κ-Braided non-commutative field theory

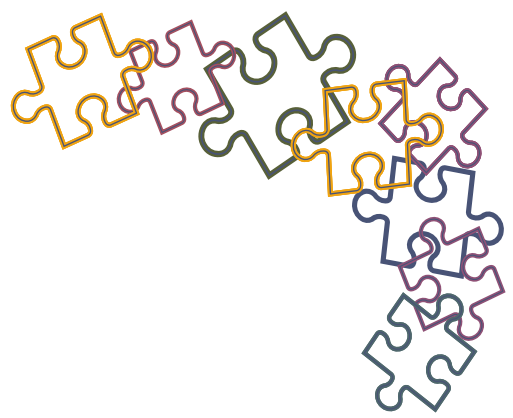## COST CA18108 Third Training School in Pałac Wojanów
18/02/2023

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[Lizzi,Mercati, PRD 103 (2021)]
[Di Luca, Mercati, arXiv:2211.11627v1]

Introduction

[Szabo, PR 378 (2003)]
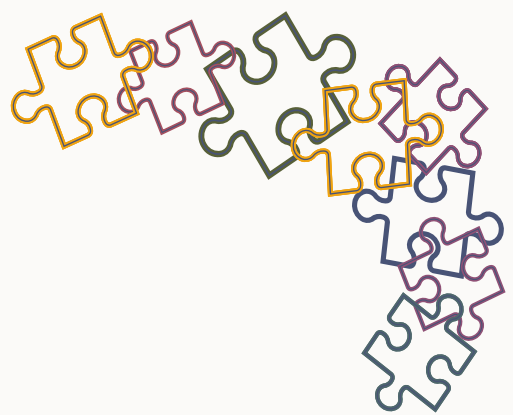[Aschieri, Dimitrijevic, Kulish, Lizzi, Wess, Springer Berlin Heidelberg (2011)]
[Chamseddine, Consani, Higson, Khalkhali, Moscovici, Yu, Springer International Publishing (2020)]

QFT coupled to classical GR

The support of a quantum field cannot be localized better than

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}}$$

κ – Minkowski

[Celeghini, Giachetti, Sorace, Tarlini, JMP 31 (1990)]
[Lukierski, Nowicki, Ruegg, PLB 264 (1991), 293 (1992)]
[Majid, Ruegg, PLB 334 (1994)] [Zakrzewski, JPA 27 (1993)]

$$[x^{\mu}, x^{\nu}] = \frac{i}{\kappa}(v^{\mu}x^{\nu} - v^{\nu}x^{\mu}), \quad \mu, \nu = 0, \ldots, d \qquad v^{\mu} \in \mathbb{R}^{d+1},$$

$$(x^{\mu})^{\dagger} = x^{\mu}, \quad [\kappa] = \ell^{-1} = (\text{length})^{-1} \text{ scale},$$

$x^{\mu} \in A$ = coordinate algebra = 'non-commutative functions' = scalar fields

κ– Poincaré

Poincaré group described through $\mathrm{C}[\mathrm{ISO}(3,1)]$

$$\Delta[\Lambda^\mu_{\ \nu}] = \Lambda^\mu_{\ \alpha} \otimes \Lambda^\alpha_{\ \nu}, \quad \Delta[a^\mu] = \Lambda^\mu_{\ \nu} \otimes a^\nu + a^\mu \otimes 1,$$

$$S[\Lambda^\mu_{\ \nu}] = (\Lambda^{-1})^\mu_{\ \nu}, \qquad S[a^\mu] = -(\Lambda^{-1})^\mu_{\ \nu} a^\nu, \qquad \epsilon[\Lambda^\mu_{\ \nu}] = \delta^\mu_\nu, \qquad \epsilon[a^\mu] = 0$$

