Quantum Euler angles and agency-dependent spacetime

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Introduction

- In classical Minkowski spacetime the connection between two inertial observers reference frame is determined by an element of the Poincaré group.

- In quantum spaces we expect deformations of classical symmetries and we focus on the case in which two reference frames are just rotated one with respect to the other.

- We will assume that the classical rotation group $SO(3)$ at quantum level is replaced by its quantum group deformation $SO_q(3)$.

SU(2) coordinatization and homomorphism with SO(3)

- In classical and quantum mechanics, rotations are described by the group $SU(2)$.

$$SU(2) \ni U = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix} \quad a, c \in \mathbb{C} \ : \quad |a|^2 + |c|^2 = 1$$

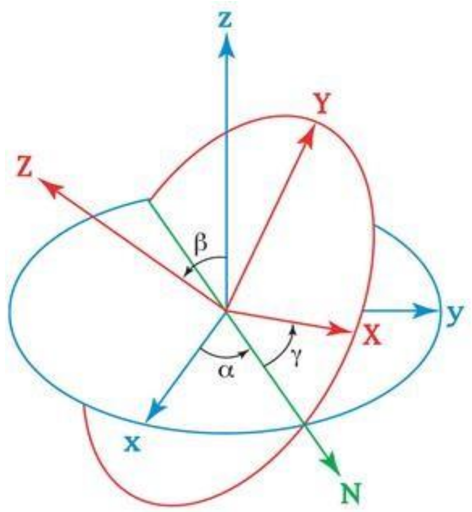
$$a = e^{i\chi} \cos\left(\frac{\theta}{2}\right) \quad c = e^{i\phi} \sin\left(\frac{\theta}{2}\right)$$

$$R = \begin{pmatrix} \frac{1}{2}(a^2 - c^2 + (a^*)^2 - (c^*)^2 & \frac{i}{2}(-a^2 + c^2 + (a^*)^2 - (c^*)^2 & a^*c + c^*a \\ \frac{i}{2}(a^2 + c^2 - (a^*)^2 - (c^*)^2 & \frac{1}{2}(a^2 + c^2 + (a^*)^2 + (c^*)^2 & -i\,(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1 - 2cc^* \end{pmatrix}$$

Classical Euler Angles

The $SU(2)$ parameters are linked to Euler angles as follows:

$$\begin{cases} \theta = \beta \\ \chi = \dfrac{\alpha+\gamma}{2} \\ \phi = \dfrac{\pi}{2} - \dfrac{\alpha-\gamma}{2} \end{cases}$$



$SU_q(2)$ algebra

- The quantum group $SU_q(2)$ is defined by considering the algebra of complex functions on $SU(2)$, denoted by $C(SU(2))$ and deforming it in a non commutative way.

- $U_q = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$ $\quad a, c \in C(SU_q(2))$ and $q \in (0,1)$

where :

- $ac = qca; \quad ac^* = qc^*a; \quad cc^* = c^*c; \quad c^*c + a^*a = 1; \quad aa^* - a^*a = (1-q^2)c^*c$

- q is a deformation parameter larger than 0 and close to 1

Homomorphism with $SO_q(3)$

- As in the classical case we can construct a homomorphism between $SU_q(2)$ and $SO_q(3)$ and we obtain the following q-deformed rotation matrix:

- $$R_q = \begin{pmatrix} \frac{1}{2}(a^2 - qc^2 + (a^*)^2 - q(c^*)^2 & \frac{i}{2}(-a^2 + qc^2 + (a^*)^2 - q(c^*)^2 & \frac{1}{2}(1+q^2)(a^*c + c^*a) \\ \frac{i}{2}(a^2 + qc^2 - (a^*)^2 - q(c^*)^2 & \frac{1}{2}(a^2 + qc^2 + (a^*)^2 + q(c^*)^2 & -\frac{i}{2}(1+q^2)\,(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1 - (1+q^2)cc^* \end{pmatrix}$$

