Solving constraints using quantum computers

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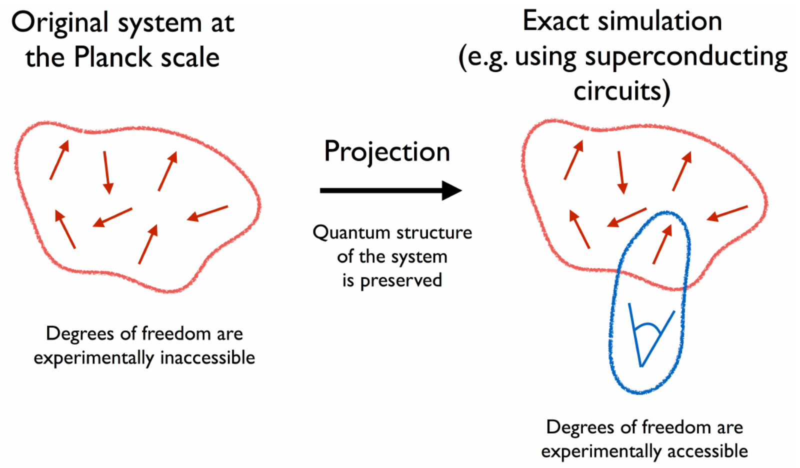
Quantum simulations of physics

Feynman, R. P. **Simulating physics with computers.** Int. J. Theor. Phys. 21, 467–488 (1982)



Quantum computers and quantum systems

for one qubit

$$\dim\left(\mathcal{H}\right) = 2$$

for N qubits

$$\dim\left(\mathcal{H}^{\otimes N}\right) = 2^N$$

Quantum computers and quantum systems

for one qubit

$$\dim(\mathcal{H}) = 2$$

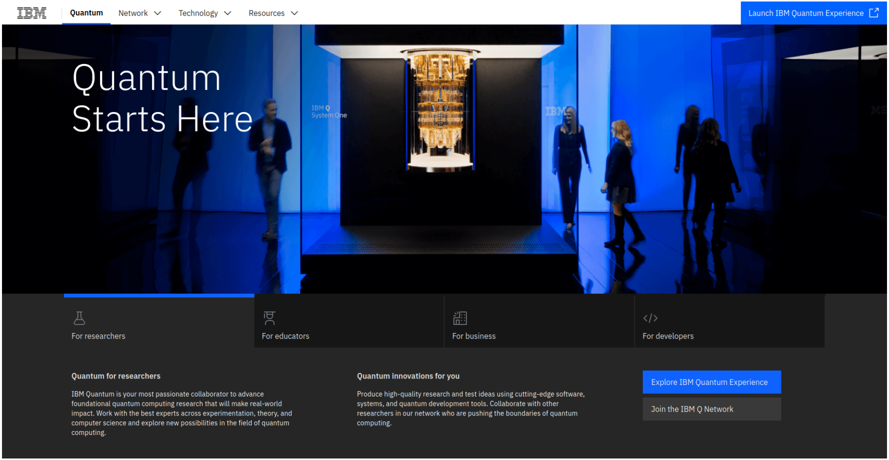
for N qubits

$$\dim\left(\mathcal{H}^{\otimes N}\right) = 2^N$$

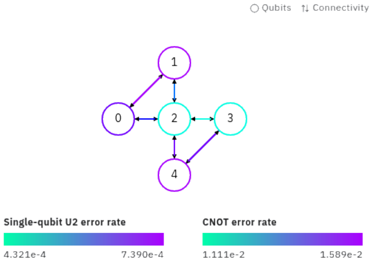
for $N = 50$

$$\dim\left(\mathcal{H}^{\otimes 50}\right) \simeq 10^{15}$$

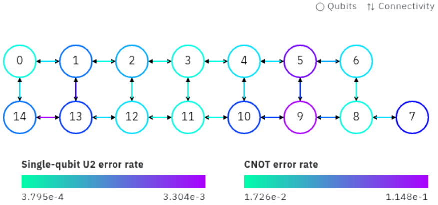
Quantum computers IBM Q



Quantum processors



Yorktown quantum processor



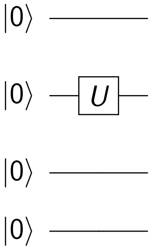
Melbourne quantum processor

Quantum circuits

Quantum register:

$$|0\rangle \;—$$
$$|0\rangle \;—$$
$$|0\rangle \;—$$
$$|0\rangle \;—$$

One-qubit gates:

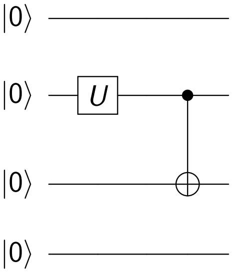


$$U_3 = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i\lambda+i\phi}\cos(\theta/2), \end{pmatrix}$$

$$U_3|0\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle$$

Quantum circuits

Two-qubits gates:

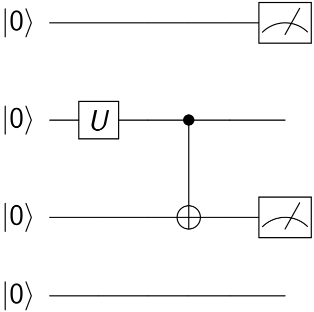


$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

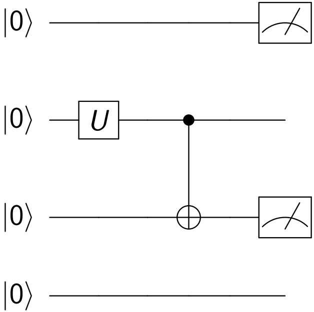
$$CX|00\rangle = |00\rangle$$
$$CX|01\rangle = |01\rangle$$
$$CX|10\rangle = |11\rangle$$
$$CX|11\rangle = |10\rangle$$

Quantum circuits

Measurements:



Measurements:



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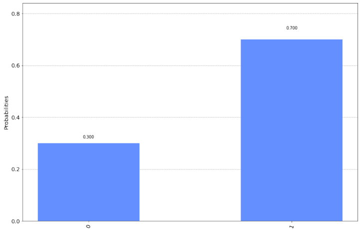


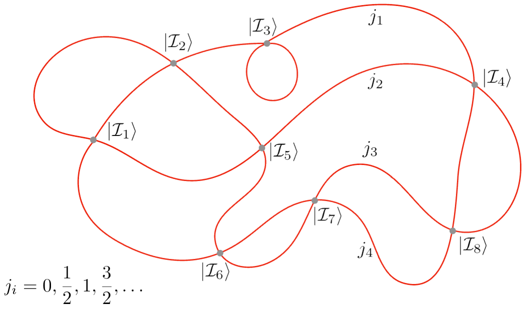
Considered cases

- Gauss constraint in loop quantum gravity
- Wheeler-DeWitt equation

Simulations of LQG

- Cohen, L., Brady, A. J., Huang, Z., Liu, H., Qu, D., Dowling, J. P., & Han, M. (2021). **Efficient simulation of loop quantum gravity: A scalable linear-optical approach.** Physical Review Letters, 126(2), 020501.
- Li, K., Li, Y., Han, M., Lu, S., Zhou, J., Ruan, D., ... & Laflamme, R. (2019). **Quantum spacetime on a quantum simulator.** Communications Physics, 2(1), 1-6.
- Zhang, P., Huang, Z., Song, C., Guo, Q., Song, Z., Dong, H., ... & Wan, Y. (2020). **Observation of two-vertex four-dimensional spin foam amplitudes with a 10-qubit superconducting quantum processor.** arXiv preprint arXiv:2007.13682.
- van der Meer, R., Huang, Z., Anguita, M. C., Qu, D., Hooijschuur, P., Liu, H., ... & Cohen, L. (2022). **Experimental Simulation of Loop Quantum Gravity on a Photonic Chip.** arXiv preprint arXiv:2207.00557.

Spin-network states



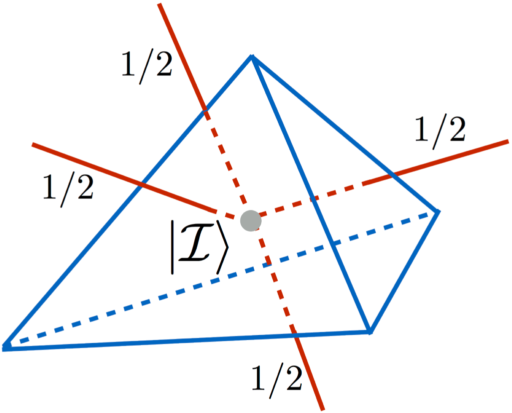
Gauss constraints:

$$|\mathcal{I}_n\rangle \in Inv_{SU(2)}\left(\mathcal{H}_{j_a} \otimes \mathcal{H}_{j_b} \otimes \mathcal{H}_{j_c} \otimes \mathcal{H}_{j_d}\right)$$

