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Testing modified teleparallel theories via scalar induced gravitational waves from primordial black holes

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Content

Main idea

- Primordial Black Holes (PBHs) form in the early universe, before star formation, out of the collapse of enhanced energy density perturbations upon horizon reentry of the typical size of the collapsing overdensity region.
- They are cosmologically relevant mainly as DM candidates and seeds for cosmic structure (eg supermassive BH) (a nice review Sasaki et al (1801.05235v1))
- SIGWs are induced from enhanced scalar perturbations collapsing to PBHs due to second-order gravitational interactions $\rightarrow$ they carry the information to potentially test the gravitational theory

Teleparallel gravity

Teleparallel Gravity (TG) is an alternative formulation of gravity based on torsion. Its dynamical variable is the tetrad field $\mathbf{e}_A(x^\mu)$ for which:

$$g_{\mu\nu} = e^A_{\ \mu}\, e^B_{\ \nu}\, \eta_{AB} \,, \tag{1}$$

with $e^A_{\ \mu} e_A^{\ \nu} = \delta^\nu_\mu$ and $e^A_{\ \mu} e_B^{\ \mu} = \delta^A_B$

(1) $\rightarrow$ the tetrad fields are only determined up to transformations of the six-parameter Lorentz group. To ensure the covariance of the theory one needs to introduce a *Lorentz* or *spin connection* :

$$\omega^A_{\ B\mu} = \Lambda^A_{\ D}(x)\partial_\mu \Lambda_B^{\ D}(x) \tag{2}$$

Weitzenböck connection

TG is characterised by the choice to formulate gravity in a particular class of frames (called proper frames) for which the spin connection is *flat*, i.e. $\omega^{A}{}_{B\mu} = 0$. This choice is facilitated by the local Lorentz invariance of TG. The corresponding spacetime-indexed connection which is the so-called Weitzenböck connection is the following:

$$\Gamma^{\rho}{}_{\mu\nu} = e_{A}{}^{\rho}\left(\partial_{\mu} e^{A}{}_{\nu} + \omega^{A}{}_{B\mu} e^{B}{}_{\nu}\right) \Rightarrow \overset{\mathbf{w}}{\Gamma}{}^{\lambda}{}_{\nu\mu} \equiv e_{A}^{\lambda}\, \partial_{\mu} e_{\nu}^{A}. \tag{3}$$

The Weitzenböck connection of TG and the Levi-Civita connection of GR, $\bar{\Gamma}^{\rho}{}_{\mu\nu}$, are related as follows

$$\Gamma^{\rho}{}_{\mu\nu} = \bar{\Gamma}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu}. \tag{4}$$

Action functional of TG

The action functional of TG is defined by

$$S = -\frac{M_{\rm Pl}^2}{2} \int \mathrm{d}^4 x \, e \, T, \tag{5}$$

with $e = \det\left(e^A_{\ \mu}\right) = \sqrt{-g}$ and $M_{\rm Pl}^2 \equiv (8\pi G)^{-1}$ being the reduced Planck mass. The torsion scalar T is defined by

$$T = S_\rho^{\ \mu\nu} \, T^\rho_{\ \mu\nu}, \tag{6}$$

with $T^\rho_{\ \mu\nu}$ being the components of the torsion tensor defined by

$$T^\rho_{\ \mu\nu} \equiv e_A^{\ \rho} \left[\partial_\mu e^A_{\ \nu} - \partial_\nu e^A_{\ \mu} + \omega^A_{\ B\mu} \, e^B_{\ \nu}, -\omega^A_{\ B\nu} \, e^B_{\ \mu}\right] \tag{7}$$

and $S_\rho^{\ \mu\nu}$ being the so-called super-potential which reads as

$$S_\rho^{\ \mu\nu} \equiv \frac{1}{2} \left(K^{\mu\nu}_{\ \ \ \rho} + \delta^\mu_{\ \rho} \, T^{\theta\nu}_{\ \ \ \theta} - \delta^\nu_{\ \rho} \, T^{\theta\mu}_{\ \ \ \theta}\right) , \tag{8}$$

with $K^{\mu\nu}_{\ \ \ \rho}$ standing for the contortion tensor defined by

$$K^{\mu\nu}_{\ \ \ \rho} \equiv -\frac{1}{2} \left(T^{\mu\nu}_{\ \ \ \rho} - T^{\nu\mu}_{\ \ \ \rho} - T_\rho^{\ \mu\nu}\right) . \tag{9}$$

Equivalence of GR and TG

From (4) and (9) : $T = -R - 2e^{-1}\partial_\mu(eT^{\nu\mu}{}_\nu) \to$ TG and GR are *equivalent theories at the level of the field equations*.

