κ-deformed complex fields, (discrete) symmetries, and charges [1,2]

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[1] M. Arzano, **A. B.**, J. Kowalski-Glikman, G. Rosati, and J. Unger, κ-deformed complex fields and discrete symmetries. Phys.Rev.D, 103:106015

[2] **A.B.**, J. Kowalski-Glikman, and W. Wislicki, κ-deformed complex scalar field: conserved charges, symmetries and their impact on physical observables. Phys.Rev.D, 105:105004.

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- CPT in the deformed context;
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κ-Minkowski spacetime and momentum space picture

Non-commutative coordinates: $[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa}\hat{x}^i$ ($\mathfrak{an}(3)$ algebra)

Physical insight: $\left[\frac{1}{\kappa}\right] = L$. However, this κ-deformed theory is intended as an effective theory modelling quantum gravitational effects $\implies \frac{1}{\kappa} \approx l_p$. '$\kappa \to \infty$' gives the 'classical' limit.

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$$\hat{x}^0 = -\frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix} \quad \hat{\mathbf{x}} = \frac{i}{\kappa}\begin{pmatrix} 0 & \epsilon^T & 0 \\ \epsilon & \mathbf{0} & \epsilon \\ 0 & -\epsilon^T & 0 \end{pmatrix}$$

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$$e^{A\hat{x}} = \sum_{n=0}^{\infty} \frac{(A\hat{x})^n}{n!} \quad \leftarrow \quad \text{Definition of exp}$$

κ-Minkowski spacetime and momentum space picture

$$\hat{e}_k = \begin{pmatrix} \frac{\bar{p}_4}{\kappa} & \frac{\mathbf{k}}{\kappa} & \frac{p_0}{\kappa} \\ \frac{\mathbf{p}}{\kappa} & \mathbf{1} & \frac{\mathbf{p}}{\kappa} \\ \frac{\bar{p}_0}{\kappa} & -\frac{\mathbf{k}}{\kappa} & \frac{p_4}{\kappa} \end{pmatrix} \qquad \begin{aligned} p_0 &= \kappa \sinh \frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa} \\ p_i &= k_i e^{k_0/\kappa} \\ p_4 &= \kappa \cosh \frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa} \end{aligned}$$

Notice that $\hat{e}_k \Leftrightarrow (p_0, p_i, p_4)^T$ and if $\mathcal{O} = (0, \ldots, 0, \kappa)^T$ then one can write $(p_0, p_i, p_4)^T = \hat{e}_k \mathcal{O}$, and importantly

$$-p_0^2 + \mathbf{p}^2 + p_4^2 = \kappa^2$$

$$p_4 > 0, \qquad p_+ := p_0 + p_4 > 0$$

κ-Minkowski spacetime and momentum space picture

Notice: both the k_a and the p_A can be interpreted as coordinates in (intrinsically curved) momentum space, and their sum is now non-trivial.

$$\hat{e}_k \hat{e}_l := \hat{e}_{k \oplus l} \quad \leftarrow \quad \text{Group property}$$

$$(k \oplus l)_0 = k_0 + l_0$$
$$(k \oplus l)_i = k_i + e^{-k_0/\kappa} l_i$$

$$(p \oplus q)_0 = \frac{p_0}{\kappa} q_+ + \frac{\mathbf{pq}}{p_+} + \frac{\kappa}{p_+} q_0$$
$$(p \oplus q)_i = \frac{\mathbf{p}_i}{\kappa} q_+ + \mathbf{q}_i$$
$$(p \oplus q)_4 = \frac{p_4}{\kappa} q_+ - \frac{\mathbf{pq}}{p_+} - \frac{\kappa}{p_+} q_0$$

For similar reasons, $-(.) \mapsto S(.)$ with $p \oplus S(p) = S(p) \oplus p = 0$.

Why p and not k? Using p, we can now work in a commutative spacetime.

In particular, using an object called Weyl map, one can send a group element $\hat{e}_k$ into a canonical plane wave e_p

$$\mathcal{W}(\hat{e}_k) = e_p \qquad e_p = e^{ip_\mu x^\mu} = e^{i(\omega t - \mathbf{p}\mathbf{x})}$$

$$\mathcal{W}(\hat{e}_{k\oplus l}) = e_{p(k)\oplus q(l)} = e_p \star e_q$$

This $\star$ product is in general non-commutative.