Spin network basis:

$$|\Gamma, j_l, \mathcal{I}_n\rangle = \bigotimes_n \mathcal{I}_n$$

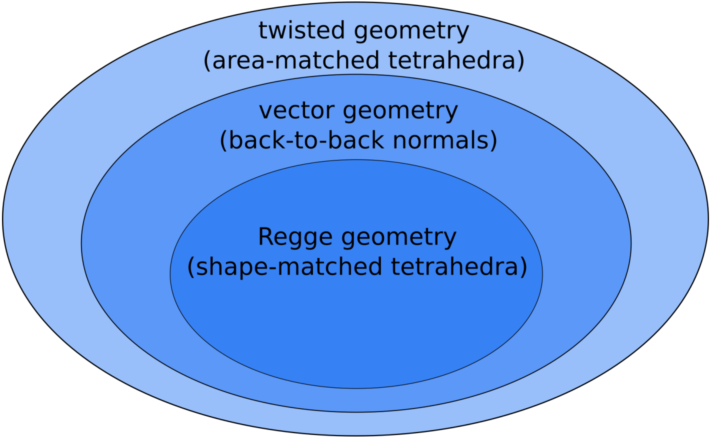
Single node



$$|\mathcal{I}\rangle \in Inv_{SU(2)}\left(\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}\right)$$

$$|\mathcal{I}\rangle = \cos\frac{\theta}{2}|\iota_0\rangle + e^{i\phi}\sin\frac{\theta}{2}|\iota_1\rangle$$

Gluing tetrahedra



Spin-network basis states are un-entangled $|\Gamma, j_l, \mathcal{I}_n\rangle = \bigotimes_n \mathcal{I}_n$

Gluing tetrahedra

Baytaș, B., Bianchi, E., & Yokomizo, N. (2018). **Gluing polyhedra with entanglement in loop quantum gravity.** Physical Review D, 98(2), 026001.

squeezed states

$$|\mathcal{B}, \lambda_l\rangle = \left(1 - |\lambda_l|^2\right) \sum_j \sqrt{2j+1}\lambda_l^{2j} |\mathcal{B}, j\rangle \tag{1}$$

singlet state of spin j, which is maximally entangled

$$|\mathcal{B}, j\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^{j-m} |j, m\rangle_s |j, -m\rangle_t$$

Gauss constraint implementation

projection on spin-network basis states

$$P_\Gamma = \sum_{j_l, \mathcal{I}_n} |\Gamma, j_l, \mathcal{I}_n\rangle \langle \Gamma, j_l, \mathcal{I}_n|$$

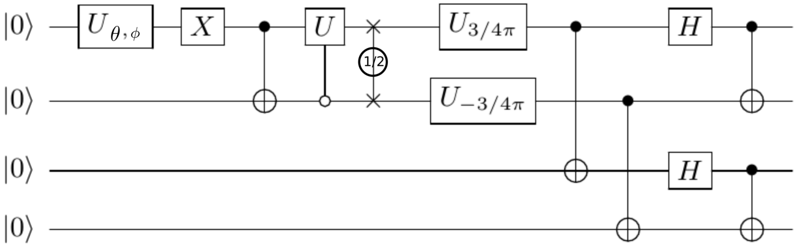
$$|\Gamma, \mathcal{B}, \lambda_l\rangle = P_\Gamma \bigotimes_l |\mathcal{B}, \lambda_l\rangle$$

Gluing tetrahedra

for $j = \frac{1}{2}$ in qubit notations:

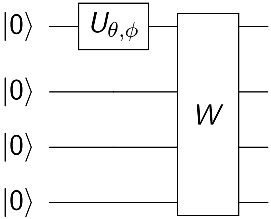
$$\left|\mathcal{B}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Quantum circuit for node



$$|\mathcal{I}\rangle = \cos\frac{\theta}{2}|\iota_0\rangle + e^{i\phi}\sin\frac{\theta}{2}|\iota_1\rangle =$$
$$\frac{c_1}{\sqrt{2}}(|0011\rangle + |1100\rangle) + \frac{c_2}{\sqrt{2}}(|0101\rangle + |1010\rangle) + \frac{c_3}{\sqrt{2}}(|0110\rangle + |1001\rangle)$$

Quantum circuit for node



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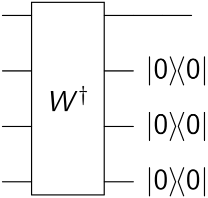
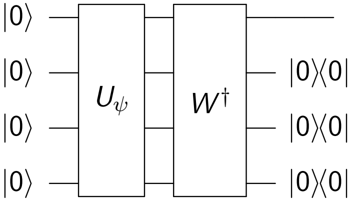
Projection



"Projection" operator on intertwiner subspace, expressed in one-qubit representation.

