Quasinormal modes in teleparallel gravity

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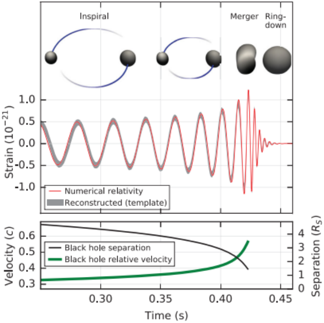
Gravitational waves and quasinormal modes

Normal modes: Classical oscillating system where energy is conserved.

$$\chi(t,x) = \sum_{n=1}^{\infty} a_n e^{i\omega_n t} \chi_n(x) \tag{1}$$

Quasinormal modes: The system loses energy and the oscillations decay in time.

Detection of gravitational waves in 2015. [1]
Three stages for the GW signal: 1) Inspiral 2) Merger 3) Ringdown



[1] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), "Observation of Gravitational Waves from a Binary Black Hole Merger", Phys. Rev. Lett. 116, 061102 (2016)

Black hole perturbations in GR [2]

Einstein equations

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{2}$$

In vacuum:

$$R_{\mu\nu} = 0 \tag{3}$$

Schwarzschild background

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{4}$$

Small perturbation of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{5}$$

Perturbed Christoffel symbols

$$\Gamma^{\rho}{}_{\mu\nu} = \bar{\Gamma}^{\rho}{}_{\mu\nu} + \delta\Gamma^{\rho}{}_{\mu\nu}, \quad \delta\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\kappa\alpha}(\bar{\nabla}_{\mu}h_{\nu\rho} + \bar{\nabla}_{\nu}h_{\mu\rho} - \bar{\nabla}_{\rho}h_{\mu\nu}) \tag{6}$$

Perturbed Ricci tensor

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}, \quad \boxed{\delta R_{\mu\nu} = \bar{\nabla}_{\rho}\delta\Gamma^{\rho}{}_{\nu\mu} - \bar{\nabla}_{\nu}\delta\Gamma^{\rho}{}_{\rho\mu}} \tag{7}$$

[2] T. Regge and J. A. Wheeler, "Stability of a Schwarzschild Singularity", Phys. Rev. 108, 1063–1069 (1957)

Metric perturbation $h_{\mu\nu}$

- Sperically symmetric backgound

✓ Decomposition into separate functions of (t, r) and (θ, ϕ).

Spatial part: Harmonic time dependence $F(t, r) = F(r)e^{-i\omega t}$

Angular part: Spherical harmonics $Y_{lm} = Y_{lm}(\theta, \phi) \quad (l = 0, 1, 2, ... - l \leq m \leq l)$

$$\nabla^2 Y_{lm} = \frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial Y_{lm}}{\partial\theta}) + \frac{1}{sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial\phi^2} = -l(l+1)Y_{lm}, \tag{8}$$

✓ We can set $m = 0$.

- Invariance under spatial rotations $(\theta, \phi) \to (\pi - \theta, \pi + \phi)$

$$h_{\mu\nu} = \begin{pmatrix} S & S & V & V \\ S & S & V & V \\ V & V & T & T \\ V & V & T & T \end{pmatrix} \tag{9}$$

Factor $(-1)^{(l+1)}$ for axial parity and $(-1)^l$ for polar parity.

- Regge-Wheeler gauge

Axial: $\xi^\mu = (0, 0, \Lambda(t, r)\epsilon^{ab}\partial_b)Y_{lm}$, Polar: $\xi^\mu = (M_0(t, r), M_1(t, r), M_2(t, r)\gamma^{ab}\partial_b)Y_{lm}$