Action, EOM, and fields

Because of the star product we have two possible orderings $\Longrightarrow$ two possible actions.

$$S_1 = \int_{\mathbb{R}^4} d^4x \ (\partial^\mu \phi)^\dagger \star (\partial_\mu \phi) - m^2 \phi^\dagger \star \phi$$

$$S_2 = \int_{\mathbb{R}^4} d^4x \ (\partial_\mu \phi) \star (\partial^\mu \phi)^\dagger - m^2 \phi \star \phi^\dagger.$$

Therefore

$$S = \frac{1}{2} \int_{\mathbb{R}^4} d^4x \ [(\partial^\mu \phi)^\dagger \star (\partial_\mu \phi) + (\partial^\mu \phi) \star (\partial_\mu \phi)^\dagger - m^2 (\phi^\dagger \star \phi + \phi \star \phi^\dagger)]$$

How to compute the EOM or the charges? Usually, one uses integration by parts. Here, however, the Leibniz rule for derivatives does not work!

$$i(p \oplus q)_\mu e_{p\oplus q} = \partial_\mu (e_p \star e_q) = (\partial_\mu e_p) \star e_q + e_p \star \partial_\mu e_q$$
$$= i(p+q) e_{p\oplus q}$$

Instead, more complicated rules need to be applied. Example:

$$\partial_0(\phi \star \psi) = \frac{1}{\kappa}(\partial_0 \phi) \star (\Delta_+ \psi) + \kappa(\Delta_+^{-1}\phi) \star (\partial_0 \psi) + i(\Delta_+^{-1}\partial_i \phi) \star (\partial_i \psi)$$

Action, EOM, and fields

The field satisfies the Klein-Gordon equations.

$$(\partial_\mu \partial^\mu - m^2)\phi = 0$$

Any complex scalar field satisfying these eom can be written as

$$\phi(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} \xi(p) a_{\mathbf{p}}\, e^{-i(\omega_p t - \mathbf{px})}$$
$$+ \int \frac{d^3p^*}{\sqrt{|2\omega_p^*|}} \xi(p) b^\dagger_{\mathbf{p}^*}\, e^{i(S(\omega_p^*)t - S(\mathbf{p}^*)\mathbf{x})}$$

Properties of the fields under C, P, T

How do κ-deformed fields transform? P and T can consistently be defined as acting like in the undeformed case (they leave $[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa}\hat{x}^i$ invariant)

$$\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^{-1} = \phi(-t,\mathbf{x}) \quad \Longrightarrow \quad \mathcal{T}a_{\mathbf{p}}\mathcal{T}^{-1} = a_{-\mathbf{p}}$$
$$\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^{-1} = \phi(t,-\mathbf{x}) \quad \Longrightarrow \quad \mathcal{P}a_{\mathbf{p}}\mathcal{P}^{-1} = a_{-\mathbf{p}}$$

Because of the presence of the antipode $S(\cdot)$ in the fields and because of the form of the action (sum of two orderings), also C can be shown to behave like in the undeformed case (in its action on $a, a^\dagger, b, b^\dagger$) when acting on fields.

$$\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^{-1} = \phi(t,\mathbf{x})^\dagger \quad \Longrightarrow \quad \boxed{\mathcal{C}a_{\mathbf{p}}\mathcal{C}^{-1} = b_{\mathbf{p}^*}}$$

The action is manifestly invariant under C, P, T **and** under (deformed) Lorentz transformations.

Charges

We need the charges. How to get them?

- Using the Noether theorem. However, difficult computations (recall integration by parts), so only the translational charges are easily obtainable in this way;

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- Using the Noether theorem. However, difficult computations (recall integration by parts), so only the translational charges are easily obtainable in this way;

- More pragmatic approach: use the canonical formalism (Noether theorem) to compute translational charges, then switch to covariant phase space formalism for the others ($-\delta_\xi \lrcorner \Omega \stackrel{!}{=} \delta Q_\xi$). Keep in mind, a kind of 'matching' is necessary!