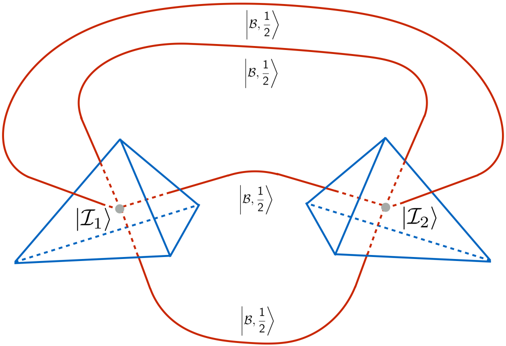
Projection



Projection of state $|\psi\rangle$ on intertwiner subspace, expressed in one-qubit representation.

$$\left(\sum_k |\iota_k\rangle \langle \iota_k| \right) |\psi\rangle$$

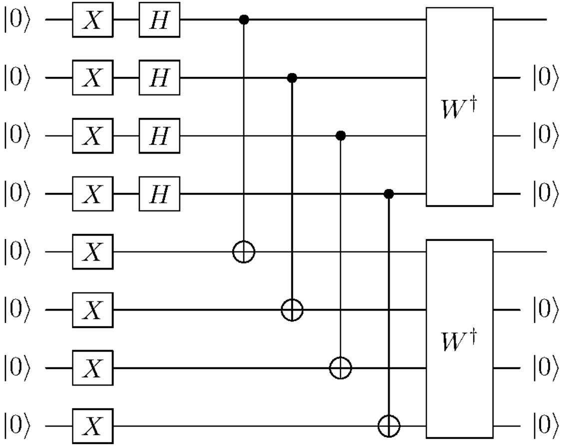
Dipole



$$\left|\Gamma_2, \mathcal{B}, \frac{1}{2}\right\rangle = P_\Gamma \bigotimes_l \left|\mathcal{B}, \frac{1}{2}\right\rangle = \sum_{k,l} \iota_{(k)}^{m_1 m_2 m_3 m_4} \iota_{(l) m_1 m_2 m_3 m_4} \left|\iota_k \iota_l\right\rangle$$

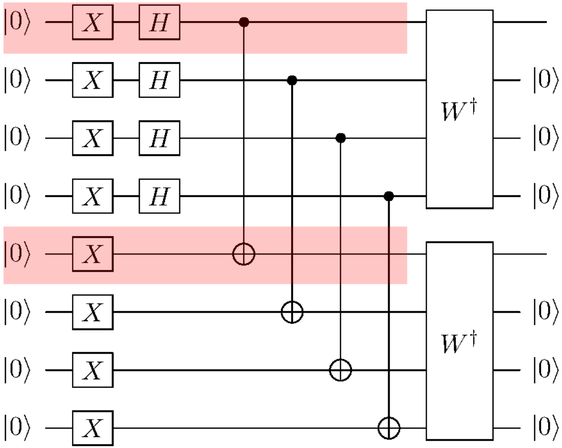
$$= \frac{1}{\sqrt{2}} \left(\left|\iota_0 \iota_0\right\rangle + \left|\iota_1 \iota_1\right\rangle\right)$$

Dipole



$$\left|\Gamma_2, \mathcal{B}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|\iota_0\iota_0\rangle + |\iota_1\iota_1\rangle\right)$$

Dipole



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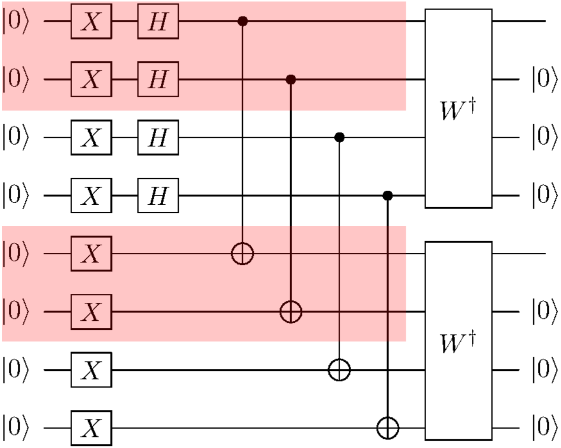
Dipole



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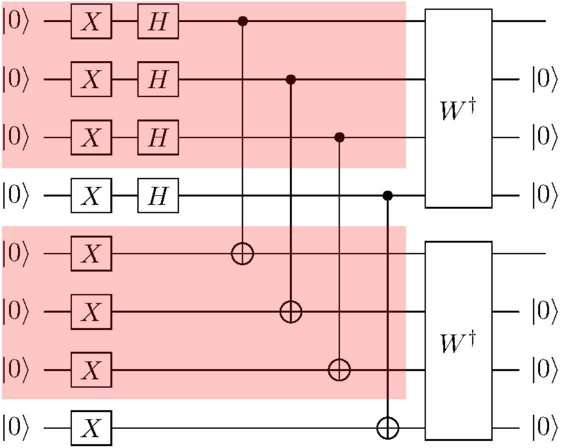
Dipole