$SU_q(2)$ representations

- The Hilbert space containing the two unique irreducible representations of the $SU_q(2)$ algebra is $H = H_\pi \oplus H_\rho$ where $H_\pi = L^2(S^1) \otimes L^2(S^1) \otimes \ell$ and $H_\rho = L^2(S^1)$

- $\rho(a)|\eta\rangle = e^{i\eta}|\eta\rangle$; $\rho(a^*)|\eta\rangle = e^{-i\eta}|\eta\rangle$; $\rho(c)|\eta\rangle = 0$; $\rho(c^*)|\eta\rangle = 0$

- $\pi(a)|n,\delta,\epsilon\rangle = e^{i\epsilon}\sqrt{(1-q^{2n})}|n-1,\delta,\epsilon\rangle$; $\pi(a^*)|n,\delta,\epsilon\rangle = e^{-i\epsilon}\sqrt{(1-q^{2n+2})}|n+1,\delta,\epsilon\rangle$;

- $\pi(c)|n,\delta,\epsilon\rangle = e^{i\delta}q^n|n,\delta,\epsilon\rangle$; $\pi(c^*)|n,\delta,\epsilon\rangle = e^{-i\delta}q^n|n,\delta,\epsilon\rangle$;

- $a = e^{i\chi}\cos\left(\frac{\theta}{2}\right)$ $c = e^{i\phi}\sin\left(\frac{\theta}{2}\right)$ (Classical case)

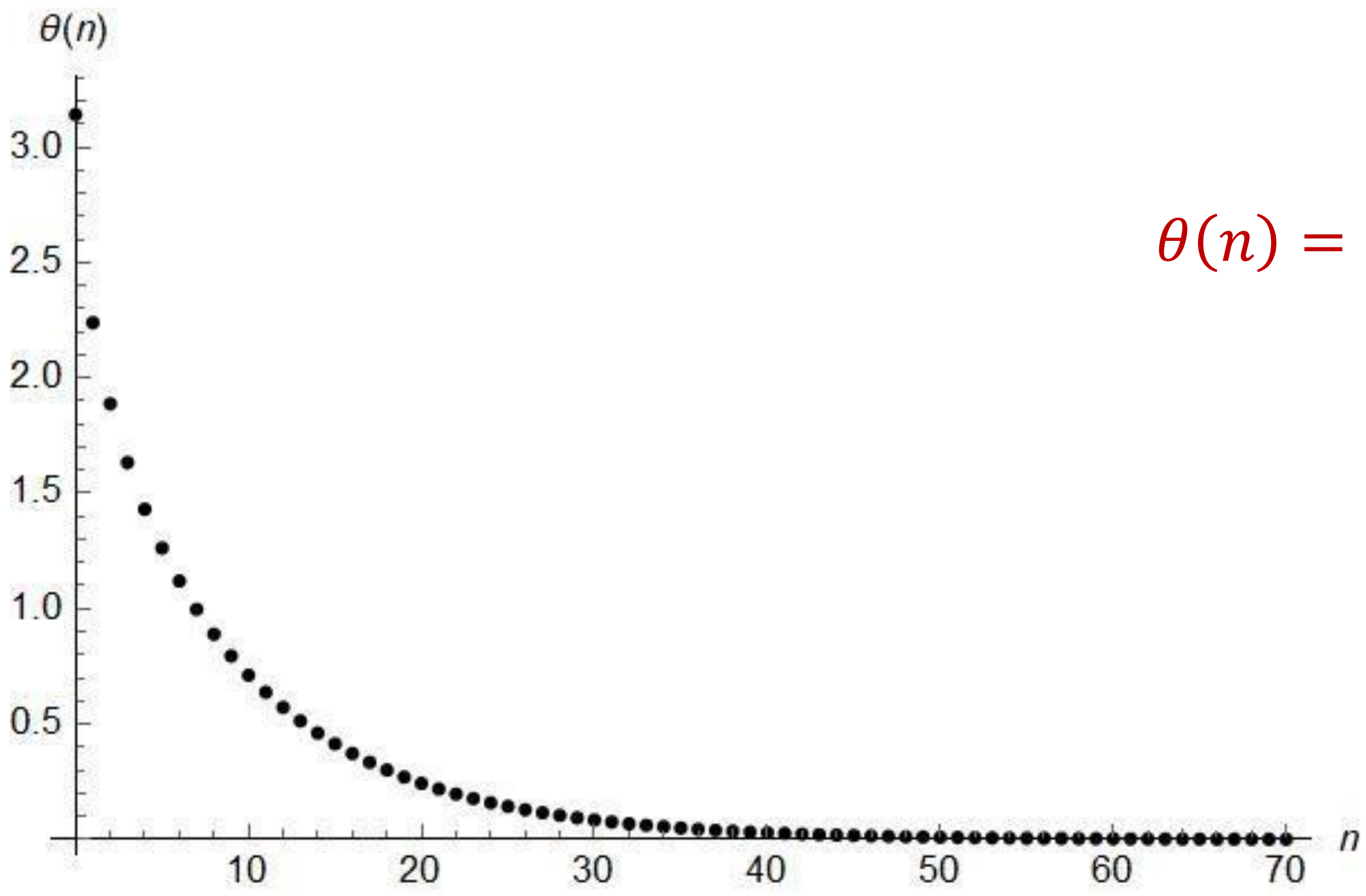
Quantum Euler angles (1)

