Total momentum and other Noether charges for particles interacting in a quantum spacetime

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Quantum spacetimes

Spacetime non-commutativity is expected at short distance scales ($\ell \simeq l_p$) in several quantum gravity proposals.

Invariance of the length scale ℓ requires a deformation of relativistic symmetries.

For phenomenological opportunities, it is important to understand the properties of the associated Noether charges, still a blurry concept in the field.

Quantum-Spacetime Phenomenology, G. Amelino-Camelia, Living Rev. Rel 16 (2013)

Spatial 2D κ-Minkowski

Non-commutativity between coordinates

$$[x_1, x_2] = i\ell x_1$$

Deformed symmetry algebra of 2D space

$$[P_1, P_2] = 0 \qquad [R, P_1] = i\left(P_2 - \ell P_2^2 + \frac{\ell}{2} P_1^2 + \frac{2}{3}\ell^2 P_2^3\right) \qquad [R, P_2] = -iP_1$$

Central element of the algebra (Casimir operator)

$$C = p_1^2 + p_2^2 + \ell p_1^2 p_2 + \ell^2 p_1^2 p_2^2 + \ell^2 p_2^4$$

can be interpreted as a deformation of the kinetic energy

Planck-scale soccer-ball problem: a case of mistaken identity, G. Amelino-Camelia, Entropy 19 (2017)

Symplectic structure

Non-commutativity between coordinates

$$[x_1, x_2] = i\ell x_1$$

Deformed symplectic structure

$$[x_1, P_1] = i \qquad [x_2, P_2] = i \qquad [x_1, P_2] = 0 \qquad [x_2, P_1] = -i\ell P_1$$

$$[R, x_1] = ix_2$$

$$[R, x_2] = -i(x_1 - \ell x_1 P_2 + \frac{\ell}{2} x_2 P_1 + \frac{\ell}{2} P_1 x_2 + \ell^2 x_1 P_2^2 + \frac{\ell^2}{4}(x_1 P_1^2 + P_1^2 x_1))$$

κ-coproduct composition law

In a process $A + B \to C + D$, the conservation law $P_1^A + P_1^B = P_1^C + P_1^D$ is not compatible with deformed rotations

In quantum spacetimes, there are several alternatives for composition laws giving rise to covariant conservation laws

The κ-coproduct composition law is non-commutative but associative

$$(p^A \oplus_\kappa p^B)_1 = p_1^A + p_1^B - \ell p_2^A p_1^B + \frac{\ell^2}{2}\left(p_2^A\right)^2 p_1^B \qquad (p^A \oplus_\kappa p^B)_2 = p_2^A + p_2^B$$

$$R^A \oplus_\kappa R^B = R^A + R^B - \ell p_2^A R^B + \frac{\ell^2}{2}\left(p_2^A\right)^2 R^B$$

Proper dS composition law

The proper dS composition law is commutative but non-associative

$$(p^A \oplus_{dS} p^B)_1 = p_1^A + p_1^B - \ell\left(p_2^A p_1^B + p_1^A p_2^B\right) + \frac{\ell^2}{2}\left[\left(p_2^A p_1^B + p_1^A p_2^B\right)\left(p_2^A + p_2^B\right) - p_1^A (p_1^B)^2 - \left(p_1^A\right)^2 p_1^B\right]$$

$$(p^A \oplus_{dS} p^B)_2 = p_2^A + p_2^B + \ell p_1^A p_1^B - \frac{\ell^2}{2}\left[\left(p_1^A\right)^2 p_2^B + (p_1^B)^2 p_2^A - p_1^A p_1^B \left(p_2^A + p_2^B\right)\right]$$

$$(R^A \oplus_{dS} R^B) = R^A + R^B$$

Both $\oplus_\kappa$ and $\oplus_{dS}$ are composition laws used to construct total charges that close the deformed symmetry algebra of 2D space.

Total momentum: which one?

For free multiparticle systems, any non-linear function of the momenta can be regarded as the conserved total charge of the system.

It is only when interactions are present that we deal with a meaningful physical scenario and can define the Noether charges for the system in an unambiguous way.