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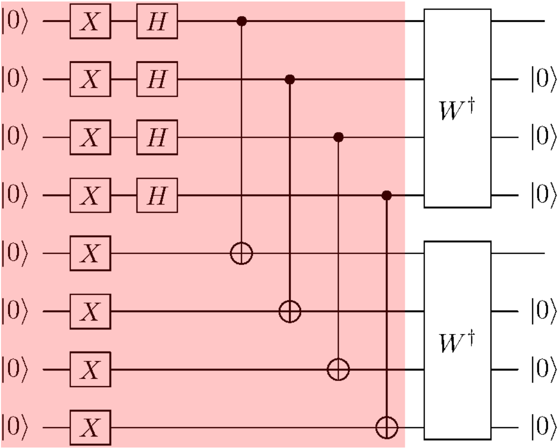
Dipole



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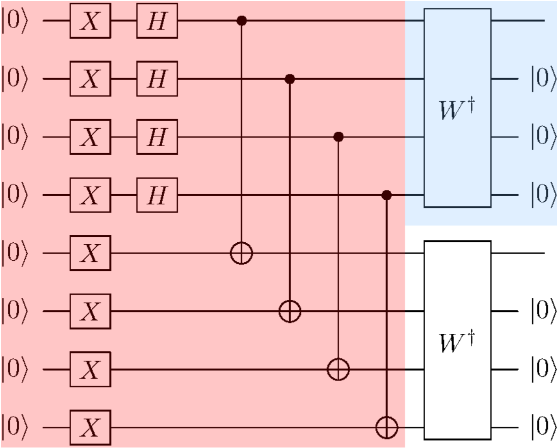
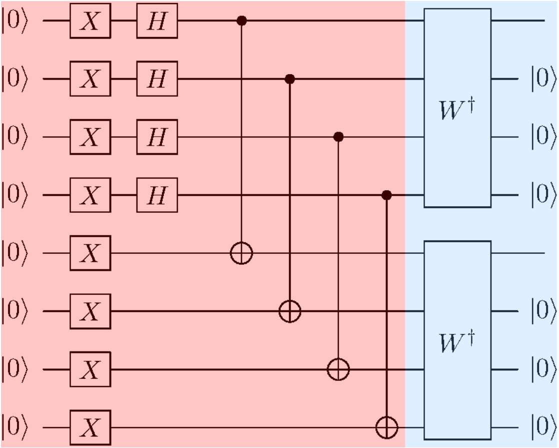
Dipole



$$\left|\Gamma_2, \mathcal{B}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|\iota_0\iota_0\rangle + |\iota_1\iota_1\rangle\right)$$

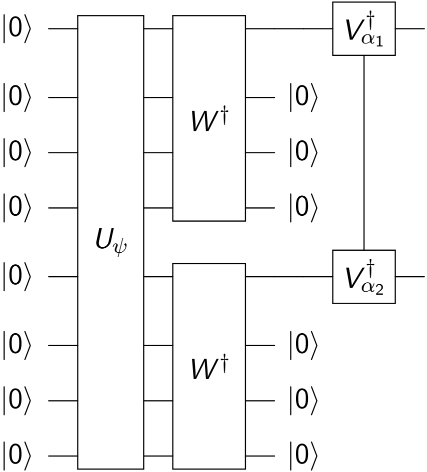
Dipole



$$\left|\Gamma_2, \mathcal{B}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left(|\iota_0\iota_0\rangle + |\iota_1\iota_1\rangle\right)$$

Dipole

Transfer of projected state on ansatz:

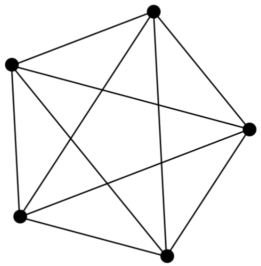


Pentagram

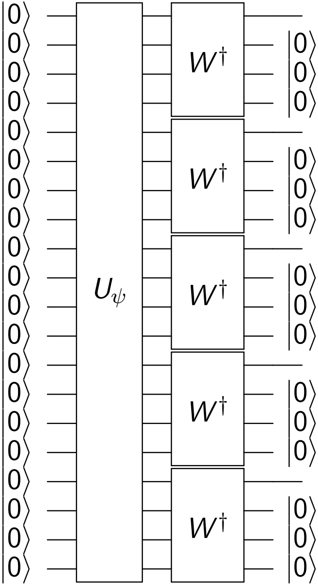
$$P_\Gamma |\psi\rangle = \sum_{\iota_{k_i}} \overline{\{15j\}} \, |\iota_{k_1}\iota_{k_2}\iota_{k_3}\iota_{k_4}\iota_{k_5}\rangle$$

where

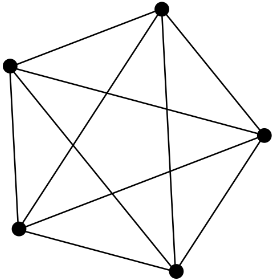
$$\{15j\} = \iota_1^{m_{12}m_{13}m_{14}m_{15}} \, \iota_{2;m_{12}}{}^{m_{13}m_{14}m_{15}} \, \iota_{3;m_{12}m_{13}}{}^{m_{14}m_{15}} \, \iota_{4;m_{12}m_{13}m_{14}}{}^{m_{15}} \, \iota_{5;m_{12}m_{13}m_{14}m_{15}}$$