- Comparing the phases of operators *a* and *c* and their classical analogues, we identify ε with χ and δ with φ.

- Then, exploiting the fact that *c* is diagonal, we are led to a significant result:

- $$q^n = \sin\left(\frac{\theta(n)}{2}\right) \Longleftrightarrow \theta(n) = 2\arcsin(q^n)$$

Quantum Euler angles (2)



$$\theta(n) = 2Arcin(q^n)$$

q=0.99

Quantum rotations

- A state $|\psi\rangle \in H$ is representative of the relative orientation between two reference frames, A and B.

- Our proposal is that the mean value of R_q on $|\psi\rangle$ will give an estimate of the entries of the rotation matrix that connects A and B

$$\langle\psi|R_q|\psi\rangle_{\mathrm{ij}}$$

- However, due to the non commutatitvity we will have in general a non vanishing variance for the matrix elements:

$$\Delta_{ij} = \sqrt{\langle\psi|R_q^2|\psi\rangle_{\mathrm{ij}} - \langle\psi|R_q|\psi\rangle_{\mathrm{ij}}^2}$$

Quantum rotations around z-axis

- Consider a state $|\chi\rangle$ in representation ρ.

- The mean value of the rotation matrix is:

$$\langle\chi|R_q|\chi\rangle_{\mathrm{ij}} = \begin{pmatrix} \cos(2\chi) & -\sin(2\chi) & 0 \\ \sin(2\chi) & \cos(2\chi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which coincides with a classical rotation matrix that describes a rotation of 2χ around the *z*-axis.

- $\Delta_{ij} = 0 \rightarrow$ Classical rotations

Quantum rotation of π around the x-axis

- $|\psi\rangle = \left|0, \frac{\pi}{2}, 0\right\rangle$

- $\langle\psi|R_q|\psi\rangle = \begin{pmatrix} 1-(1-q) & 0 & 0 \\ 0 & -1+(1-q) & 0 \\ 0 & 0 & -1+2(1-q) \end{pmatrix} + \mathrm{o}(1-\mathrm{q})$

- $\langle\psi|\Delta R_q|\psi\rangle = \begin{pmatrix} \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2(1-q)} & \sqrt{2(1-q)} & 0 \end{pmatrix} + o(1-q)$