We will constructively deduce the conserved charges by making use of the spatial 2D κ-Minkowski scenario in a first quantization scheme

A case study: the elastic potential

In standard QM, a system of two particles interacting via an elastic potential is characterized by conservation of total momentum and total angular momentum

$$H_0 = \frac{(\vec{p}^A)^2}{2m} + \frac{(\vec{p}^B)^2}{2m} + \frac{1}{2} g(\vec{q}^A - \vec{q}^B)^2 \qquad \left[q_i^I, p_j^J\right] = i\delta^{IJ}\delta_{ij}$$

The conserved charges read

$$\vec{P} = \vec{p}^A + \vec{p}^B \qquad R_0 = R_0^A + R_0^B \qquad R_0^I = q_1^I p_2^I - q_2^I p_1^I$$

Indeed

$$[H_0, R_0] = \left[H_0, \vec{P}\right] = 0$$

Our objective will be to deduce the conserved charges starting from a generalization of the elastic interaction Hamiltonian

The deformed Hamiltonian

The deformed kinetic term expanded at order ℓ^2 is inspired by the Casimir

$$H_K^I = \frac{(p_1^I)^2}{2m} + \frac{(p_2^I)^2}{2m} + \ell \frac{(p_1^I)^2 p_2^I}{2m} + \frac{\ell^2 (p_1^I)^2 (p_2^I)^2}{4m} + \frac{\ell^2 (p_2^I)^4}{24m}$$

while the most general ansatz for the quadratic potential for two particles reads

$$V^{AB} = \frac{1}{2} g (\vec{x}^A - \vec{x}^B)^2 + \ell g \sum \alpha_{ijk}^{IJK} p_i^I x_j^J x_k^K + \ell^2 g \sum \beta_{ijkh}^{IJKH} p_i^I p_j^J x_k^K x_h^H$$

For the three-particle potential we consider a similar ansatz

$$V^{ABC} = \frac{1}{2} g (\vec{x}^A - \vec{x}^B)^2 + \frac{1}{2} g (\vec{x}^A - \vec{x}^C)^2 + \frac{1}{2} g (\vec{x}^C - \vec{x}^B)^2 + \ell g \sum \tilde{\alpha}_{ijk}^{IJK} p_i^I x_j^J x_k^K + \ell^2 g \sum \tilde{\beta}_{ijkh}^{IJKH} p_i^I p_j^J x_k^K x_h^H$$

For the two particle-case we consider Hamiltonians of the type $H^{AB} = H_K^A + H_K^B + V^{AB}$

Proper dS: two particle dynamics

The coefficients α and β in the potential ansatz need to satisfy the constraints

$$[(p^A \oplus_{dS} p^B)_1, H^{AB}] = 0 \qquad [(p^A \oplus_{dS} p^B)_2, H^{AB}] = 0 \qquad [R^A + R^B, H^{AB}] = 0$$

for the quantities $P_i^{dS} = (p^A \oplus_{dS} p^B)_i$ and $R^{dS} = R^A + R^B$ to be conserved charges.

Proper dS: two particle dynamics

$$
\begin{aligned}
H_{dS}^{AB} &= H_K^A + H_K^B + V_{dS}^{AB} \\
&= H_K^A + H_K^B + \frac{g}{2}\{(\vec{x}^A - \vec{x}^B)^2 + 2\ell\left[-p_2^A\left(x_1^A\right)^2 + \frac{1}{2}p_1^A x_1^A x_2^A + \frac{1}{2}x_2^A x_1^A p_1^A + p_2^A x_1^A x_1^B - x_2^A p_1^A x_1^B + (A \leftrightarrow B)\right] \\
&+ \frac{\ell^2}{2}[(p_1^B)^2\left(-2\left(x_2^A\right)^2 + 6x_2^A x_2^B - 2(x_2^B)^2\right) + 4p_1^B p_2^B x_1^A x_2^A - p_1^A p_1^B x_2^A x_2^B + p_1^B p_2^A x_1^A x_2^B - 6p_1^B x_1^A x_2^B p_2^B \\
&- 2p_1^B x_1^B\left(p_2^A x_2^A - x_2^B p_2^B\right) - 2(p_2^B)^2\left(\left(x_1^A\right)^2 - x_1^A x_1^B - (x_1^B)^2\right) + p_2^B p_1^A x_2^A x_!^B \ - 2p_2^B p_2^A x_1^A x_1^B - 3p_2^B x_2^A x_1^B p_1^B \\
&+ 2x_1^A p_1^A\left(p_1^A x_1^A - p_1^A x_!^B + p_2^A x_2^A - \frac{3}{2}p_2^A x_2^B + \frac{3}{2}x_2^B p_2^B\right) + x_2^A p_2^A x_1^B p_1^B + (A \leftrightarrow B)]\}
\end{aligned}
$$

This Hamiltonian is invariant under the exchange of particles A and B, a property that reflects the commutativity of the proper dS composition law.