$$h_{\mu\nu} = h_{\mu\nu}^{(axial)} + h_{\mu\nu}^{(polar)} \tag{10}$$

$$h_{\mu\nu}^{(axial)} = \begin{pmatrix} 0 & 0 & -h_0(t,r)\frac{1}{sin\theta}\partial_\phi & h_0(t,r)sin\theta\partial_\theta \\ 0 & 0 & -h_1(t,r)\frac{1}{sin\theta}\partial_\phi & h_1(t,r)sin\theta\partial_\theta \\ \text{Sym} & \text{Sym} & h_2(t,r)(\frac{1}{sin\theta}\partial^2_{\theta\phi} - \frac{cos\theta}{sin^2\theta}\partial_\phi) & -h_2(t,r)sin\theta W_{\theta\phi} \\ \text{Sym} & \text{Sym} & \text{Sym} & -h_2(t,r)sin\theta X_{\theta\phi} \end{pmatrix} Y_{LM} \tag{11}$$

$$h_{\mu\nu}^{(polar)} = \begin{pmatrix} B(r)H_0(t,r) & H_1(t,r) & c_0(t,r)\partial_\theta & c_0(t,r)\partial_\phi \\ \text{Sym} & \frac{H_2(t,r)}{B(r)} & c_1(t,r)\partial_\theta & c_1(t,r)\partial_\phi \\ \text{Sym} & \text{Sym} & r^2(K(r) + G(t,r)\partial_{\theta\theta}) & \frac{1}{2}r^2 G(t,r)X_{\theta\phi} \\ \text{Sym} & \text{Sym} & \text{Sym} & r^2 sin^2\theta(K(t,r) + G(t,r)t(\partial_{\theta\theta} - W_{\theta\phi}) \end{pmatrix} Y_{LM} \tag{12}$$

Harmonic time dependence, $m = 0$, Regge-Wheeler gauge $\rightarrow$

$$h_{\mu\nu}^{(axial)} = \begin{pmatrix} 0 & 0 & 0 & h_0(r)sin\theta\partial_\theta \\ 0 & 0 & 0 & h_1(r)sin\theta\partial_\theta \\ 0 & 0 & 0 & 0 \\ \text{Sym} & \text{Sym} & 0 & 0 \end{pmatrix} e^{-i\omega t} Y_{lm} \tag{13}$$

$$h_{\mu\nu}^{(polar)} = \begin{pmatrix} B(r)H_0(r) & H_1(r) & 0 & 0 \\ \text{Sym} & \frac{H_2(r)}{B(r)} & 0 & 0 \\ 0 & 0 & r^2 K(r) & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta K(r) \end{pmatrix} e^{-i\omega t} Y_{lm} \tag{14}$$

Non-trivial field equations

Axial

$$\delta R_{23} = B\frac{dh_1}{dr} + \frac{i\omega}{B}h_0 + \frac{2M}{r^2}h_1 = 0$$

$$\delta R_{13} = \frac{i\omega}{B}\frac{dh_0}{dr} - \frac{2i\omega}{rB}h_0 + \left(\frac{l(l+1)}{r^2} - \frac{2}{r^2} - \frac{\omega^2}{B}\right)h_1 = 0$$

$$\delta R_{03} = B\frac{d^2h_0}{dr^2} + i\omega B\frac{dh_1}{dr} - \left(\frac{l(l+1)}{r^2} - \frac{4M}{r^3}\right)h_0 + \frac{2i\omega B}{r}h_1 = 0 \tag{15}$$

Polar

$$\delta R_{01} = \frac{dK}{dr} + \frac{l(l+1)}{2i\omega r^2}H_1 + \frac{r-3M}{r^2B}K - \frac{1}{r}H = 0$$

$$\delta R_{02} = B\frac{dH_1}{dr} + \frac{2M}{r^2}H_1 + i\omega(K+H) = 0$$

$$\delta R_{12} = B\frac{dH}{dr} - B\frac{dK}{dr} + i\omega H_1 + \frac{2M}{r^2}H = 0$$

$$\delta R_{00} = B^2\frac{d^2H}{dr^2} + 2i\omega B\frac{dH_1}{dr} - \frac{2MB}{r^2}\frac{dK}{dr} + \frac{2B}{r}\frac{dH}{dr} + 2i\omega(2r-3M)r^2H_1 - 2\omega^2K - \left(\omega^2 + \frac{l(l+1)B}{r^2}\right)H = 0$$