Non-commutative deformation of $\mathrm{C}[\mathrm{ISO}(3,1)] \dashrightarrow \mathrm{C}_\kappa[\mathrm{ISO}(3,1)]$

$$\left[\Lambda^\mu_{\ \nu}, a^\gamma\right] = \frac{i}{\kappa}\left[\left(\Lambda^\mu_{\ \alpha} v^\alpha - v^\mu\right)\Lambda^\gamma_{\ \nu} + \left(\Lambda^\alpha_{\ \nu} g_{\alpha\beta} - g_{\nu\beta}\right)v^\beta g^{\mu\gamma}\right]$$

$$\left[\Lambda^\mu_{\ \nu}, \Lambda^\alpha_{\ \beta}\right] = 0, \ \ [a^\mu, a^\nu] = \frac{i}{\kappa}(v^\mu a^\nu - v^\nu a^\mu),$$

$$\Lambda^\mu_{\ \alpha} \Lambda^\nu_{\ \beta}\, g^{\alpha\beta} = g^{\mu\nu}, \Lambda^\mu_{\ \alpha} \Lambda^\nu_{\ \beta}\, g_{\mu\nu} = g_{\alpha\beta}$$

Hopf algebra $\quad \forall g_{\mu\nu}: \quad g_{\mu\nu}g^{\nu\gamma} = \delta^\gamma_\mu$ $\qquad$ Different models $\leftrightarrow \boxed{v^\mu v^\nu g_{\mu\nu}}$

Relativity principle

Non-commutative Poincaré transformation

Map $\overline{(\cdot)} :\ A \to C_\kappa[ISO(3,1)] \otimes A$

$$\bar{x}^{\,\mu} = \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad \left[x^\mu, \Lambda^\rho{}_\sigma\right] = [x^\mu, a^\nu] = 0$$

Given the $C_\kappa[ISO(3,1)]$ algebra

$$\left[\bar{x}^\mu, \bar{x}^\nu\right] = \frac{i}{\kappa}\left(v^\mu \bar{x}^\nu - v^\nu \bar{x}^\mu\right) = \overline{\left[x^\mu, x^\nu\right]}$$

the commutation relations are the same to all observers.

Field theory on κ-Minkowski

Multilocal functions

QFT requires the concept of N-point functions. In the commutative case, given the Abelian algebra of coordinates F, two-point functions are elements of $F \otimes F$, generated by

$$x_1^\mu = x^\mu \otimes 1, \quad x_2^\mu = 1 \otimes x^\mu, \quad 1 = 1 \otimes 1$$

Starting from the non-abelian algebra A, the canonical algebra structure on the tensor product $A \otimes A$ is

$$\left[x_1^\mu, x_1^\nu\right] = \frac{i}{\kappa}\left(v^\mu x_1^\nu - v^\nu x_1^\mu\right), \quad \left[x_2^\mu, x_2^\nu\right] = \frac{i}{\kappa}\left(v^\mu x_2^\nu - v^\nu x_2^\mu\right),$$

$$\left[x_1^\mu, 1\right] = \left[x_2^\mu, 1\right] = 0, \quad \boxed{\left[x_1^\mu, x_2^\nu\right] = 0} \longrightarrow \text{Not covariant!}$$
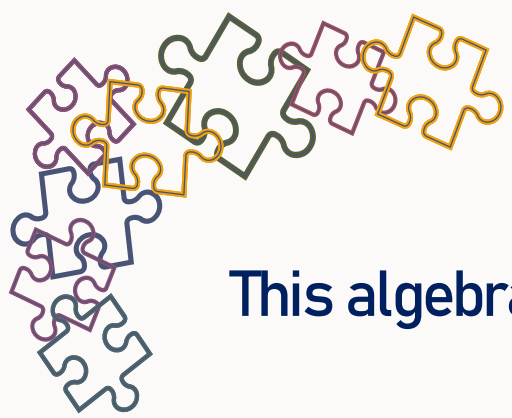
Braided tensor product algebra

We look for a deformation $A \otimes_\kappa A$ of the tensor product algebra, such that:

$$\left[x_1^\mu, x_1^\nu\right] = \frac{i}{\kappa}\left(v^\mu x_1^\nu - v^\nu x_1^\mu\right), \ \left[x_2^\mu, x_2^\nu\right] = \frac{i}{\kappa}\left(v^\mu x_2^\nu - v^\nu x_2^\mu\right),$$

$$\left[x_1^\mu, 1\right] = \left[x_2^\mu, 1\right] = 0, \ \ \left[\bar{x}_1^\mu, \bar{x}_2^\nu\right] = \overline{\left[x_1^\mu, x_2^\nu\right]}$$

