$$\underline{\mathfrak{h}}^{\mu\nu\alpha}_{LL} = \frac{1}{2}(-\underline{\mathfrak{g}})^{-\frac{1}{2}}\,\delta^{\mu\nu}_{\gamma\rho}\,\underline{\mathring{\nabla}}_{\beta}\left(-\mathfrak{g}\, g^{\rho\alpha} g^{\gamma\beta}\right) \tag{18}$$

- Bergmann and Thomson superpotential [3]

$$\underline{\mathfrak{h}}^{\mu\nu\alpha}_{BT} = \sqrt{\underline{g}/g}\;\underline{\mathfrak{h}}^{\mu\nu\alpha}_{LL} \tag{19}$$

- Papapetrou superpotential [4]

$$\underline{\mathfrak{h}}^{\mu\nu\alpha}_{P} = \delta^{\mu\nu}_{\gamma\lambda}\,\delta^{\alpha\rho}_{\beta\sigma}\,\underline{g}^{\lambda\beta}\sqrt{-\mathfrak{g}}\left(\frac{1}{4}g^{\gamma\sigma}g^{\tau\delta} - \frac{1}{2}g^{\gamma\tau}g^{\sigma\delta}\right)\underline{\mathring{\nabla}}_{\rho}\, g_{\tau\sigma} \tag{20}$$

- Weinberg superpotential [5]

$$\underline{\mathfrak{h}}^{\mu\nu\alpha}_{W} = \delta^{\mu\nu}_{\gamma\lambda}\,\delta^{\alpha\rho}_{\beta\sigma}\,\underline{g}^{\lambda\beta}\sqrt{-\underline{\mathfrak{g}}}\left(\frac{1}{4}\underline{g}^{\gamma\sigma}\underline{g}^{\tau\delta} - \frac{1}{2}\underline{g}^{\gamma\tau}\underline{g}^{\sigma\delta}\right)\underline{\mathring{\nabla}}_{\rho}\, g_{\tau\sigma}\,. \tag{21}$$

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[1] P. Freud, Annals of Mathematics 40 (1939) 417–419.

[2] L. D. Landau and E. M. Lifschits, The Classical Theory of Fields, vol. Volume 2 of Course of Theoretical Physics. Pergamon Press, Oxford, 1975.

[3] P. G. Bergmann, R. Thomson, Phys. Rev. 89 (1953) 400–407

[4] A. Papapetrou, Proc. Roy. Irish Acad. A 52 (1948) 11–23.

[5] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. John Wiley and Sons, New York, 1972.

After taking $\underline{g}_{\mu\nu} \rightarrow g_{\mu\nu}$ and $\underline{\mathring{\nabla}}_\alpha \rightarrow \nabla_\mu$, we can show that the previous expressions are equivalent

$$h^{\mu\nu}_{vF\,\alpha} = h^{\mu\nu}_{LL\,\alpha} = h^{\mu\nu}_{BT\,\alpha} = h^{\mu\nu}_{P\ \ \alpha} = h^{\mu\nu}_{W\,\alpha} = h^{\mu\nu}{}_{\alpha}\,. \tag{22}$$

The inertial frame is characterised by the vanishing of the energy-momentum associated with the metric field, i.e. $t^\mu{}_\nu = 0$. In this case, $\nabla_\alpha \mathfrak{h}^{\alpha\mu}{}_\nu = \mathfrak{T}^\mu{}_\nu$.

formulation	constraints	superpotential	canonical frame
symm. tele$_\parallel$	$R^\alpha{}_{\beta\mu\nu} = T^\alpha{}_{\mu\nu} = 0$	$m_P^{-2} h^{\mu\nu}{}_\alpha = \delta^{[\mu}_\alpha \tilde{Q}^{\nu]} - \delta^{[\mu}_\alpha Q^{\nu]} - Q^{[\mu\nu]}{}_\alpha$	$t^\mu{}_\nu = q^\mu{}_{\alpha\beta} Q_\nu{}^{\alpha\beta} - \frac{1}{2}\delta^\mu_\nu q^\alpha{}_{\beta\gamma} Q_\alpha{}^{\beta\gamma} = 0$
metric tele$_\parallel$	$R^\alpha{}_{\beta\mu\nu} = Q_\alpha{}^{\mu\nu} = 0$	$m_P^{-2} t_\alpha{}^{\mu\nu} = \frac{1}{2} T^\mu{}_\alpha{}^\nu + T^{[\mu\nu]}{}_\alpha + 2\delta^{[\mu}_\alpha T^{\nu]}$	$t^\mu{}_\nu = 2 t_\alpha{}^{\beta\mu} T^\alpha{}_{\nu\beta} - \frac{1}{2}\delta^\mu_\nu t_\alpha{}^{\beta\gamma} T^\alpha{}_{\beta\gamma} = 0$
Palatini	–	$m_P^{-2} h^{\mu\nu}_{K\ \alpha} = \mathring{\nabla}^{[\mu} \delta^{\nu]}_{\hat{\alpha}}$	$t^\mu_{K\nu} \stackrel{?}{=} \frac{m_P^2}{2}\left[\left(\mathring{R} - \mathring{\Box}\right)\delta^\mu_{\hat{\nu}} + \left(2\mathring{\nabla}^\mu \mathring{\nabla}_\alpha - \mathring{\nabla}_\alpha \mathring{\nabla}^\mu\right)\delta^\alpha_{\hat{\nu}}\right]$

Charged black hole

We have $C_\mu = 4\pi m_P^2 V r \ell_\mu$, where where V is a scalar function of r and ℓ_μ is a null geodesic vector. If we will consider the Schwarzschild-Reissner- Nordström-de Sitter, then

$$V(r) = \frac{m_S}{4\pi m_P^2 r} - \frac{q^2}{8\pi m_P^2 r^2} + \frac{\Lambda}{3} r^2 \,. \tag{23}$$

We have

$$C_0 = m_S - \frac{q^2}{2r} + \frac{4}{3}\pi r^3 \rho_\Lambda \,, \tag{24}$$

which can be interpreted as the gravitational energy.
In the inertial frame, $C_0^{vF} = C_0^{LL} = C_0^{BT} = C_0^{P} = C_0^{W} = C_0$.

Entropy

- Wald formula (diffeomorphism charge) [6]

$$S = \frac{2\pi}{\kappa} \oint_{\mathcal{H}} \mathrm{d}^{n-2}\sigma_{\mu\nu}\mathfrak{h}^{\mu\nu}{}_{\alpha} v^{\alpha} \,. \tag{25}$$

- Center of mass momentum charge (Lorentz charge)[7]

$$S = -2\pi \int_{\mathcal{C}} \mathrm{d}^2 x \mathfrak{r}_{\alpha}{}^{\beta\mu\nu} n^{\alpha}{}_{\beta} n_{\mu\nu} \,, \tag{26}$$

Both expressions lead to the correct area law.

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[6] R. M. Wald, Phys. Rev. D 48 (1993) 3427–3431,

[7] J. B, Jimenez, T. S. Koivisto, Phys. Rev. D 105 (2022) L021502

Conclusions

- In STGR, the Noether charge for diffeomorphisms yields the modified Komar expression;
- In the inertial frame, the ambiguity of the energy-momentum is eliminated;
- We obtain the desired results for the entropy.

Thank you!