However, when one extends TG by introducing a non-minimally coupled matter field, for instance a scalar field, or by adding into the action non-linear terms in the torsion scalar T, as for example in $f(T)$ gravity, one obtains new classes of modified gravity theories with interesting phenomenology which are *not equivalent* to their corresponding curvature based counterparts

Generalised scalar-torsion theories

By extending the gravitational sector to an arbitrary function of T and ϕ, the corresponding action functional is

$$S = \int \mathrm{d}^4x\, e\, \left[f(T,\phi) + P(\phi)X\right], \tag{10}$$

with X being the so-called canonical kinetic term defined by $X \equiv -\partial_\mu\phi\partial^\mu\phi/2$. (TG with a scalar field potential $V(\phi)$ is recovered when $f(T,\phi) = -M_{\mathrm{Pl}}^2 T/2 - V(\phi)$).

The corresponding field equations are:

$$f_{,T}G_{\mu\nu} + S_{\mu\nu}{}^{\rho}\partial_\rho f_{,T} + \frac{1}{4}g_{\mu\nu}\left(f - Tf_{,T}\right) + \frac{P}{4}\left(g_{\mu\nu}X + \partial_\mu\phi\partial_\nu\phi\right) = 0, \tag{11}$$

with $G_A^{\mu} \equiv e^{-1}\partial_\nu\left(ee_A^{\sigma}S_\sigma{}^{\mu\nu}\right) - e_A^{\sigma}T^{\lambda}_{\rho\sigma}S_\lambda^{\rho\mu} + e_B^{\lambda}S_\lambda^{\rho\mu}\omega^B{}_{A\rho} + \frac{1}{4}e_A^{\mu}T$ and $G^{\mu}_{\nu} = e_A^{\mu}G^A_{\nu}$ being the Einstein tensor

Local Lorentz invariance violation

The action (10) is ***not*** locally Lorentz invariant.

- Consider an infinitesimal Lorentz transformation to the tetrads as follows: $e'^A_{\ \mu} = e^A_{\ \mu} + \xi_B^{\ A} e^B_{\ \mu}$, with $\xi^{AB} = -\xi^{BA}$.
- The effect of this transformation on the action is

$$\delta S = \int d^4 x e \partial_\rho f_{,T} S^\rho_{\mu\nu} e_A^{\ \mu} e_B^{\ \nu} \xi^{AB}. \tag{12}$$

- For $\delta S = 0$ for arbitrary ξ^{AB}, the following equation needs to be satisfied

$$\partial_\rho f_{,T} S_{[\mu\nu]}{}^\rho = 0. \tag{13}$$

- Since equation (13) is not satisfied in general $\rightarrow$ (10) is not Lorentz invariant locally. (For the special case of TG, $f \sim T \rightarrow \partial_\rho f_{,T} = 0 \rightarrow$ (13) is satisfied.)

Cosmological framework

To apply this general formulation into a cosmological setting, one needs to impose the standard flat, homogeneous and isotropic FLRW geometry

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\,\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{14}$$

which corresponds to the following tetrads

$$e^A_{\ \mu} = \mathrm{diag}(1, a(t), a(t), a(t)), \tag{15}$$

with $a(t)$ being the scale factor. By substituting the tetrad field (15) into the field equations (11) one obtains

$$f(T,\phi) - P(\phi)X - 2Tf_{,T} = 0, \tag{16}$$

$$f(T,\phi) + P(\phi)X - 2Tf_{,T} - 4\dot{H}f_{,T} - 4H\dot{f}_{,T} = 0, \tag{17}$$

$$-P_{,\phi}X - 3P(\phi)H\dot{\phi} - P(\phi)\ddot{\phi} + f_{,\phi} = 0, \tag{18}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and a dot denotes derivative with respect to t. Additionally, from Eq. (6) one obtains $T = 6H^2$.