Assuming that $\left[x_1^\mu, x_2^\nu\right]$ goes to zero as $v^\mu \to 0$ and that it is linear in x_a^μ, we find a unique solution:

$$\left[x_1^\mu, x_2^\nu\right] = \frac{i}{\kappa}\left[v^\mu x_1^\nu - v^\nu x_2^\mu - g^{\mu\nu} g_{\rho\sigma} v^\rho (x_1^\sigma - x_2^\sigma)\right]$$
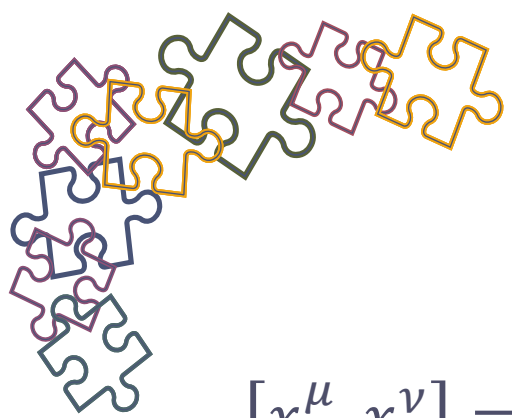
Braided tensor product algebra
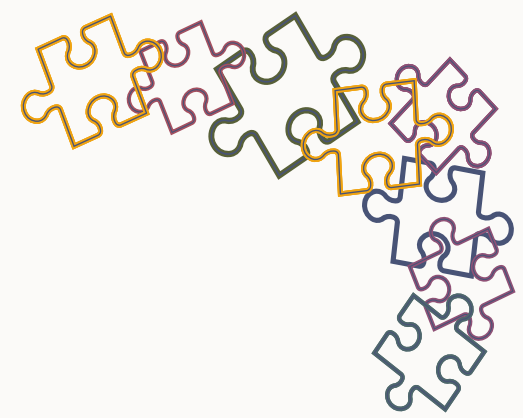
This algebra can be immediately extended to N points, $A^{\otimes_\kappa^N}$:

$$\left[x_a^\mu, x_b^\nu\right] = \frac{i}{\kappa}\left[v^\mu x_a^\nu - v^\nu x_b^\mu - g^{\mu\nu}g_{\rho\sigma}v^\rho\left(x_a^\sigma - x_b^\sigma\right)\right], \quad a, b = 1, \dots, N$$
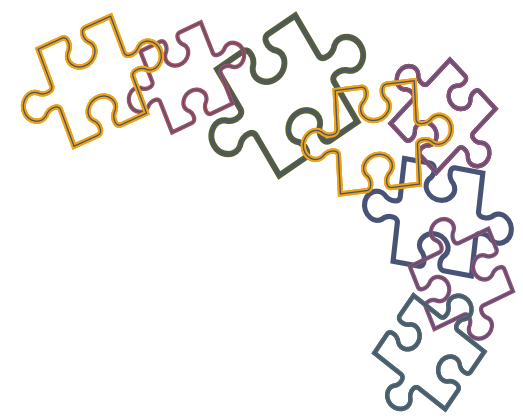
If we impose the Jacobi rule, we find

$$\left[x_a^\mu, \left[x_b^\nu, x_c^\rho\right]\right] + \left[x_b^\nu, \left[x_c^\rho, x_a^\mu\right]\right] + \left[x_c^\rho, \left[x_a^\mu, x_b^\nu\right]\right] =$$

$$= -\frac{g_{\alpha\beta}v^\alpha v^\beta}{\kappa^2}\left(g^{\nu\rho}\left(x_c^\mu - x_b^\mu\right) + g^{\rho\mu}(x_a^\nu - x_c^\nu) + g^{\mu\nu}\left(x_b^\rho - x_a^\rho\right)\right) = 0$$

and this is satisfied only if $g_{\alpha\beta}v^\alpha v^\beta = 0$, i.e. the so-called *lightlike*-κ-Minkowski non-commutativity.

$A^{\otimes^N_\kappa}$ is a mostly-commutative algebra

$$\left[x_a^\mu, x_b^\nu\right] = \frac{i}{\kappa}\left[v^\mu x_a^\nu - v^\nu x_b^\mu - g^{\mu\nu} g_{\rho\sigma} v^\rho (x_a^\sigma - x_b^\sigma)\right]$$

The coordinate differences $\delta x_{ab}^\mu = x_a^\mu - x_b^\mu$ close an Abelian subalgebra:

$$\left[\delta x_{ab}^\mu, \delta x_{cd}^\nu\right] = 0$$

κ– Poincaré invariant field theory

QFT requires the concept of N-point functions, i.e. , in the commutative case, Poincaré-invariant distributions which admit a Fourier representation.