$$\delta R_{11} = B^2\frac{d^2H}{dr^2} - 2B^2\frac{d^2K}{dr^2} + 2i\omega B\frac{dH_1}{dr} - \frac{2B(2r-3M)}{r^2}\frac{dK}{dr} + \frac{2}{r}\frac{dH}{dr} + \frac{2i\omega M}{r^2}H_1 - \left(\omega^2 - \frac{l(l+1)B}{r^2}\right)H = 0$$

$$\delta R_{22} = r^2B\frac{d^2K}{dr^2} + (4r-6M)\frac{dK}{dr} - 2rB\frac{dH}{dr} - 2i\omega rH_1 + \left(2 + \frac{\omega^2r^2}{B} - l(l+1)\right)K - 2H = 0 \tag{16}$$

Metric affine theories of gravity

An alternative to dark matter and dark energy.

Metric $g_{\mu\nu}$ and affine connection $\Gamma^{\rho}{}_{\mu\nu}$

Curvature

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho} \tag{17}$$

Torsion

$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} - \Gamma^{\mu}{}_{\rho\nu} \tag{18}$$

Non-metricity.

$$Q_{\mu\nu\rho} = \nabla_{\mu} g_{\nu\rho} = \partial_{\mu} - \Gamma^{\sigma}{}_{\nu\mu} g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu} g_{\nu\sigma} \tag{19}$$

In GR: The gravitational field is mediated by the curvature

$$R^{\mu}{}_{\nu\rho\sigma} \neq 0, \quad T^{\mu}{}_{\nu\rho} = 0, \quad Q_{\mu\nu\rho} = 0 \tag{20}$$

In Metric Teleparallel Gravity: The gravitational field is mediated by the torsion

$$T^{\mu}{}_{\nu\rho} \neq 0, \quad R^{\mu}{}_{\nu\rho\sigma} = 0, \quad Q_{\mu\nu\rho} = 0 \tag{21}$$

Metric Teleparallel Gravity

Fundamental variables: Tetrad $\theta^a{}_\mu$, inverse e_a and spin connection $\omega^A{}_{B\mu}$

$$\theta^a = \theta^a{}_\mu dx^\mu, \quad e_a = e_a{}^\mu \partial_\mu, \quad \omega^A{}_{B\mu} = \Lambda^a{}_c \partial_\mu \Lambda_b^c \tag{22}$$

$$\theta^a{}_\mu e_b{}^\mu = \delta^a_b, \quad \theta^a{}_\mu e_a{}^\nu = \delta^\nu_\mu \tag{23}$$

$$g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu \theta^b{}_\nu, \quad g^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu \tag{24}$$

Connection

$$\Gamma^\rho{}_{\mu\nu} = e_a{}^\rho(\partial_\nu \theta^a{}_\mu + \omega^a{}_{b\nu}\theta^b{}_\mu) \tag{25}$$

Weitzenbock gauge $\omega^A{}_{B\mu} = 0$

$$\Gamma^\rho{}_{\mu\nu} = e_a{}^\rho \partial_\nu \theta^a{}_\mu \tag{26}$$

Irreducible decomposition of torsion

$$T_{\rho\mu\nu} = \frac{2}{3}(\mathfrak{t}_{\rho\mu\nu} - \mathfrak{t}_{\rho\nu\mu}) + \frac{1}{3}(g_{\rho\mu}\mathfrak{v}_\nu - g_{\rho\nu}\mathfrak{v}) + \epsilon_{\rho\mu\nu\kappa}\mathfrak{a}^\kappa \tag{27}$$

$$\mathfrak{v}_\mu = T^\rho{}_{\rho\mu}, \quad \mathfrak{a}_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma} \quad \mathfrak{t}_{\mu\nu\rho} = T_{(\mu\nu)\rho} + \frac{1}{3}(g_{\rho(\mu}\mathfrak{v}_{\nu)} - g_{\mu\nu}\mathfrak{v}_\rho) \tag{28}$$