After some (very long an tedious) computations one finally gets to the following translation charges (from **Noether theorem**).

$$\mathcal{P}_0 = \int d^3p\alpha(p)\left\{-a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\, S(\omega_p) + b^\dagger_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\, \omega_p\right\}$$
$$\xrightarrow{\kappa\to\infty} \int d^3p\omega_p\left\{a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}} + b^\dagger_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\right\}$$

$$\mathcal{P}_i = \int d^3p\,\alpha\left\{-a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\, S(\mathbf{p})_i + b^\dagger_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\, \mathbf{p}_i\right\}$$

Notice that in the limit '$\kappa \to \infty$' one gets the canonical $\mathcal{P}_0, \mathcal{P}_i$. First hint: particles and antiparticles behave differently (but same mass!).

Charges

Using **covariant phase space formalism** and after some matching we obtain the other charges. For example, the boost charge is

$$\mathcal{N}_i = -\frac{1}{2}\int d^3p\,\alpha\left\{S(\omega_p)\left[\frac{\partial a_{\mathbf{p}}^\dagger}{\partial S(\mathbf{p})^i}a_{\mathbf{p}} - a_{\mathbf{p}}^\dagger\frac{\partial a_{\mathbf{p}}}{\partial S(\mathbf{p})^i}\right]\right.$$
$$\left. + \omega_p\left[b_{\mathbf{p}}\frac{\partial b_{\mathbf{p}}^\dagger}{\partial \mathbf{p}^i} - \frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i}b_{\mathbf{p}}^\dagger\right]\right\}.$$

Notice: all the deformed charges satisfy the undeformed Poincaré algebra (checked by direct long computations).

Relation between CPT and boost charges

Although the undeformed Poincaré algebra is satisfied, non trivial relations arise! One can show that

$$[\mathcal{N}_i, C] \neq 0 \qquad [\mathcal{N}_i, C] \xrightarrow{\kappa\to\infty} 0$$

This is due to $p \neq -S(p)$, and translates (for example) into a **difference of decay times** for particles and antiparticles in a boosted frame.

$$\mathcal{P}_{\text{part}}(t) = \frac{\Gamma E}{M} \exp\left(-\Gamma \frac{E}{M} t\right)$$

$$\mathcal{P}_{\text{apart}}(t) = \Gamma\left(\frac{E}{M} - \frac{\mathbf{p}^2}{\kappa M}\right) \exp\left[-\Gamma\left(\frac{E}{M} - \frac{\mathbf{p}^2}{\kappa M}\right) t\right]$$

where $\mathcal{P}$ = decay probability density function, and $\Gamma = 1/\tau$

Relation between CPT and boost charges

The Greenberg's theorem relates CPT invariance with Lorentz invariance of a theory. According to the theorem,

$$\text{Lorentz invariance} \;\Leftrightarrow \text{CPT invariance}$$

In our case, the action is manifestly invariant under deformed Lorentz invariance and CPT transformations. However, CPT symmetry is broken in a more subtle way (see previous slide).

Greenberg's theorem does not work: $p \neq -S(p)$.

Conclusions and future works

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- We have a well defined theory $\implies$ propagator and n-point functions (**soon on arXiv**: propagator and imaginary part to the loop correction to it);

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- Immediate generalization: higher spins.

Thank you

Backup slides

Geometric approach to conserved charges

How to compute a general charge of a given theory? As previously said, we will now concentrate on the more general case: geometric approach.

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Assuming that the charges come from a symmetry described by some continuous vector field ξ in spacetime, then

$$-\delta_\xi \lrcorner \Omega \overset{!}{=} \delta Q_\xi$$

where δ is the exterior derivative in phase space, Q_ξ is the charge associated to the vector ξ. $\delta_\xi A$ measures the infinitesimal variation of the object A in phase space due to the symmetry of the action along ξ in spacetime.

Undeformed case: time translations

Example: Translation charge in undeformed context.

$$\Omega^U = i \int d^3p \, (\delta a_{\mathbf{p}} \wedge \delta a^\dagger_{\mathbf{p}} - \delta b^\dagger_{\mathbf{p}^*} \wedge \delta b_{\mathbf{p}^*})$$

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We know that after a time translation we have

$$a_{\mathbf{p}} \mapsto e^{i\epsilon\omega_p} a_{\mathbf{p}} = a_{\mathbf{p}} + i\epsilon\omega_p a_{\mathbf{p}}$$

and therefore

$$\delta_{\partial_0} a_{\mathbf{p}} = i\epsilon\omega_p a_{\mathbf{p}} \quad \Leftrightarrow \quad \delta_{\partial_0} a^\dagger_{\mathbf{p}} = -i\epsilon\omega_p a^\dagger_{\mathbf{p}}$$