In the non-commutative case, we can choose an ordering prescription for the x_a^μ and try to write a function of N points as
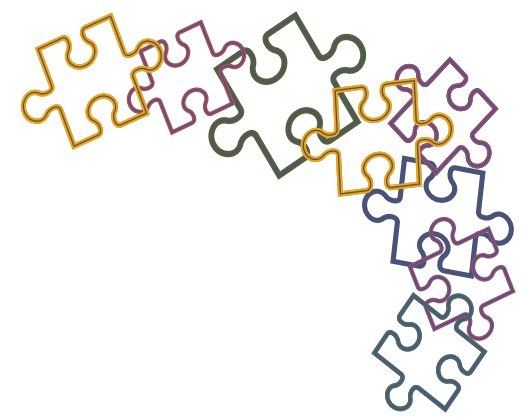
$$f(x_a^\mu) = \int d^4 k^1 \dots d^4 k^N \ \tilde{f}(k_\mu^a) \ e^{ik_\mu^1 x_1^\mu} \dots e^{ik_\mu^N x_N^\mu}$$

and we can prove that κ-Poincaré invariance implies that $f\left(x_a^\mu\right)$ only depends upon coordinate differences.
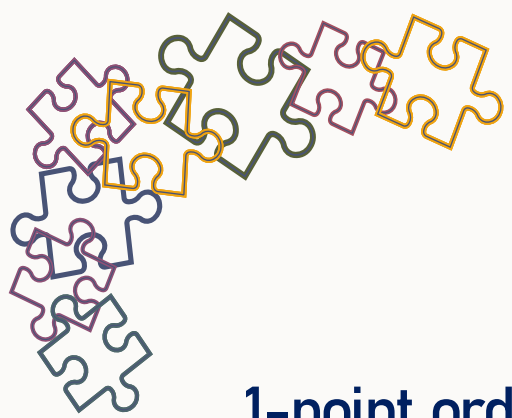
N-point functions are commutative!

1+1D κ – deformed field theory

Let us set w.l.o.g. $\mathrm{g}^{\mu\nu} = \mathrm{diag}(-1,1) \implies v^\mu = (1,1)$.

The commutation relations are conveniently written in terms of lightcone coordinates:

$$\mathrm{x}_a^{\pm} = \mathrm{x}_a^0 \pm \mathrm{x}_a^1 \implies \left[x_a^+, x_b^+\right] = 2i(x_a^+ - x_b^+), \ \left[x_a^+, x_b^-\right] = 2ix_b^-, \ \left[x_a^-, x_b^-\right] = 0,$$

1+1D κ– deformed field theory

1-point ordered plane waves $e_a[k_\mu] = e^{ik_- x_a^-} e^{ik_+ x_a^+}, \ k_\pm \in \mathbb{R}$, form a Lie group under noncommutative multiplication.
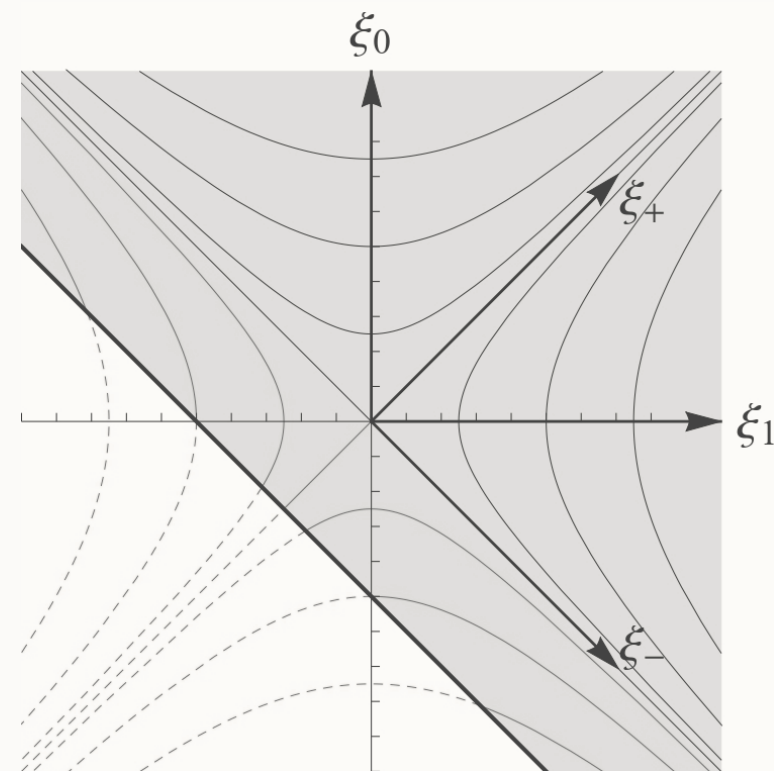
The group manifold is a **half** Minkowski space