Gravitational Lagrangian

$$\mathcal{L}_G = c_v \mathfrak{v}^\mu \mathfrak{v}_\mu + c_a \mathfrak{a}^\mu \mathfrak{a}_\mu + c_t t^{\rho\mu\nu} t_{\rho\mu\nu} \tag{29}$$

Field equations

$$E_{\mu\nu} = 8\pi T_{\mu\nu} \tag{30}$$

Most general spherically symmetric background metric [3]

$$g_{00} = C_3^2 - C_1^2, \quad g_{11} = C_4^2 - C_2^2, \quad g_{22} = C_5^2 + C_6^2, \quad g_{33} = (C_5^2 + C_6^2) sin^2\theta,$$
$$g_{01} = g_{10} = C_3 C_4 - C_1 C_2 \tag{31}$$

Schwarzschild metric

$$C_3 = \sqrt{C_1^2 - F^2}, \quad C_4 = \frac{|C_1|}{F^2}, \quad C_6 = \sqrt{r^2 - C_5^2}, \quad C_3 C_4 = C_1 C_2 \tag{32}$$

where $C_1, ..., C_6 = C(r)$ and $F = \sqrt{1 - \frac{2M}{r}}$.

Non-trivial background field equations

$$E_{00} = E_{10} = E_{11} = E_{22} = E_{33} = E_{23} = 0 \tag{33}$$

$$E_{11} = \frac{2\delta c_a F^2}{9r^4 C_{51}^2}(rFC_{52} + 2C_{51}^2)(rFC_{52} - 2C_{51}^2) = 0 \tag{34}$$

where

$$C_{51} = \sqrt{r^2 - C_5^2}, \quad C_{52} = r\frac{dC_5}{dr} - C_5 \tag{35}$$

Branch 1: $C_5 = \pm r$
Branch 2: $C_1 = \pm F, C_5 = ...$
Branch 3: $C_5 = r cos a$ with $a \neq n\pi$

[3] M. Hohmann et al., "Modified teleparallel theories of gravity in symmetric spacetimes", Phys. Rev. D 100, 084002 (2019)

Perturbations in metric teleparallel gravity [4]

Small perturbation of the tetrad

$$\theta^a{}_\mu = \bar{\theta}^a{}_\mu + \tau^a{}_\mu \tag{36}$$

Decomposition into symmetric and antisymmetric part

$$\tau_{\mu\nu} = h_{\mu\nu} + a_{\mu\nu}, \quad h_{\mu\nu} = 2\tau_{(\mu\nu)}, \quad a_{\mu\nu} = 2\tau_{[\mu\nu]} \tag{37}$$

Perturbed connection

$$\Gamma^\rho{}_{\mu\nu} = \bar{\Gamma}^\rho{}_{\mu\nu} + \delta\Gamma^\rho{}_{\mu\nu}, \quad \delta\Gamma^\rho{}_{\mu\nu} = \bar{g}^{\rho\kappa}\bar{\nabla}_\nu\tau_{\kappa\mu} \tag{38}$$

Perturbed torsion

$$T^\rho{}_{\mu\nu} = \bar{T}^\rho{}_{\mu\nu} + \delta T^\rho{}_{\mu\nu}, \quad \delta T^\rho{}_{\mu\nu} = \bar{g}^{\rho\kappa}(\bar{\nabla}_{[\mu}h_{\nu]\kappa} - \bar{\nabla}_{[\mu}a_{\nu]\kappa}) \tag{39}$$

Perturbed vector, axial and tensor irreducible components

$$\mathfrak{u}_\mu = \bar{\mathfrak{u}}_\mu + \delta\mathfrak{u}_\mu \tag{40}$$

$$\mathfrak{a}_\mu = \bar{\mathfrak{a}}_\mu + \delta\mathfrak{a}_\mu \tag{41}$$

$$\mathfrak{t}_{\mu\nu\rho} = \bar{\mathfrak{t}}_{\mu\nu\rho} + \delta\mathfrak{t}_{\mu\nu\rho} \tag{42}$$