Slow roll parameters

In order to describe slow-roll inflation, one needs to introduce the following slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \ \ \delta_{PX} \equiv -\frac{P(\phi)X}{2H^2 f_{,T}}, \ \ \ \delta_{f_{,T}} \equiv \frac{\dot{f}_{,T}}{f_{,T}H}, \tag{19}$$

such as that from equations (16) and (17) one can write ϵ as

$$\epsilon = \delta_{PX} + \delta_{f_{,T}}. \tag{20}$$

Furthermore, it is useful to split the parameter $\delta_{f_{,T}}$ as

$$\delta_{f_{,T}} = \delta_{f\dot{H}} + \delta_{fX}, \text{with} \ \ \delta_{f\dot{H}} \equiv \frac{f_{,TT}\dot{T}}{Hf_{,T}}, \ \ \delta_{fX} \equiv \frac{f_{,T\phi}\dot{\phi}}{Hf_{,T}}. \tag{21}$$

Therefore, from expressions (19) and (20):

$$\delta_{f\dot{H}} = -\frac{2\mu}{1+2\mu}\left(\delta_{PX} + \delta_{fX}\right), \ \delta_{f_{,T}} = \frac{1}{1+2\mu}\left(\delta_{fX} - 2\mu\delta_{PX}\right), \tag{22}$$

$$\epsilon = \frac{1}{1+2\mu}\left(\delta_{PX} + \delta_{fX}\right), \text{with } \mu \equiv Tf_{,TT}/f_{,T} \tag{23}$$

Scalar Perturbations

To describe scalar perturbations, we shall employ the ADM decomposition of the tetrad field where

$$e^0_{\ \mu} = (N, \mathbf{0}) \,, \quad e^a_{\ \mu} = (N^a, h^a_{\ i}) \,,$$
$$e_0^{\ \mu} = \left(1/N, -N^i/N\right) \,, \quad e_a^{\ \mu} = \left(0, h_a^{\ i}\right) \,, \tag{24}$$

with N being the *lapse* function and N^i the *shift* vector, which is defined by $N^i \equiv h_a^{\ i} N^a$ and $h^a_{\ i}$ being the induced tetrad field satifying the *orthonormality condition*, i.e. $h^a_{\ j} h_a^{\ i} = \delta^i_j$.
Choosing the uniform field gauge, or otherwise called comoving gauge, i.e. $\delta\phi = 0$, a convenient ansatz is

$$N = 1 + A, \quad N^a = a^{-1} e^{-\mathcal{R}} \delta^a_{\ i} \partial^i \psi, \quad h^a_{\ i} = a e^{\mathcal{R}} \delta^a_{\ j} \delta^j_{\ i}, \tag{25}$$

which gives rise to the corresponding perturbed metric

$$\mathrm{d}s^2 = -\left[(1+A)^2 - a^{-2} e^{-2\mathcal{R}} \left(\partial\psi\right)^2\right] \mathrm{d}t^2 + 2\partial_i \psi \mathrm{d}t \mathrm{d}x^i + a^2 e^{2\mathcal{R}} \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j. \tag{26}$$

Expanding the action to 2^{nd} order

- Now one needs to expand the action (10) up to second order in the perturbation variables of the perturbed tetrad (25).
- In order to accomplish this, one needs to address the fact that the action is not Lorentz invariant locally.
- The standard procedure for that essentially consists in adding the additional six Lorentz degrees of freedom, directly into the perturbed tetrad field (Golovnev, Koivisto (1808 .05565)) (25) .
- Afterwards, once a particular perturbed tetrad frame is chosen, these extra modes can be absorbed into Goldstone modes of the Lorentz symmetry breaking, by performing a Lorentz rotation of the tetrad field.