Undeformed case: time translations

$$
\begin{aligned}
-\delta_{\partial_0} \lrcorner\, \Omega^U &= -i \int d^3p\, (\delta_{\partial_0} a_{\mathbf{p}} \delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} \delta_{\partial_0} a^\dagger_{\mathbf{p}} \\
&\quad - \delta_{\partial_0} b^\dagger_{\mathbf{p}^*} \delta b_{\mathbf{p}^*} + \delta b^\dagger_{\mathbf{p}^*} \delta_{\partial_0} b_{\mathbf{p}^*})
\end{aligned}
$$

$$
\begin{aligned}
&= -i \int d^3p\, [i\epsilon\omega_p a_{\mathbf{p}} \delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} (-i\epsilon\omega_p a^\dagger_{\mathbf{p}}) \\
&\quad - (-i\epsilon\omega_p b^\dagger_{\mathbf{p}^*}) \delta b_{\mathbf{p}^*} + \delta b^\dagger_{\mathbf{p}^*} i\epsilon\omega_p b_{\mathbf{p}^*}]
\end{aligned}
$$

$$
= \epsilon\delta \int d^3p\, \omega_p (a_{\mathbf{p}} a^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}^*} b_{\mathbf{p}^*})
$$

Undeformed case: boosts

Example: Boost charges in undeformed context.

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$$\delta^B a_{\mathbf{p}} = i\omega_{\mathbf{p}} \, \lambda^j \frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^j} + i \, a_{\mathbf{p}} \lambda^j \frac{\mathbf{p}_j}{2\omega_{\mathbf{p}}},$$

$$\delta^B a^\dagger_{\mathbf{p}} = i\omega_{\mathbf{p}} \, \lambda^j \frac{\partial a^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^j} + i \, a^\dagger_{\mathbf{p}} \lambda^j \frac{\mathbf{p}_j}{2\omega_{\mathbf{p}}}$$

Undeformed case: boosts

$$
\begin{aligned}
-\delta^B \lrcorner\, \Omega^{\kappa\to\infty} &= i \int d^3p \, (\delta^B a^\dagger_{\mathbf{p}} \delta a_{\mathbf{p}} - \delta a^\dagger_{\mathbf{p}} \delta^B a_{\mathbf{p}} - \delta^B b_{\mathbf{p}} \delta b^\dagger_{\mathbf{p}} + \delta b_{\mathbf{p}} \delta^B b^\dagger_{\mathbf{p}}) \\
&= \lambda^i \int d^3p \, \omega_{\mathbf{p}} \left(\frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i} \delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} \frac{\partial a^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i} - \frac{\partial b^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i} \delta b_{\mathbf{p}} + \delta b^\dagger_{\mathbf{p}} \frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i} \right) \\
&+ \lambda^i \int d^3p \, \frac{\mathbf{p}_i}{2\omega_{\mathbf{p}}} \left(a_{\mathbf{p}} \delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} a^\dagger_{\mathbf{p}} + \delta b^\dagger_{\mathbf{p}} b_{\mathbf{p}} - b^\dagger_{\mathbf{p}} \delta b_{\mathbf{p}} \right)
\end{aligned}
$$

Apparent issue! Recall that $-\delta_\xi \lrcorner\, \Omega \stackrel{!}{=} \delta Q_\xi$

Solution:

$$\frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i}\delta a_{\mathbf{p}}^{\dagger} - \delta a_{\mathbf{p}}\frac{\partial a_{\mathbf{p}}^{\dagger}}{\partial \mathbf{p}^i} = \frac{1}{2}\delta\left(\frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i}a_{\mathbf{p}}^{\dagger} - a_{\mathbf{p}}\frac{\partial a_{\mathbf{p}}^{\dagger}}{\partial \mathbf{p}^i}\right) + \frac{1}{2}\frac{\partial}{\partial \mathbf{p}^i}\left(a_{\mathbf{p}}\delta a_{\mathbf{p}}^{\dagger} - \delta a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}\right).$$

The second term on the RHS after integration by parts becomes

$$-\frac{\mathbf{p}_i}{2\omega_{\mathbf{p}}}\left(a_{\mathbf{p}}\delta a_{\mathbf{p}}^{\dagger} - \delta a_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}\right)$$

which cancels with the second term in the previous expression.