Pentagram

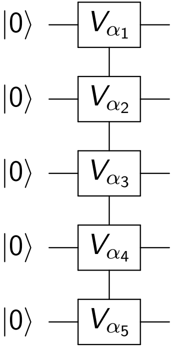


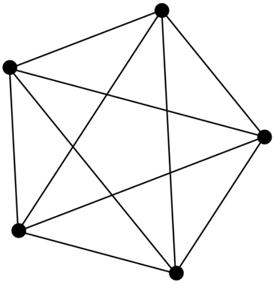
$|0\rangle$
U_ψ
$W^\dagger$
$W^\dagger$
$W^\dagger$
$W^\dagger$
$W^\dagger$
$|0\rangle$



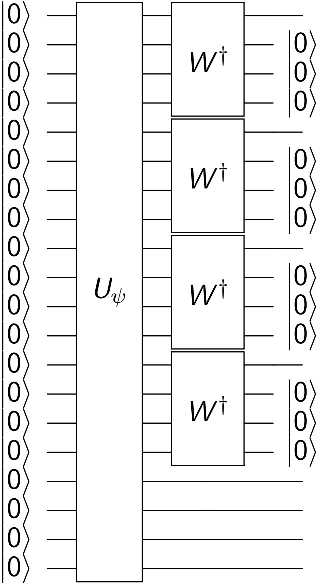
Pentagram

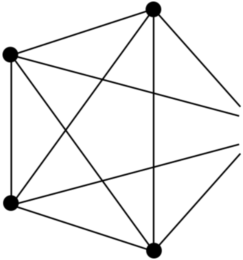
Ansatz:





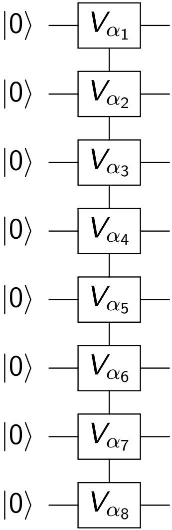
Open pentagram

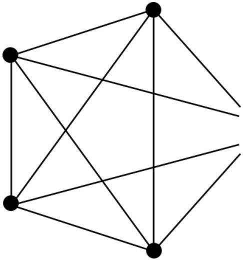




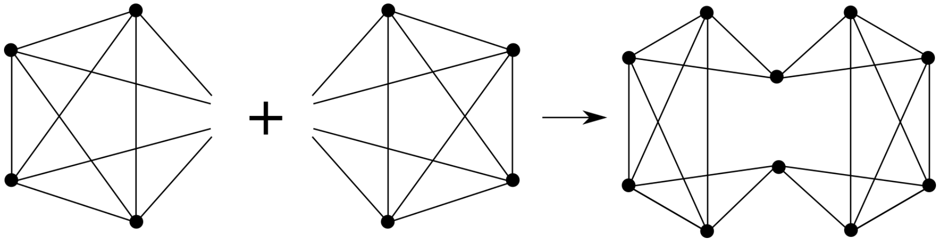
Open pentagram

Ansatz:



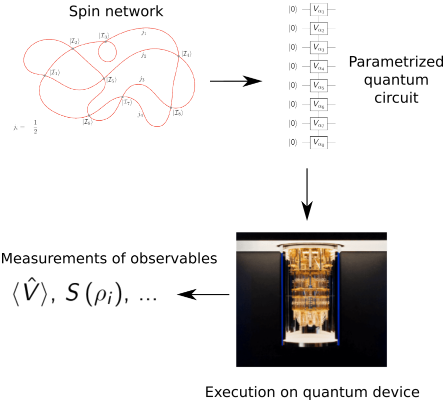


Decagram



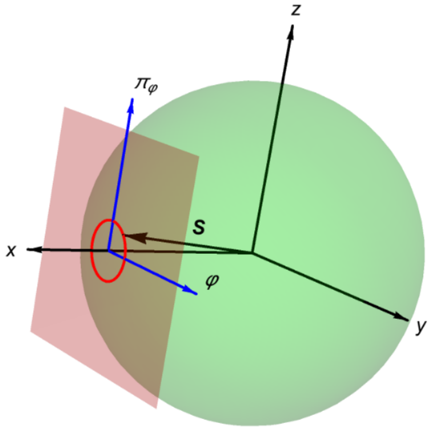
8 qubits + 8 qubits → 10 qubits

Simulations of spin networks



Czelusta, G. (2022, October). Quantum computations in loop quantum gravity. In Particle Physics Summer Student Alumni Conference 2022 (Kraków, 9–10 July 2022) (p. 45).