└──→ Lorentz breaking?

Nonlinear action of the κ –Poincaré group
on momentum space:

$$\bar{e}_a[k] = e^{ik_- \bar{x}_a^-} e^{ik_+ \bar{x}_a^+} = e_a[\lambda(k, \omega)] e^{ik_- a^-} e^{ik_+ a^+}$$

$$\lambda(k, \omega) = \left(e^{-\omega} k_-, \frac{1}{2} \ln \left[1 + e^\omega \left(e^{\frac{2k_+}{\kappa}} - 1 \right) \right] \right)$$
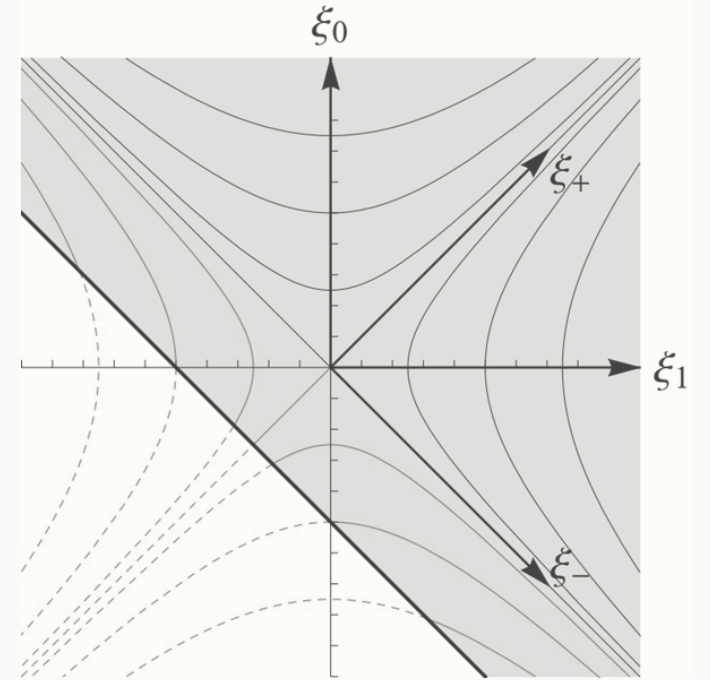
The missing jigsaw pieces

For $k_+ < 0$, $e^\omega > \dfrac{1}{1 - e^{\frac{2k_+}{\kappa}}}$,

$$\lambda(k, \omega)_+ = \frac{1}{2} \ln\left[e^\omega \left(1 - e^{\frac{2k_+}{\kappa}}\right) - 1 \right] \boxed{+ \frac{i\pi}{2} + n\pi i}$$

[Arzano, Bevilacqua, Kowalski-Glikman, Rosati, Unger, PRD 103 (2021)]

$$e_a[k] \to \epsilon_a[p] \boxed{e^{-n\pi x_a^+}} = e^{ip_- x_a^-} e^{ip_+ x_a^+} e^{-\frac{\pi}{2} x_a^+} e^{-n\pi x_a^+}$$

