Noncommutative gravity using twists

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$\underbrace{\text{Noncommutative}}_{\text{braided}}$ $\underbrace{\text{gravity}}_{\text{differential geometry}}$ using $\overset{\text{Drinfel'd}}{\underbrace{\text{twists}}_{\text{Hopf algebras}}}$

Main idea: transform the commutative algebra of functions $C(M)$ into noncommutative $C_\star(M)$.

A Hopf algebra recipe

$$\underline{\mathfrak{g}} \xrightarrow{\text{embed}} \underline{\mathcal{U}(\mathfrak{g})} \xrightarrow{\text{twist by } \mathcal{F}} \underline{\mathcal{U}^{\mathcal{F}}(\mathfrak{g})} =: H$$

$$[\ ,\ ] \hookrightarrow m: \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$$

$$\underline{\Delta}: \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \dashrightarrow \underline{\Delta^{\mathcal{F}}}(x) = \mathcal{F}\Delta(x)\mathcal{F}^{-1}$$

Corresponding module (representation)

$$\underline{C(M)} = \underline{C(M)} \xrightarrow{\mathcal{F}} \underline{C_\star(M)}$$

$$(fg)(x) = f(x)\,g(x) = \cdot(f \otimes g) \qquad (f \star g)(x) = f(x)g(x) + \dots$$

Leibniz "rule" (module algebra)

$$e^{a^\mu\partial_\mu}(fg) = (e^{a^\mu\partial_\mu} f)(e^{a^\mu\partial_\mu} g) \ldots$$

$$fg(x+a) = f(x+a)\,g(x+a)$$

$$\Big\downarrow \text{ first order}$$

$$\partial_\mu(f\cdot g) = (\partial_\mu f)\cdot g + f\cdot(\partial_\mu g) = \cdot\underbrace{(\partial_\mu \otimes 1 + 1 \otimes \partial_\mu)}_{\Delta\partial_\mu \ldots}(f\otimes g)$$

$$\Delta : H \to H \otimes H$$

$$\boxed{\begin{array}{c} \partial_\mu(f\cdot g) = \cdot(\Delta\partial_\mu)(f\otimes g) \\ \Big\downarrow F \\ \partial_\mu(f\star g) = \star(\Delta^F\partial_\mu)(f\otimes g) \end{array}}$$

Properties of the $\star$-product

- as a vector space $C_\star(M) \cong C(M)$
- non-local (involves derivatives)
- braided: $f \star g = \underbrace{(\bar{R}^\alpha \triangleright g) \star (\bar{R}_\alpha \triangleright f)}_{R\text{-matrix}}$ $\quad R^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha \in H \otimes H$
- associative

Example (Moyal product): $f \star g = f \exp(\Theta^{\alpha\beta} \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta) g$

$$\Downarrow$$

$$[x^\mu, x^\nu]_\star = \Theta^{\mu\nu}$$

$\star$-geometry

Explicit form: $g \star h = \cdot\left[\mathcal{F}^{-1} \triangleright (g \otimes h)\right] = (\bar{f}^{\alpha} \triangleright g)(\bar{f}_{\alpha} \triangleright h)$

$\mathcal{F}^{-1} = \bar{f}^{\alpha} \otimes \bar{f}_{\alpha} \in H \otimes H$

$\star$-Lie derivative: $\mathcal{L}^{\star}_{u}(g) := \mathcal{L}_{\bar{f}^{\alpha} \triangleright u}(\bar{f}_{\alpha} \triangleright g)$

$h, g \in C_{\star}(M)$

$u \in \mathfrak{X}_{\star}(M)$

Properties: $\mathcal{L}^{\star}_{h \star u}(g) = h \star \mathcal{L}^{\star}_{u}(g)$

$$\mathcal{L}^{\star}_{u}(g \star h) = \mathcal{L}^{\star}_{u}(g) \star h + (\bar{R}^{\alpha} \triangleright g) \star \mathcal{L}^{\star}_{\bar{R}_{\alpha} \triangleright u}(h)$$

$$\mathrm{Riemann}^\star(u,v,w) = -\mathrm{Riemann}^\star(\bar{R}^\alpha \triangleright u, \bar{R}_\alpha \triangleright v, w)$$

$$\mathfrak{X}_\star \otimes \mathfrak{X}_\star \otimes \mathfrak{X}_\star \to \mathfrak{X}_\star$$

Vacuum Einstein equation:

$$R^\star_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \star R^\star = 0$$

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