Compact phase spaces



$$S_x := S\cos\frac{\pi_\phi}{R_2}\cos\frac{\phi}{R_1}$$

$$S_y := S\cos\frac{\pi_\phi}{R_2}\sin\frac{\phi}{R_1}$$

$$S_z := S\sin\frac{\pi_\phi}{R_2}$$

$$S_x^2 + S_y^2 + S_z^2 = S^2$$

$$R_1 R_2 = S$$

$$\omega = \cos\frac{\pi_\phi}{R_2} d\pi_\phi \wedge d\phi$$

De Sitter model

$$\omega = dp \wedge dq$$

$$|q| := V_0 a^3$$

$$H_{GR} = Nq \left(-\frac{3}{4}\kappa p^2 + \frac{\Lambda}{\kappa} \right)$$

$$\kappa = 8\pi G = 8\pi l_{Pl}^2$$

De Sitter model

$$\omega = dp \wedge dq$$

$$|q| := V_0 a^3$$

$$H_{GR} = Nq\left(-\frac{3}{4}\kappa p^2 + \frac{\Lambda}{\kappa}\right)$$

$$\kappa = 8\pi G = 8\pi l_{Pl}^2$$

$$S_x := S\cos\frac{p}{R_1}\cos\frac{q}{R_2}$$

$$S_y := S\cos\frac{q}{R_2}\sin\frac{p}{R_1}$$

$$S_z := -S\sin\frac{q}{R_2}$$

$$p \to p_S := \frac{S_y}{R_2}$$

$$q \to q_S := -\frac{S_z}{R_1}$$

$$H_S = N\frac{S_z}{R_1}\left(\frac{3}{4}\kappa\frac{S_y^2}{R_2^2} - \frac{\Lambda}{\kappa}\right)$$

D. Artigas, J. Mielczarek, C. Rovelli, Minisuperspace model of compact phase space gravity, Phys. Rev. D 100, 043533

Cosmological constraints

$$\hat{H}_S = N\hat{C}$$

$$\frac{4S^2}{3\kappa R_1}\hat{C} = \frac{1}{3}\left(\hat{S}_z\hat{S}_y\hat{S}_y + \hat{S}_y\hat{S}_z\hat{S}_y + \hat{S}_y\hat{S}_y\hat{S}_z\right) - \delta\hat{S}^2\hat{S}_z$$

$$\delta := \frac{4}{3}\frac{\Lambda}{R_1^2\kappa^2}$$

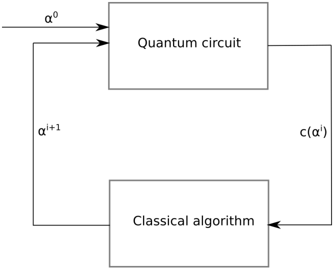
Constraints solving

$$\hat{C}|\psi_0\rangle = 0 \leftrightarrow \langle\psi_0|\hat{C}^\dagger \hat{C}|\psi_0\rangle = 0$$

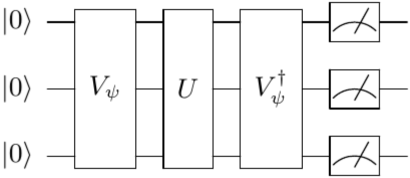
$$\langle\psi|\hat{C}^\dagger \hat{C}|\psi\rangle \geq 0$$

cost function:

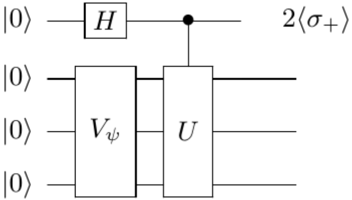
$$c(\alpha) = \langle\psi(\alpha)|\hat{C}^\dagger \hat{C}|\psi(\alpha)\rangle$$