Unique selection of conserved charges

General ansatz for the conserved charges

$$P_1^{tot} = \sum p_1^I + \ell \gamma_{ij}^{IJ} p_i^I p_j^J + \ell^2 \Gamma_{ijk}^{IJK} p_i^I p_j^J p_k^K$$
$$P_2^{tot} = \sum p_2^I + \ell \theta_{ij}^{IJ} p_i^I p_j^J + \ell^2 \Theta_{ijk}^{IJK} p_i^I p_j^J p_k^K$$
$$R^{tot} = \sum R^I + \ell \phi_i^{IJ} p_i^I R^J + \ell^2 \Phi_{ij}^{IJK} p_i^I p_j^J R^k$$

By requiring them to commute with H_{dS}^{AB}, the coefficients are **uniquely** fixed yielding

$$P_i^{tot} = (p^A \oplus_{dS} p^B)_i \qquad R^{tot} = R^A + R^B$$

Proper dS: three particle dynamics

The three particle charges

$$\tilde{P}_i^{dS} = \left((p^A \oplus_{dS} p^B) \oplus_{dS} p^C\right)_i \qquad \tilde{R}^{dS} = R^A + R^B + R^C$$

are compatible with the Hamiltonian $H_{dS}^{ABC} = H_K^A + H_K^B + H_K^C + V_{dS}^{ABC}$, with

$$V_{dS}^{ABC} = V_{dS}^{AB} + V_{dS}^{BC} + V_{dS}^{CA} + V_{dS(\star)}^{ABC}$$

and

$$V_{dS(\star)}^{ABC} = \frac{g\ell^2}{2}\,[(p_1^C p_2^B - p_2^C p_1^B)\left(x_1^C x_2^A - x_2^C x_1^A\right) - p_1^C p_2^A x_2^C x_1^B - p_2^C p_1^A x_1^C x_2^B + (p_1^B p_1^A x_2^C\left(2x_2^C - x_2^A - x_2^B\right) - p_1^B p_2^A x_2^C\left(2x_1^C - x_1^B\right) - 2p_2^B p_1^A x_1^C x_2^C + p_2^B p_2^A x_1^C\left(2x_1^C - x_1^A - x_1^B\right) + p_1^A x_1^C p_2^B x_2^B + x_1^C p_1^C p_2^A x_2^B + x_2^C p_2^C p_1^A x_1^B + x_1^A p_1^A p_2^B x_2^C + x_2^A p_2^A p_1^B x_1^C]$$

Unique selection of conserved charges

General ansatz for the conserved charges

$$\widetilde{P_1}^{tot} = \sum p_1^I + \ell \tilde{\gamma}_{ij}^{IJ} p_i^I p_j^J + \ell^2 \tilde{\Gamma}_{ijk}^{IJK} p_i^I p_j^J p_k^K$$
$$\widetilde{P_2}^{tot} = \sum p_2^I + \ell \tilde{\theta}_{ij}^{IJ} p_i^I p_j^J + \ell^2 \tilde{\Theta}_{ijk}^{IJK} p_i^I p_j^J p_k^K$$
$$\tilde{R}^{tot} = \sum R^I + \ell \tilde{\phi}_i^{IJ} p_i^I R^J + \ell^2 \tilde{\Phi}_{ij}^{IJK} p_i^I p_j^J R^k$$

By requiring them to commute with H_{dS}^{ABC}, the coefficients are **uniquely** fixed yielding

$$\widetilde{P}_i^{tot} = \left((p^A \oplus_{dS} p^B) \oplus_{dS} p^C \right)_i \qquad R^{tot} = R^A + R^B + R^C$$

Proper dS: Hamiltonian symmetrization

Consider the symmetrization of the 3 particle proper dS Hamiltonian

$$H^{ABC}_{dS(sym)} = H^A_K + H^B_K + H^C_K + V^{AB}_{dS} + V^{BC}_{dS} + V^{AC}_{dS} + \frac{1}{6}\sum_{\pi(A,B,C)} V^{ABC}_{dS(\star)}$$

Requiring $H^{ABC}_{dS(sym)}$ to commute with the most general charges, we find that the <u>**uniquely**</u> selected conserved quantities are

$$\widetilde{P}_i^{tot} = \frac{1}{3}[(p^A \oplus_{dS} p^B) \oplus_{dS} p^C + p^A \oplus_{dS} (p^B \oplus_{dS} p^C) + (p^A \oplus_{dS} p^C) \oplus_{dS} p^B]_i$$

$$R^{tot} = R^A + R^B + R^C$$

Even with a composition law that is not invariant under particle exchange (when more than two particles are involved), dynamics can select conserved charges invariant under particle exchange.