Undeformed case: boosts

The final boost charge is given by

$$\mathcal{N}_i^{\kappa\to\infty} = \frac{1}{2}\int d^3p\,\omega_{\mathbf{p}}\left(\frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i}a_{\mathbf{p}}^\dagger - a_{\mathbf{p}}\frac{\partial a_{\mathbf{p}}^\dagger}{\partial \mathbf{p}^i} - \frac{\partial b_{\mathbf{p}}^\dagger}{\partial \mathbf{p}^i}b_{\mathbf{p}} + b_{\mathbf{p}}^\dagger\frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i}\right).$$

in accordance with the literature.

Deformed case: translations

We now start from the following transformations describing time translations.

$$\delta^T a_{\mathbf{p}} = i\epsilon^\mu p_\mu a_{\mathbf{p}}, \quad \delta^T a^\dagger_{\mathbf{p}} = i\epsilon^\mu S(p)_\mu a^\dagger_{\mathbf{p}},$$
$$\delta^T b^\dagger_{\mathbf{p}} = i\epsilon^\mu S(p)_\mu b^\dagger_{\mathbf{p}}, \quad \delta^T b_{\mathbf{p}} = i\epsilon^\mu p_\mu b_{\mathbf{p}}.$$

Notice the antipode! Therefore, a naive application of the previous procedure would not give a consistent result (no quantity Q_ξ such that $-\delta_\xi \lrcorner \Omega \stackrel{!}{=} \delta Q_\xi$). "Matching" with the direct computation needed! We will need to introduce the antipode in the contraction of a vector field with a 2-form.

Deformed case: translations

We postulate the following rule

$$\delta_\xi \lrcorner (\delta a^\dagger_{\mathbf{p}} \wedge \delta a_{\mathbf{p}}) = (\delta_\xi a^\dagger_{\mathbf{p}})\delta a_{\mathbf{p}} + \delta a^\dagger_{\mathbf{p}}[S(\delta_\xi) a_{\mathbf{p}}]$$

which solves the issue!

$$\begin{aligned}
&- \delta^T \lrcorner \Omega \\
&= i \int d^3p\, \alpha\, (\delta^T a^\dagger_{\mathbf{p}} \delta a_{\mathbf{p}} + \delta a^\dagger_{\mathbf{p}} S(\delta^T) a_{\mathbf{p}} - \delta^T b_{\mathbf{p}} \delta b^\dagger_{\mathbf{p}} - \delta b_{\mathbf{p}} S(\delta^T) b^\dagger_{\mathbf{p}}) \\
&= -\epsilon^\mu \delta \left(\int d^3p\, \alpha\, [S(p)_\mu a^\dagger_{\mathbf{p}} \delta a_{\mathbf{p}} - p_\mu b^\dagger_{\mathbf{p}} \delta b_{\mathbf{p}} \right)
\end{aligned}$$

$$\mathcal{P}_\mu = \int d^3p\, \alpha\, [-S(p)_\mu a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + p_\mu b^\dagger_{\mathbf{p}} b_{\mathbf{p}}],$$

Deformed case: boosts

In the case of boosts, we need to assume the following transformations for the creation/annihilation transformations

$$\delta^B a_{\mathbf{p}} = -i\lambda^j\,\omega_{\mathbf{p}}\left[\frac{\partial}{\partial \mathbf{p}^j} + \frac{1}{2}\frac{1}{\omega_{\mathbf{p}}}\frac{\partial[\omega_{\mathbf{p}} S(\alpha)]}{\partial \mathbf{p}^j}\right] a_{\mathbf{p}},$$

$$\delta^B a^\dagger_{\mathbf{p}} = -i\lambda^j\,S(\omega_{\mathbf{p}})\left[\frac{\partial}{\partial S(\mathbf{p})^j} + \frac{1}{2}\frac{1}{S(\omega_{\mathbf{p}})}\frac{\partial[S(\omega_{\mathbf{p}})\alpha]}{\partial S(\mathbf{p})^j}\right] a^\dagger_{\mathbf{p}},$$