Expectation value of unitary operator U:



$$\langle\psi|U|\psi\rangle = \langle 0|V_\psi^\dagger U V_\psi|0\rangle$$



$$\langle\psi|U|\psi\rangle = 2\langle\sigma_+\rangle$$

Constraints solving

$$\hat{C} = \sum_i \bigotimes_j \hat{\sigma}_{ij}^k \quad k = x, y, z$$

Constraints solving

$$\hat{C} = \sum_i \bigotimes_j \hat{\sigma}_{ij}^k \quad k = x, y, z$$

$$\hat{C} = \frac{1}{8}\Bigg((1-\delta)\, P_n(\sigma_z, \sigma_y, \sigma_y) - \delta P_n(\sigma_z, \sigma_x, \sigma_x) - \delta P_n(\sigma_z, \sigma_z, \sigma_z) + \left(n - \frac{2}{3} - \delta\,(5n-2)\right) P_n(\sigma_z) \Bigg)$$

$$P_n(\sigma_i) := \sum_j \mathbb{I}^1 \otimes \ldots \mathbb{I}^{j-1} \otimes \sigma_i^j \otimes \mathbb{I}^{j+1} \otimes \ldots \mathbb{I}^n, \tag{2}$$

$$P_n(\sigma_i, \sigma_j, \sigma_p) := \sum_{k,l,q,k\neq l,k\neq q,l\neq q} \mathbb{I}^1 \otimes \ldots \sigma_i^k \otimes \ldots \sigma_j^l \otimes \ldots \sigma_p^q \otimes \ldots \mathbb{I}^n \tag{3}$$

Additional condition

$$C|\psi_0\rangle = 0 \tag{4}$$

$$D|\psi_0\rangle = \lambda|\psi_0\rangle \tag{5}$$

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Additional condition

Subspace of spin s:

$$c(\alpha) + s(s+1) - \langle\psi(\alpha)|\hat{\vec{S}}^2|\psi(\alpha)\rangle \tag{8}$$

$$S^2|\psi_0\rangle = s(s+1)|\psi_0\rangle \tag{9}$$

Gradient

Shift rule:

$$f(\theta) = \langle U \rangle_{\psi_\theta} = \langle 0 V_\theta^\dagger U V_\theta 0 \rangle \tag{10}$$

$$V_\theta = A_{\theta_0,\ldots,\theta_i-1} G_{\theta_i} B_{\theta_i+1,\ldots,\theta_n} \tag{11}$$

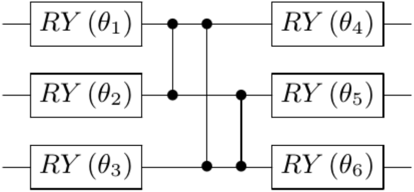
where

$$G_{\theta_i} = e^{-i\theta_i \mathcal{G}} \tag{12}$$

$$\partial_{\theta_i} f = r\left(f(\theta_i + s) - f(\theta_i - s)\right) \tag{13}$$

where $s = \frac{\pi}{4r}$ i $-r, +r$ are eigenvalues of $\mathcal{G}$.

Ansatz for $|\psi(\theta)\rangle$:



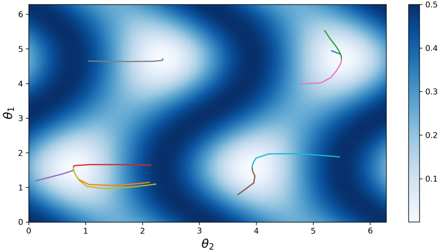
$$RY(\theta) = \exp\left(-i\frac{\theta}{2}Y\right) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

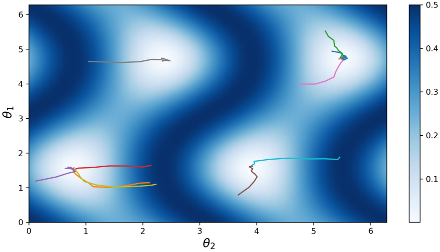
Constraints solving

Results for $s = 1$

Minimization of cost function:



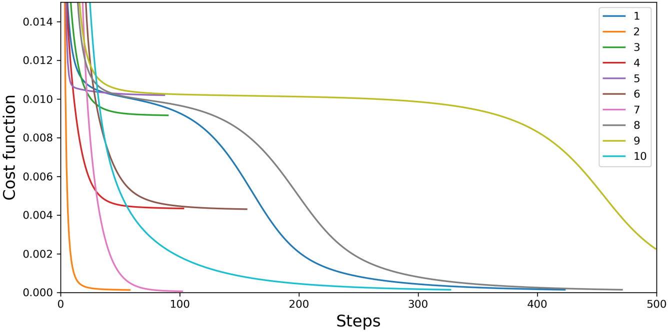
Classical simulator



Quantum computer

Constraints solving

Results for $s = 2$
Minimization of cost function:



Classical simulator

Summary

- we can prepare a broad class of spin-network states on a quantum register by applying Gauss constraint
- we solve regularized cosmological constraints
- measurements can be performed both on simulator and real quantum processor

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Thank you for your attention