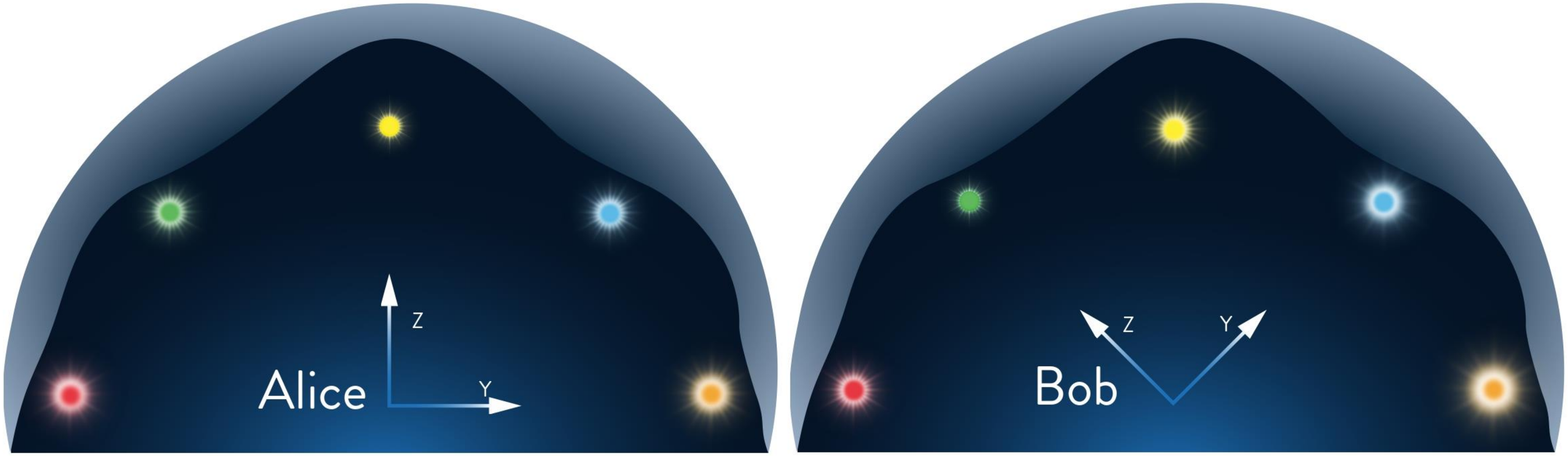
- As $q \to 1$, these reproduce a rotation of π around the x-axis with null uncertainty.

Quantum rotation around the x-axis

- The previous example shows that rotations around the x-axis are not classical unlike rotations around the z-axis, but they are q-deformed.

- Since the variances of the entries of a rotation matrix around the x-axis do not vanish, two observers connected by such a rotation cannot know their relative orientation with arbitrary precision.

Agency dependent properties of spacetime



- Fuzziness of spacetime points depends on the choices made by the observer: ***different skies may originate from the observation of the same stars***.

Conclusions

Describing deformed spatial rotations using the quantum group $SU_q(2)$ we obtain:

- **Quantization** of one of the *Euler angles*;

- **Agency dependent space**: space is reconstructed as a collection of fuzzy points, exclusive to each agent. Two agents making different choices will thus observe the same points with different degrees of fuzziness.

Thank you!

«Semi-classical» states construction

- To effectively describe rotations' deformations, we demand that our states of geometry $|\psi\rangle$ satisfy

$$\left\langle\psi(\theta,\phi,\chi)\middle|(R_q)_{ij}\middle|\psi(\theta,\phi,\chi)\right\rangle_{ij} = R_{ij}(\theta,\phi,\chi)\text{+O(1-q)} \qquad \text{and} \qquad \Delta_{ij}^2 = \text{O}(1-\text{q})$$

where R_{ij} are the entries of a classical rotation matrix.

- Since (ϕ,χ) behave as in the classical case, to properly describe a rotation deformation of Euler angles $(\phi,\chi,\theta(\bar{n}))$ we must look for states of the form

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n,\phi,\chi\rangle$$

heavily weighted around $\bar{n}$ and which satisfy the criteria above.