2^{nd} order action

After this procedure, a new massive term is generated and the corresponding action is (Gonzalez-Espinoza, Otalora (1808 .05565))

$$S^{(2)} = \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3 x \left[(v')^2 - (\partial v)^2 - M^2 v^2 \right], \tag{27}$$

where we defined the usual Mukhanov-Sasaki (MS) variable

$$v \equiv z\mathcal{R}, \text{ with } z^2 \equiv 2a^2 Q_s \text{ and } Q_s \equiv \frac{PX}{H^2}, \tag{28}$$

The M is an effective mass parameter defined by

$$M^2 \equiv a^2 m^2 - \frac{z''}{z}, \tag{29}$$

where $m^2 = 3H^2 \eta_{\mathcal{R}}$ and $\eta_{\mathcal{R}}$ is given by

$$\eta_{\mathcal{R}} = \frac{m^2}{3H^2} = \delta_{f_{,T}} \left[1 + \left(1 + \frac{\delta_{fX}}{\delta_{PX}} \right) \frac{\delta_{f_{,T}}}{\delta_{f\dot{H}}} \right]. \tag{30}$$

The parameter m is a new explicit mass term, which arises due to the effects of local Lorentz-symmetry breaking mentioned earlier.

Mukhanov- Sasaki equation

By varying the action (27) and using the Fourier expansion of the MS variable

$$v(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} v_{\mathbf{k}}(\tau) e^{i\mathbf{k}.\mathbf{x}}, \tag{31}$$

one obtains the following field equation

$$v_k'' + \left(k^2 + M^2\right) v_k = 0, \tag{32}$$

which is the corresponding Mukhanov-Sasaki equation within the modified teleparallel gravity setup. Rewriting in terms of $\mathcal{R}$

$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + \left(k^2 + a^2 m^2\right) \mathcal{R}_k = 0. \tag{33}$$

Tensor perturbations

To describe the tensor perturbations we shall adopt again the uniform field gauge, $\delta\phi = 0$, so from our earlier ADM decomposition of the tetrad field from equations (24) we get that

$$N = 1, \quad N^a = 0, \quad h^a_{\ i} = a(\delta^a_{\ i} + \frac{1}{2}\gamma^a_{\ i}). \tag{34}$$

Then we can define the induced 3−metric

$$g_{ij} = \eta_{ab} h^a_{\ i} h^b_{\ j} = a^2 \left[\delta_{ij} + h_{ij} + \frac{1}{4}\gamma_{ki}\gamma^k_{\ j}\right], \tag{35}$$

where we defined the spatial tensor modes by

$$h_{ij} = \frac{1}{2}\eta_{ab}\left(\delta^a_{\ i}\gamma^b_{\ j} + \delta^b_{\ j}\gamma^a_{\ j}\right) = \frac{1}{2}\left(\gamma_{ij} + \gamma_{ji}\right), \text{ with } \gamma^a_{\ j} = \gamma^i_{\ j}\delta^a_{\ i} \tag{36}$$

Tensor perturbations

Substituting and neglecting the γ^2 terms

$$S_{\mathrm{T}}^{(2)} = \int \mathrm{d}\tau \mathrm{d}^3 x a^2 Q_T \left[(h'_\lambda)^2 - (\partial h_\lambda)^2 \right], \tag{37}$$

with $Q_T \equiv -f_{,T}/2$ and $\lambda = (+)$ or $(\times)$ accounting for two polarisation states of the tensor modes. At the end, minimising the aforementioned second-order action for the tensor modes and Fourier transforming h_λ one obtains the following equation of motion for $h_{\boldsymbol{k}}^\lambda$

$$h_{\boldsymbol{k}}^{\lambda,\prime\prime} + 2\mathcal{H}(1-\gamma_T) h_{\boldsymbol{k}}^{\lambda,\prime} + k^2 h_{\boldsymbol{k}}^\lambda = 0, \tag{38}$$

with

$$\gamma_T \equiv -\frac{f'_T}{2\mathcal{H} f_T}. \tag{39}$$

Our specific $f(T,\phi)$ gravity models

We will work with specific $f(T,\phi)$ gravity models with canonical kinetic terms ($P(\phi)=1$) and without explicit non-minimal matter-gravity couplings ($f_{,T\phi}=0$): $f(T,\phi)=f(T)+X-V(\phi)$