$= (-1)^n$ in the self-adjoint representation of $A^{\otimes_\kappa^N}$: always appears to an even power in N-point functions
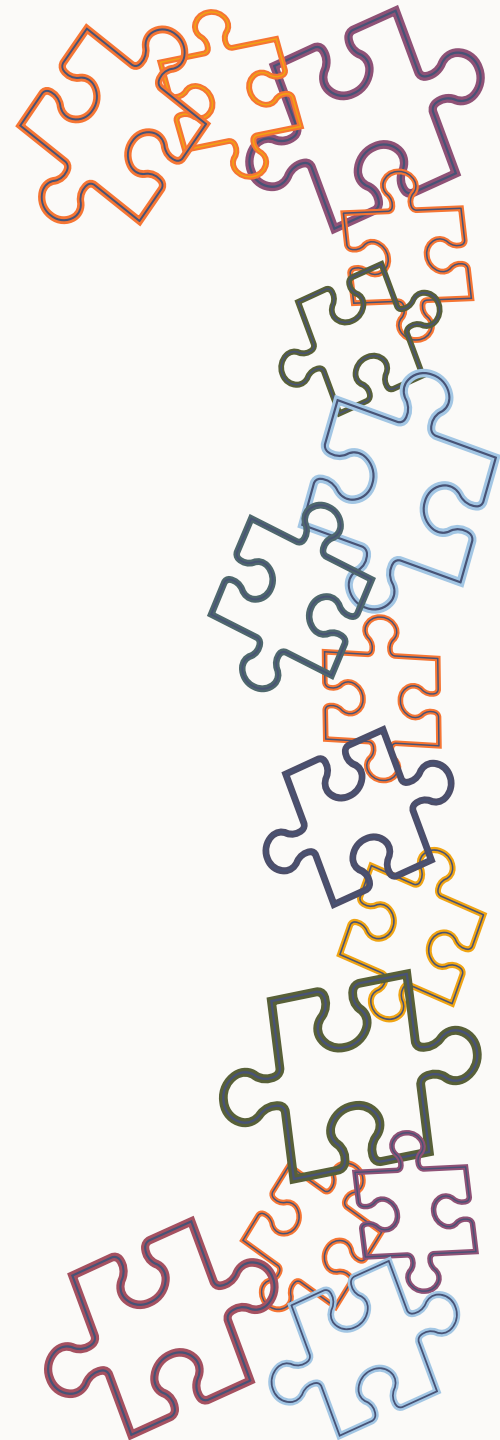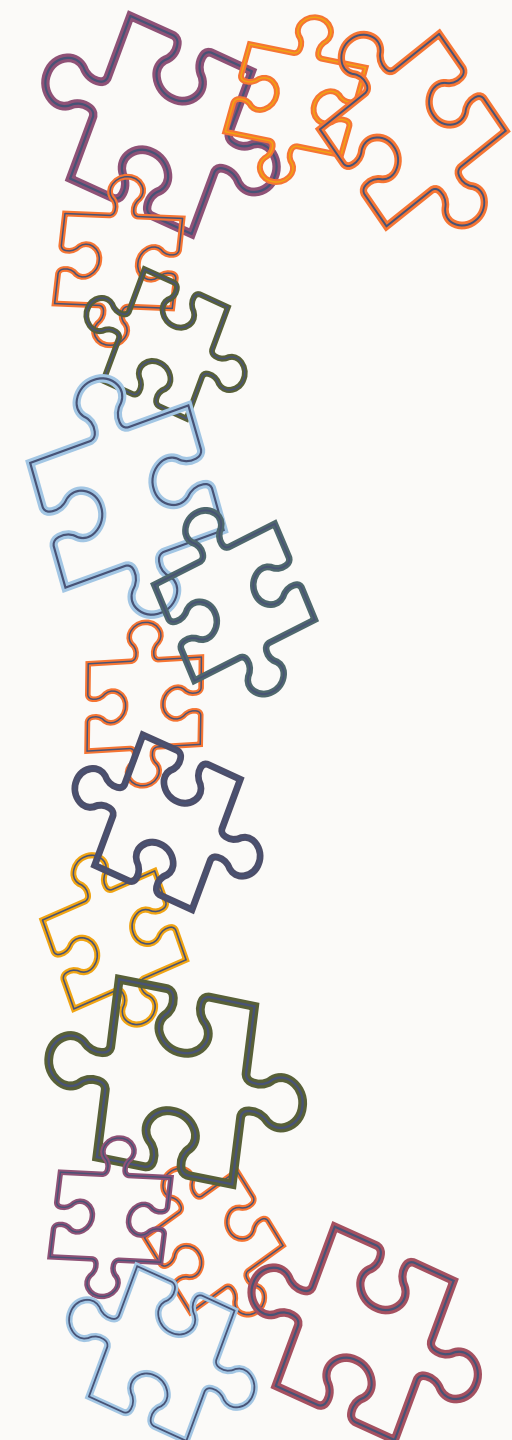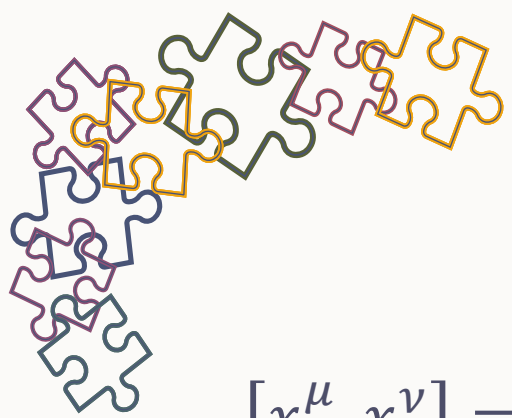
Conclusions

