Spherical harmonic decomposition + first encounter with scattering equation

1. The following equation describes perturbations induced by a massless scalar field on the Schwarzschild background:

$$\frac{d^2\psi}{dr_\star^2} + \left[\omega^2 - f\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)\right]\psi = S \ , \tag{1}$$

where $r_\star = r + 2M\ln(r/2M - 1)$ is the tortoise coordinate and $f = 1 - 2M/r$. In this case S is given by a gaussian packet, i.e.

$$S = fe^{-\frac{(r_\star - r_0)^2}{\sigma^2}} \quad , \quad (r_0 = 8, \sigma = 5) \ . \tag{2}$$

Integrate the previous sourced equation for the parameters given by the problem, and find the power at infinity for the $l = 1$ mode as a function of the frequency ω. In particular:

(a) Consider the homogeneous problem associated to eq. (1). At the horizon and at infinity the solution of the master equation can be written as a power series of the form:

$$\psi_h = e^{-i\omega r_\star}[a + b(r - 2M)] + \mathcal{O}(r - 2M)^2 \ , \tag{3}$$

and

$$\psi_\infty = e^{i\omega r_\star}(c + d/r) + \mathcal{O}(1/r^2) \ . \tag{4}$$

Find the coefficient (a, b, c, d).

(b) Find two solutions $\psi_{1,2}$ of the associated homogeneous problem, which can be obtained: (i) one starting from the horizon and integrating outward with boundary conditions given by ψ_h; (ii) one starting from infinity integrating inward with boundary conditions given by ψ_∞.

(c) Compute the general solution at infinity, i.e.

$$\psi(\omega, r) = e^{i\omega r_\star}\int_{-\infty}^{\infty}\frac{\psi_1 S}{W}dr_\star \ , \tag{5}$$

where W is the wronskian of the two solutions $\psi_{1,2}$.

(d) Compute the power $P = \omega^2|\psi(\omega, r)|^2$ as a function of the frequency ω, and plot it in the interval $[0.1, 0.5]$.

2. The stress-energy tensor of a test particle orbiting around a BH can be expanded in a complete set of tensor harmonics, as:

$$\mathbf{T} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\mathcal{A}^{(0)}_{\ell m}\mathbf{a}^{(0)}_{\ell m} + \mathcal{A}^{(1)}_{\ell m}\mathbf{a}^{(1)}_{\ell m} + \mathcal{A}_{\ell m}\mathbf{a}_{\ell m} + \mathcal{B}^{(0)}_{\ell m}\mathbf{b}^{(0)}_{\ell m} + \mathcal{B}_{\ell m}\mathbf{b}_{\ell m}\right.$$
$$\left. + \mathcal{Q}^{(0)}_{\ell m}\mathbf{c}^{(0)}_{\ell m} + \mathcal{Q}_{\ell m}\mathbf{c}_{\ell m} + \mathcal{D}_{\ell m}\mathbf{d}_{\ell m} + \mathcal{G}_{\ell m}\mathbf{g}_{\ell m} + \mathcal{F}_{\ell m}\mathbf{f}_{\ell m}\right] \ . \tag{6}$$

where (ℓ, m) are the multipole numbers. The *axial* $(\mathbf{c}^{(0)}_{\ell m}, \mathbf{c}_{\ell m}, \mathbf{d}_{\ell m},)$ and *polar* $(\mathbf{a}^{(0)}_{\ell m}, \mathbf{a}^{(1)}_{\ell m}, \mathbf{a}_{\ell m}, \mathbf{b}^{(0)}_{\ell m}, \mathbf{b}_{\ell m}, \mathbf{g}_{\ell m}, \mathbf{f}_{\ell m})$ tensor harmonics basis are expressed in terms of the usual spherical harmonics $Y_{\ell m}(\theta, \phi)$ and their derivatives. For example we have:

$$\mathbf{a}_{\ell m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{\ell m}(\theta,\phi) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \ \mathbf{a}^{(0)}_{\ell m} = \begin{pmatrix} Y_{\ell m}(\theta,\phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{7}$$

$$\mathbf{a}^{(1)}_{\ell m} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & Y_{\ell m}(\theta,\phi) & 0 & 0 \\ Y_{\ell m}(\theta,\phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{8}$$

$$\mathbf{b}^{(0)}_{\ell m} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & (\partial/\partial_\theta) Y_{\ell m}(\theta,\phi) & 0 \\ 0 & 0 & 0 & 0 \\ (\partial/\partial_\theta) Y_{\ell m}(\theta,\phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{9}$$

while the expansion coefficients $(\mathcal{A}^{(0)}_{\ell m}, \mathcal{A}^{(1)}_{\ell m}, \ldots, \mathcal{F}_{\ell m})$ can be computed exploiting the orthonormality properties of the tensor harmonics:

$$(A, B) = \int\int \eta^{\mu\rho}\eta^{\nu\sigma} A^{\star}_{\mu\nu} B_{\rho\sigma} d\Omega \ , \tag{10}$$

where the superscript $\star$ denotes complex conjugation and the inverse metric $\eta^{\mu\nu}$ is given by $\mathrm{diag}(-1, 1, \frac{1}{r^2}, \frac{1}{r^2 \sin^2\theta})$. For example, $\mathcal{A}^{(1)} = (\mathbf{a}^{(1)}, \mathbf{T})$.

We now consider the stress-energy tensor of a point particle of mass m_p, moving on a geodesics $x^\mu(\tau)$ of the Schwarzschild metric, being τ the proper time:

$$\begin{aligned} T^{\mu\nu} &= m_p \int u^\mu u^\nu \frac{\delta^{(4)}(x^\beta - y^\beta_p)}{\sqrt{-g}} d\tau \\ &= m_p \frac{dt}{d\tau} \frac{v^\mu v^\nu}{r^2 |\sin\theta_p|} \delta(r - r_p)\delta(\theta - \theta_p)\delta(\phi - \phi_p) \\ &= m_p \gamma \frac{v^\mu v^\nu}{r^2} \delta(r - r_p)\delta(\cos\theta - \cos\theta_p)\delta(\phi - \phi_p) \\ &= m_p \gamma \frac{v^\mu v^\nu}{r^2} \delta(r - r_p)\delta^{(2)}(\Omega - \Omega_p) \end{aligned} \tag{11}$$

where $v^\mu = dy^\mu_p/dt$, and γ the relativistic boost factor.

Compute the first four coefficients the expansion (6) for the stress-energy tensor (11), i.e. $(\mathcal{A}_{\ell m}, \mathcal{A}^{(0)}_{\ell m}, \mathcal{A}^{(1)}_{\ell m}, \mathcal{B}^{(0)}_{\ell m})$ [You can compare your results with Table I of [1]]. In oder to get the exact same functional form of the reference, note that $v^\phi = d\phi/dt = \frac{i}{m}\frac{dY_{\ell m}}{dt}/Y^\star_{\ell m}$. Remember also that the coefficients have to be multiplied by -1 if the norm of the corresponding tensor harmonics is negative [for example $(\mathbf{a}^{(1)}, \mathbf{a}^{(1)}) = -1$].

Bibliography

[1] N. Sago, H. Nakano, and M. Sasaki. Gauge problem in the gravitational self-force: Harmonic gauge approach in the Schwarzschild background. *Physical Review D*, 67(10):3457–14, May 2003.