κ-coproduct: two particle case

Repeating the analysis for the κ-coproduct composition law, imposing

$$[(p^A \oplus_\kappa p^B)_1, H^{AB}] = 0 \quad [(p^A \oplus_\kappa p^B)_2, H^{AB}] = 0 \quad [(R^A \oplus_\kappa R^B), H^{AB}] = 0$$

a compatible Hamiltonian is

$$\begin{aligned} H_\kappa^{AB} &= H_K^A + H_K^B + V_\kappa^{AB} = \\ &= H_K^A + H_K^B \\ &+ \frac{g}{2}\Big\{(\vec{x}_A - \vec{x}_B)^2 + 2\ell\left[-p_2^A\left(x_1^A\right)^2 + x_1^A p_2^A x_1^B + \frac{1}{2}x_2^A p_1^A x_1^A + \frac{1}{2}x_2^A x_1^A p_1^A - x_2^B p_1^A x_1^A - \frac{1}{2}x_2^B p_1^A x_1^A\right] \\ &+ 2\ell^2\left[(p_2^A)^2(x_1^A)^2 + \frac{1}{2}x_1^A(p_1^A)^2 x_1^A - \frac{1}{2}x_1^A(p_2^A)^2 x_1^B\right]\Big\} \end{aligned}$$

Returning to the ansatz for the two particle charges, H_κ^{AB} **uniquely** selects charges $(p^A \oplus_\kappa p^B)_i$ and $(R^A \oplus_\kappa R^B)$

κ-coproduct: three particle case

The Hamiltonian $H_\kappa^{ABC} = H_K^A + H_K^B + H_K^C + V_\kappa^{AB} + V_\kappa^{AC} + V_\kappa^{BC} + V_{\kappa(\star)}^{ABC}$, with

$$V_{\kappa(\star)}^{ABC} = g\ell\left[p_1^B\left(x_1^C x_2^A - x_2^C x_1^A + x_1^A x_2^B - x_1^C x_2^B\right) + p_2^B x_1^B\left(x_1^C - x_1^A\right)\right] + \frac{g\ell^2}{2}\left[(p_1^B)^2\left(x_1^C x_1^A - x_1^B x_1^C + x_1^B x_1^A\right) - (p_2^B)^2\left(x_1^C x_1^A - 2x_1^A x_1^B\right) + p_1^B x_1^C\left(p_1^A x_1^A - p_2^B x_2^B\right) + p_1^B p_2^A\left(x_2^C x_1^A - x_1^A x_2^B\right) + p_2^B\left(p_1^B x_1^A x_2^C + p_2^A x_1^A x_1^B\right)\right]$$

<u>**uniquely**</u> selects total charges

$$\widetilde{P}_i^{\,tot} = (p^A \oplus_\kappa p^B \oplus_\kappa p^C)_i \qquad \tilde{R}^{tot} = (R^A \oplus_\kappa R^B \oplus_\kappa R^C)$$

κ-coproduct: Hamiltonian symmetrization?

H_κ^{AB} is not invariant under the exchange of particles A and B. A symmetrization of the form

$$H_{\kappa(sym)}^{AB} = \frac{1}{2}(H_\kappa^{AB} + H_\kappa^{BA})$$

<u>does not select</u> any charges, when resorting to the general charge ansatz.

In the three-particle case, the Hamiltonian

$$H_{\kappa(sym)}^{ABC} = H_K^A + H_K^B + H_K^C + \frac{1}{6}\sum_{\pi(A,B,C)} V_\kappa^{ABC}$$

also **<u>does not select</u>** any conserved charges. The κ-coproduct composition law is not suitable for constructing conserved charges that are symmetric under the exchange of partilces .

Conclusions

- Take-home message: in non-commutative spaces with non-linear transformation laws, interactions are essential in defining conserved Noether charges, as opposed to ordinary special relativity
- Even though the non-linear composition laws may not be symmetric under particle exchanges, suitable dynamics can select conserved charges which are symmetric under such exchanges

Thanks for the attention