$$\phi(x_a) = \int_{\mathbb{R}_+} dk_+ \frac{e^{2k_+}}{e^{2k_+} - 1}\Big(a(k_+)e_a(k_+) + e^{-2k_+}b^*(k_+)e_a^\dagger(k_+)\Big) +$$

$$\int_{\mathbb{R}_+} dk_+ \frac{e^{2k_+}}{e^{2k_+} + 1}\Big(\alpha(k_+)\epsilon_a(k_+) + e^{-2k_+}\beta^*(k_+)\epsilon_a^\dagger(k_+)\Big)$$
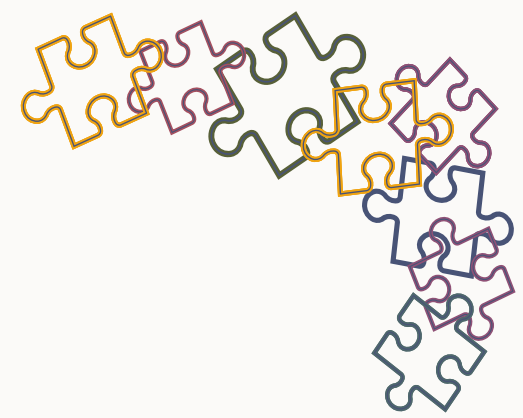
- All 2-point functions are identical to the commutative ones;

- $\big[\hat\phi(x_1), \hat\phi^\dagger(x_2)\big] = i\Delta_{PJ}(x_1 - x_2), \ \ \big[\hat\phi(x_1), \hat\phi(x_2)\big] = \big[\hat\phi^\dagger(x_1), \hat\phi^\dagger(x_2)\big] = 0;$

- consistent and κ-Poincaré invariant deformed h.o. algebra.

- Future perspectives: N-point functions and interacting fields.

Thank you

$A^{\otimes^N_\kappa}$ is a mostly-commutative algebra

$$\left[x_a^\mu, x_b^\nu\right] = \frac{i}{\kappa}\left[v^\mu x_a^\nu - v^\nu x_b^\mu - g^{\mu\nu}g_{\rho\sigma}v^\rho(x_a^\sigma - x_b^\sigma)\right]$$

The coordinate differences $\delta x_{ab}^\mu = x_a^\mu - x_b^\mu$ close an Abelian subalgebra:

$$\left[\delta x_{ab}^\mu, \delta x_{cd}^\nu\right] = 0$$

All the noncommutativity is concentrated on the center-of-mass degrees of freedom

$$x_{cm}^\mu = \frac{1}{N}\Sigma_a x_a^\mu, \quad \left[x_{cm}^\mu, x_{cm}^\nu\right] = \frac{i}{\kappa}(v^\mu x_{cm}^\nu - v^\nu x_{cm}^\mu)$$

$$y_a^\mu = x_a^\mu - x_{cm}^\mu, \quad \left[x_{cm}^\mu, y_a^\nu\right] = \frac{i}{\kappa}\left(g^{\mu\nu}g_{\rho\sigma}v^\rho y_a^\sigma - v^\nu y_a^\mu\right), \quad \left[y_a^\mu, y_b^\nu\right] = 0$$

A representation of $A^{\otimes^\kappa}$

The component of x_{cm}^μ along v^μ, $x_{cm}^- = g_{\mu\nu} v^\mu x_{cm}^\nu$, commutes with all the y_a^μ:

$$\left[x_{cm}^-, y_a^\mu \right] = 0, \quad \left[x_{cm}^\mu, x_{cm}^- \right] = \frac{i}{\kappa} v^\mu x_{cm}^-,$$

but the other components are irreducibly noncommutative: the maximal Abelian subalgebra is generated by x_{cm}^- and y_a^μ (with the constraint $\Sigma_a y_a^\mu = 0$).