- The power-law model:

$$f(T) = -\frac{M_{\mathrm{Pl}}^2}{2}\left(T+\alpha T^{\beta}\right), \tag{40}$$

$$\alpha = (6H_0^2)^{1-\beta}\frac{\Omega_{F0}}{2\beta-1},\ \Omega_{F0}=1-\Omega_{m0}-\Omega_{r0},\ -0.3<\beta<0.3$$

- The exponential model:

$$f(T) = -M_{\mathrm{Pl}}^2/2\left[T+\alpha T_0\left(1-e^{-T/(\beta T_0)}\right)\right], \tag{41}$$

$$\alpha = \frac{\Omega_{F0}}{1-(1+\frac{2}{\beta})e^{-\frac{1}{\beta}}},\ 0.02<\beta<0.2$$

Inflationary potential

We will work with inflationary setups with inflection points giving rise to an *ultra slow-roll (USR) phase*. We choose α-attractor inflationary models naturally motivated by supergravity setups (Dalianis et al (1808 .05565)).

- Chaotic inflationary model

$$V(\phi) = V_0 \left\{ \tanh\left(\frac{\phi}{\sqrt{6\alpha}}\right) + A_\phi \sin\left[\tanh\left(\frac{\phi}{\sqrt{6\alpha}}\right)/f_\phi\right]\right\}^2 ,$$

- Polynomial inflationary superpotential

$$V(\phi) = V_0 \left[c_0 + c_1 \tanh\left(\frac{\phi}{\sqrt{6\alpha}}\right) + c_2 \tanh^2\left(\frac{\phi}{\sqrt{6\alpha}}\right) + c_3 \tanh^3\left(\frac{\phi}{\sqrt{6\alpha}}\right)\right]^2 .$$

Table 1 [1]

α	A_ϕ	f_ϕ	V_0	c_0	c_1	c_2	c_3
1	0.130383	0.129576	$2 \cdot 10^{-10}$	0.16401	0.3	-1.426	2.20313

[1] At the end, as it was checked numerically, our following quantitative results turn out to be independent of the choices of these parameters.

The scalar induced tensor perturbations

To describe the 2^{nd} order SIGW we will use the equation (Papanikolaou et al (2205.06094)):

$$h_{\boldsymbol{k}}^{\lambda,\prime\prime} + 2\mathcal{H}(1-\gamma_T)h_{\boldsymbol{k}}^{\lambda,\prime} + k^2 h_{\boldsymbol{k}}^{\lambda} = 4S_{\boldsymbol{k}}^{\lambda}\,, \tag{42}$$

where $\lambda = (+), (\times)$ and the source term $S_{\boldsymbol{k}}^{\lambda}$ reads as:

$$S_{\boldsymbol{k}}^{\lambda} = \int \frac{\mathrm{d}^3 q}{(2\pi)^{3/2}} e^{\lambda}(\boldsymbol{k},\boldsymbol{q}) F(\boldsymbol{q},|\boldsymbol{k}-\boldsymbol{q}|,\eta)\phi_{\boldsymbol{q}}\phi_{\boldsymbol{k}-\boldsymbol{q}}, \tag{43}$$

with Φ being the Bardeen potential and $\Phi_k(\eta) = T_\Phi(k\eta)\phi_k$, $T_\Phi(k\eta)$ the transfer function, ϕ_k the value of Φ at the horizon crossing time, $e^{\lambda}(\boldsymbol{k},\boldsymbol{q}) \equiv e^{s}_{ij}(\boldsymbol{k})q_i q_j$ the polarisation tensors and

$$\begin{aligned} F(\boldsymbol{q},|\boldsymbol{k}-\boldsymbol{q}|,\eta) \equiv\, & 2T_\Phi(q\eta)T_\Phi\left(|\boldsymbol{k}-\boldsymbol{q}|\eta\right) + \frac{4}{3(1+w)}\left[\mathcal{H}^{-1}qT_\Phi'(q\eta) + \right. \\ & \left. + T_\Phi(q\eta)\right] \times \left[\mathcal{H}^{-1}|\boldsymbol{k}-\boldsymbol{q}|T_\Phi'\left(|\boldsymbol{k}-\boldsymbol{q}|\eta\right) + T_\Phi\left(|\boldsymbol{k}-\boldsymbol{q}|\eta\right)\right]. \end{aligned} \tag{44}$$