Perturbed field equations

$$E_{\mu\nu} = \bar{E}_{\mu\nu} + \delta E_{\mu\nu} = 0 \tag{43}$$

Decomposition in symmetric and antisymmetric parts

$$\delta E_{\mu\nu} = \delta E_{(\mu\nu)} + \delta E_{[\mu\nu]} \tag{44}$$

[4] Helen Asuküla, "Quasinormal modes of Schwarzschild black holes in 1-parameter New General Relativity" (2021) (Master Thesis, Institute of Physics, University of Tartu)

Tetrad perturbation $\tau_{\mu\nu}$

The symmetric part $h_{\mu\nu}$ is the same as in GR.

$$a_{\mu\nu} = a_{\mu\nu}^{(axial)} + a_{\mu\nu}^{(polar)} \tag{45}$$

$$a_{\mu\nu}^{(axial)} = \begin{pmatrix} 0 & 0 & 0 & a_0(r) sin\theta \partial_\theta \\ 0 & 0 & 0 & a_1(r) sin\theta \partial_\theta \\ 0 & 0 & 0 & -a_2(r) sin\theta \partial_\theta \\ \text{ASym} & \text{ASym} & \text{ASym} & 0 \end{pmatrix} e^{-i\omega t} Y_{lm} \tag{46}$$

$$a_{\mu\nu}^{(polar)} = \begin{pmatrix} 0 & A_0(r) & A_1(r)\partial_\theta & 0 \\ \text{ASym} & 0 & A_2(r)\partial_\theta & 0 \\ \text{ASym} & \text{ASym} & 0 & 0 \\ \text{ASym} & \text{ASym} & 0 & 0 \end{pmatrix} e^{-i\omega t} Y_{lm} \tag{47}$$

Non-trivial perturbed field equations (Branch 1: $C_5 = r$)

Axial

$$\delta E_{(03)} = \delta E_{(13)} = 0 \quad \begin{cases} \delta R_{03} = 0 \\ \delta R_{13} = 0 \end{cases}$$

$$\delta E_{(23)} = \delta R_{23} = 0$$

$$\delta E_{[03]} = \delta E_{[13]} = \delta E_{[23]} = 0 \quad \rightarrow \quad a_0, a_1, a_2 = F(h_0, h_1)$$

Polar

$$\begin{aligned} \delta E_{(01)} &= \delta R_{01} = 0 \\ \delta E_{(02)} &= \delta R_{02} = 0 \\ \delta E_{(12)} &= \delta R_{12} = 0 \\ \delta E_{(00)} &= \delta R_{00} = 0 \\ \delta E_{(11)} &= \delta R_{11} = 0 \\ \delta E_{(22)} &= \delta R_{22} = 0 \\ \delta E_{(33)} &= \delta R_{33} = 0 \end{aligned}$$

$$\delta E_{[01]} = -\tfrac{\delta c_a}{r^2}\left[\tfrac{dA_1}{dr} - A_0 + \tfrac{2C_1 - F^2 - 1}{2rF^2} A_1 + \left(i\omega - \tfrac{rC_{12} + C_{11}^2}{rC_{11}}\right) A_2\right] = 0$$

Summary

QNM in GR for a Schwarzschild background at linear order.

QNM in Metric TG for a Schwarschild background at linear order for the simplest branch $C_5 = r$.

For the symmetric part in TG, we obtain the same QNM as in GR.

The axial antisymmetric part in TG is fully determined by the symmetric one. No extra modes.

For the polar antisymmetric part in TG, we need extra constraints.

In GR and TG for the branch $C_5 = r$, the axial and polar parts do not mix.

Future work paths

- Study of the other branches
- Non-Schwarzschild spacetimes
- f(T) gravity
- Higher order perturbations
- Symmetric TG, f(Q) gravity

References

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Thank you!