In 3+1 dimensions, choosing w.l.o.g. $g^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, $\quad v^\mu = (1,1,0,0)$, $x_{cm}^- = x_{cm}^0 - x_{cm}^1$ and y_a^μ are multiplicative operators with real spectrum and

A representation of $\mathbb{A}^{\otimes_\kappa^N}$

$x_{cm}^+ = x_{cm}^0 + x_{cm}^1$, x_{cm}^2, x_{cm}^3 can be represented as sums of Lorentz matrices and a x_{cm}^-–dilatation:
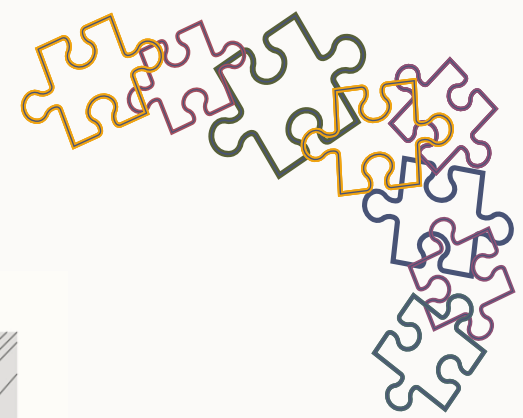
$$x_{cm}^2 = M^{12} - M^{02}, \quad x_{cm}^3 = M^{13} - M^{03},$$

$$x_{cm}^+ = 2M^{10} + 2i x_{cm}^- \frac{\partial}{\partial x_{cm}^-} + i,$$

$$M^{\mu\nu} = i \sum_{a=1}^{N-1} y_a^\mu g^{\nu\rho} \frac{\partial}{\partial y_a^\rho} - y_a^\nu g^{\mu\rho} \frac{\partial}{\partial y_a^\rho}$$ Generators of rigid Lorentz transformations of N points
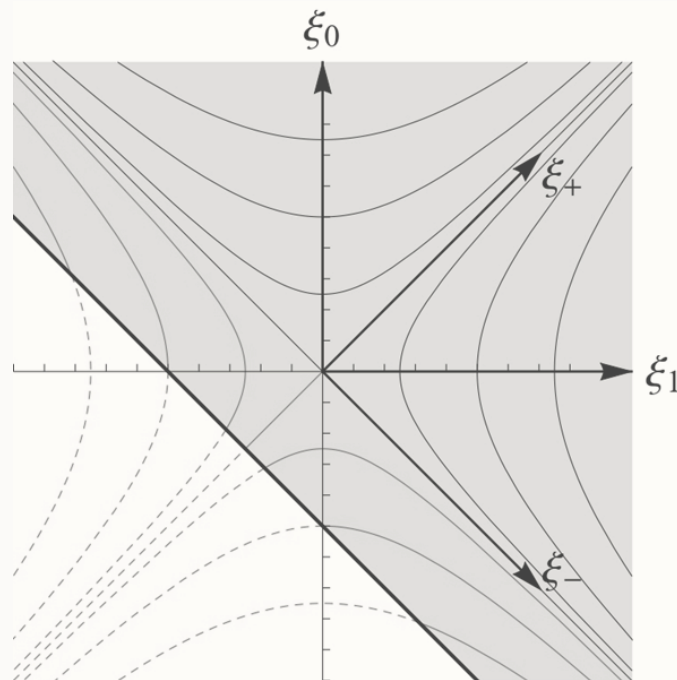
Geometry of momentum space

Left-invariant metric:

$$ds^2 = d\xi_- d\xi_+,$$
$$\xi = \left(k_-, \frac{e^{2k_+} - 1}{2}\right) \to \xi_+ > -\frac{1}{2}$$

Right-invariant metric:

$$ds^2 = d\chi_- d\chi_+,$$
$$\chi = \left(-e^{2k_+} k_-, \frac{e^{-2k_+} - 1}{2}\right) \to \chi_+ > -\frac{1}{2}$$

κ - **Lorentz transformation**
$$\xi_\pm = e^{\pm\omega} \xi_\pm$$
$$\chi_\pm = e^{\mp\omega} \chi_\pm$$