The scalar induced tensor perturbations

In the absence of entropic perturbations, the time evolution of Φ can be approximated by

$$\Phi_k'' + \frac{6(1+w)}{1+3w}\frac{1}{\eta}\Phi_k' + wk^2\Phi_k = 0\,. \tag{45}$$

Eq. (42) can be solved by virtue of the Green's function formalism

$$h^{\lambda}_{\boldsymbol{k}}(\eta) = \frac{4}{a(\eta)}\int_{\eta_\mathrm{d}}^{\eta}\mathrm{d}\bar{\eta}\, G^{\lambda}_{\boldsymbol{k}}(\eta,\bar{\eta})a(\bar{\eta})S^{\lambda}_{\boldsymbol{k}}(\bar{\eta}), \tag{46}$$

where the Green's function $G^{\lambda}_{k}(\eta,\bar{\eta})$ is the solution of the homogeneous equation.

At the end, one can extract the tensor power spectrum $\mathcal{P}_h(k)$ defined as the equal time correlation function of the tensor perturbations

$$\langle h^{\lambda}_{\boldsymbol{k}_1}(\eta) h^{\rho,*}_{\boldsymbol{k}_2}(\eta)\rangle \equiv \delta^{(3)}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\delta^{\lambda\rho}\frac{2\pi^2}{k_1^3}\mathcal{P}^{(\lambda)}_h(\eta, k_1), \tag{47}$$

The gravitational wave spectral abundance

In the subhorizon region one can use the flat spacetime approximation and (42) reduces to a free-wave equation, thus

$$\Omega_{\rm GW}(\eta, k) \equiv \frac{1}{\bar{\rho}_{\rm tot}} \frac{\mathrm{d}\rho_{\rm GW}(\eta, k)}{\mathrm{d}\ln k} = \frac{1}{24}\left(\frac{k}{\mathcal{H}(\eta)}\right)^2 \overline{\mathcal{P}_h^{(\lambda)}(\eta, k)}, \qquad (48)$$

with the bar standing for an averaging over the sub-horizon oscillations of the tensor field

One then can account for the Universe expansion and derive the GW energy density contribution today (details in the appendix)

$$\Omega_{\rm GW}(\eta_0, k) = \Omega_r^{(0)} \frac{g_{*\rho,*}}{g_{*\rho,0}} \left(\frac{g_{*\mathrm{S},0}}{g_{*\mathrm{S},*}}\right)^{4/3} \Omega_{\rm GW}(\eta_*, k), \qquad (49)$$

where $g_{*\rho}$ and $g_{*\mathrm{S}}$ stand for the energy and entropy relativistic degrees of freedom.

Teleparallel gravity modifications of the gravitational wave signal

We shall now investigate the relevant modifications of $f(T)$ theories at the level of the *source* and the *propagation* of the GWs which can potentially render the GW distinctive with respect to the one within GR.