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Charges

If we translate the creation/annihilation operators transformations in terms of the field we get (at first order in $1/\kappa$)

$$\delta^B \phi(x) = i\lambda_i \, x^i \frac{\partial}{\partial t} \phi(x)$$
$$- i\lambda_i \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}} \left\{ \frac{\mathbf{p}_i}{\kappa} \left(\frac{m^2}{4\omega_{\mathbf{p}}^2} - \frac{1}{2} \right) a_{\mathbf{p}} e^{-i(\omega_{\mathbf{p}} t - \mathbf{p}\mathbf{x})} \right.$$
$$\left. + \frac{\mathbf{p}_i}{\kappa} \left(-\frac{m^2}{4\omega_{\mathbf{p}}^2} - 1 \right) b^\dagger_{\mathbf{p}} e^{-i(S(\omega_{\mathbf{p}})t - S(\mathbf{p})\mathbf{x})} \right\},$$

Analogous relation for $\phi^\dagger$, which means that particles and antiparticles receive an additional shift under boost.

One particle states

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We now have all the tools to show that indeed particles and antiparticles behave differently.

How to see it?

- Since we have the translation charges (i.e. the operators $\mathcal{P}_\mu$), we can apply them to the a-particle and b-particle states and get their eigenvalues. We will see that they are different;

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- Use $\mathcal{C}$ to link the a-particle to the b-particle state. We will see that C switches a particle with its antiparticle with *different* momentum.

One particle states

Define the vacuum by $a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}*}|0\rangle = 0$. We then define one-particle and one-antiparticle state by

$$a^{\dagger}_{\mathbf{p}}|0\rangle := |\mathbf{p}\rangle_a \qquad b^{\dagger}_{\mathbf{p}*}|0\rangle := |\mathbf{p}\rangle_b$$

Now we want to know $\mathcal{P}_{\mu}|\mathbf{p}\rangle_a$ and $\mathcal{P}_{\mu}|\mathbf{p}\rangle_b$.

One particle states

$$\left[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}\right] = \frac{1}{\alpha}\delta(\mathbf{p}-\mathbf{q})$$

$$\begin{aligned}\mathcal{P}_i|\mathbf{q}\rangle_a &= \int d^3p\alpha \left\{-a_{\mathbf{p}}^{\dagger}\, a_{\mathbf{p}}\, S(\mathbf{p})_i + b_{\mathbf{p}^*}^{\dagger}\, b_{\mathbf{p}^*}\, \mathbf{p}_i\right\} a_{\mathbf{q}}^{\dagger}|0\rangle \\ &= \int d^3p\alpha \left\{-a_{\mathbf{p}}^{\dagger}\, \frac{1}{\alpha}\delta(\mathbf{p}-\mathbf{q})S(\mathbf{p})_i + a_{\mathbf{p}}^{\dagger}\, a_{\mathbf{q}}^{\dagger} a_{\mathbf{p}}\, \mathbf{p}_i\right\} |0\rangle + 0 \\ &= -S(\mathbf{q})_i|\mathbf{q}\rangle_a\end{aligned}$$

Doing the same thing for all $\mathcal{P}_\mu$ we have

$$\mathcal{P}_i|\mathbf{p}\rangle_a = -S(\mathbf{p})_i|\mathbf{p}\rangle_a \qquad \mathcal{P}_i|\mathbf{p}\rangle_b = \mathbf{p}_i|\mathbf{p}\rangle_b$$

$$\mathcal{P}_0|\mathbf{p}\rangle_a = -S(\omega_p)|\mathbf{p}\rangle_a \qquad \mathcal{P}_0|\mathbf{p}\rangle_b = \omega_p|\mathbf{p}\rangle_b$$

Notice: $\mathbf{p} \neq -S(\mathbf{p})$ and $\omega_p \neq -S(\omega_p)$, but $p_\mu p^\mu = m^2$ and $S(p)_\mu S(p)^\mu = m^2$, so a-particle and b-particle have same mass.

One particle states

We can use $\mathcal{C}$ to relate $|\mathbf{p}\rangle_a$ and $|\mathbf{p}\rangle_b$

$$\mathcal{C}|\mathbf{p}\rangle_b = \mathcal{C}b^\dagger_{\mathbf{p}*}\mathcal{C}^{-1}\mathcal{C}|0\rangle = a^\dagger_{\mathbf{p}}|0\rangle = |\mathbf{p}\rangle_a$$

Very easy steps due to the simplicity of the C transformation of our deformed field!

Therefore C (and CPT) transforms a particle into an anti-particle with different momentum, and vice versa.