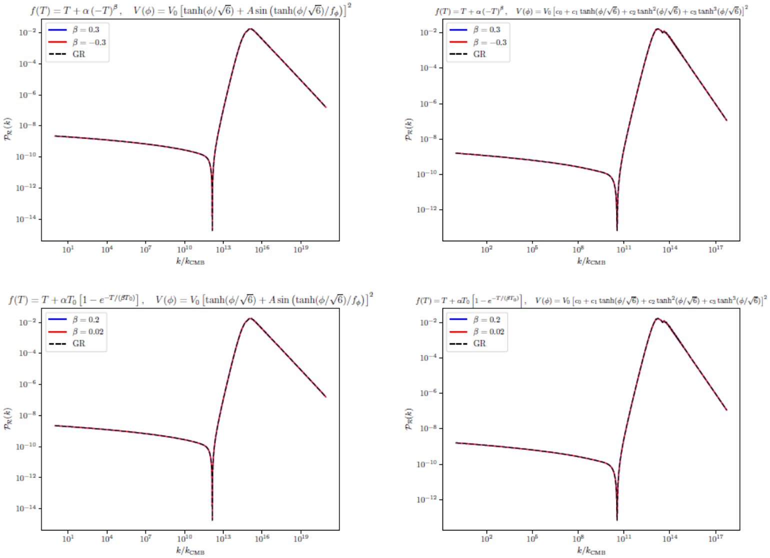
The effect on the source: It is captured by $\mathcal{P}_{\mathcal{R}}$ which sources $\mathcal{P}_h$ via:

$$\mathcal{P}_h^{(\lambda)}(\eta, k) = 4\int_0^\infty \mathrm{d}v \int_{|1-v|}^{1+v} \mathrm{d}u \left[\frac{4v^2 - (1+v^2-u^2)^2}{4uv}\right]^2 \times I^2(u,v,x)\mathcal{P}_{\mathcal{R}}(kv)\mathcal{P}_{\mathcal{R}}(ku)\,, \tag{50}$$

with

$$I(u,v,x) = \int_{x_0}^{x} \mathrm{d}\bar{x}\, \frac{a(\bar{x})}{a(x)}\, k\, G_k(x,\bar{x}) F_k(u,v,\bar{x}). \tag{51}$$

Results



$f(T) = T + \alpha(-T)^{\beta}, \quad V(\phi) = V_0\left[\tanh(\phi/\sqrt{6}) + A\sin\left(\tanh(\phi/\sqrt{6})/f_\phi\right)\right]^2$
$\beta = 0.3$
$\beta = -0.3$
GR
$\mathcal{P}_{\mathcal{R}}(k)$
$k/k_{\rm CMB}$
$f(T) = T + \alpha(-T)^{\beta}, \quad V(\phi) = V_0\left[c_0 + c_1\tanh(\phi/\sqrt{6}) + c_2\tanh^2(\phi/\sqrt{6}) + c_3\tanh^3(\phi/\sqrt{6})\right]^2$
$\beta = 0.3$
$\beta = -0.3$
GR
$\mathcal{P}_{\mathcal{R}}(k)$
$k/k_{\rm CMB}$
$f(T) = T + \alpha T_0\left[1 - e^{-T/(\beta T_0)}\right], \quad V(\phi) = V_0\left[\tanh(\phi/\sqrt{6}) + A\sin\left(\tanh(\phi/\sqrt{6})/f_\phi\right)\right]^2$
$\beta = 0.2$
$\beta = 0.02$
GR
$\mathcal{P}_{\mathcal{R}}(k)$
$k/k_{\rm CMB}$
$f(T) = T + \alpha T_0\left[1 - e^{-T/(\beta T_0)}\right], \quad V(\phi) = V_0\left[c_0 + c_1\tanh(\phi/\sqrt{6}) + c_2\tanh^2(\phi/\sqrt{6}) + c_3\tanh^3(\phi/\sqrt{6})\right]^2$
$\beta = 0.2$
$\beta = 0.02$
GR
$\mathcal{P}_{\mathcal{R}}(k)$
$k/k_{\rm CMB}$

Results

As is apparent from the plots: $\left|\frac{\mathcal{P}_{\mathcal{R}}^{f(T)}(k)-\mathcal{P}_{\mathcal{R}}^{\mathrm{GR}}(k)}{\mathcal{P}_{\mathcal{R}}^{\mathrm{GR}}(k)}\right| \sim 10^{-18}$.

To get some intuition, we derive $\mathcal{P}_{\mathcal{R}}(k)$ at linear order in the slow roll regime, namely when $\epsilon, \eta \ll 1$: After appropriate approximations, we can write

$$|\mathcal{R}_k| \simeq \frac{H_k}{2\sqrt{k^3 Q_{sk}}}\left[1 + \eta_{\mathcal{R}} \ln\left(\frac{k}{aH}\right)\right], \tag{52}$$

where H_k and Q_{sk} are the values of H and Q_s evaluated at horizon crossing time, namely when $k = aH$.

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}\left|\mathcal{R}_k(\tau)\right|^2 \simeq \frac{H_k^2}{8\pi^2 Q_{sk}}\left[1 + 2\eta_{\mathcal{R}} \ln\left(\frac{k}{aH}\right)\right]. \tag{53}$$

Results

$$n_{\mathrm{s}} - 1 \equiv \left.\frac{d \ln \mathcal{P}_{\mathcal{R}}(k)}{d \ln k}\right|_{k=aH} = -2\epsilon - \eta + 2\eta_{\mathcal{R}}, \tag{54}$$

from which we see the deviation from GR due to the presence of the term $2\eta_{\mathcal{R}}$, which carries the effects of the local Lorentz violation.

$$(30) \Leftrightarrow \eta_{\mathcal{R}} = (\delta_{fX} - 2\mu\epsilon)\left[1 - \frac{1+2\mu}{(1+2\mu)\epsilon - \delta_{fX}}\frac{\delta_{fX} - 2\mu\epsilon}{2\mu}\right]. \tag{55}$$

for $\delta_{fX} = 0$, i.e. in the absence of matter-gravity coupling, $\eta_{\mathcal{R}} = -4\mu\epsilon \ll 1$, since $\epsilon < 1$ and $\mu = Tf_{,TT}/f_{,T} \ll 1$ for viable $f(T)$ models (Nesseris et al (1308.6142)) and $Q_{sk} = \epsilon_k$

$$\text{Therefore} \quad \mathcal{P}_{\mathcal{R}}^{f(T)}(k) \simeq \mathcal{P}_{\mathcal{R}}^{\mathrm{GR}}(k) \tag{56}$$

We now study the effect of the underlying telleparallel gravity theory at the level of the GW propagation $\rightarrow G_k(\eta, \bar{\eta})$

$$G_{\boldsymbol{k}}^{\lambda,\prime\prime}(\eta,\bar{\eta}) - 2\mathcal{H}\gamma_T G_{\boldsymbol{k}}^{\lambda,\prime}(\eta,\bar{\eta}) + \left(k^2 - \frac{a''}{a} + 2\mathcal{H}^2\gamma_T\right) G_{\boldsymbol{k}}^{\lambda}(\eta,\bar{\eta}) = \delta\left(\eta - \bar{\eta}\right),$$

$$\left|\frac{G_{\boldsymbol{k}}''(\eta,\bar{\eta})}{2\mathcal{H}\gamma_T G_{\boldsymbol{k}}'(\eta,\bar{\eta})}\right| \simeq \left.\frac{1}{2\mathcal{H}\gamma_T}\right|_{\eta=\eta_{\mathrm{eq}}} \gg 1 \quad \text{and} \quad \left.\frac{k^2}{2\mathcal{H}^2\gamma_T}\right|_{k=k_{\mathrm{evap}},\eta=\eta_{\mathrm{eq}}} \gg 1,$$

where k_{evap} is the comoving scale exiting the Hubble radius at PBH evaporation time, thus being the largest scale considered here.

$$\text{Therefore} \quad G_{\boldsymbol{k}}^{f(T)}(\eta,\bar{\eta}) \simeq G_{\boldsymbol{k}}^{\mathrm{GR}}(\eta,\bar{\eta}), \quad \text{with} \quad f_{,T\phi} = 0. \qquad (57)$$

Conclusions

- PBH besides addressing major issues of modern cosmology (like dark matter) they can also give rise to unique GW signals, like SIGW.
- Such signals constitute a novel probe to test and constrain gravitational theories.
- In this work we investigated SIGW in the context of $f(T, \phi)$ theories and found no distinction wrt GR $\rightarrow$ need of non-minimal torsion-matter coupling
- We are currently studying such cases and also working on extending this formalism to other MG theories

Appendix: Calculation of today's GW energy density

$$\Omega_{\rm GW}(\eta_0,k)=\frac{\rho_{\rm GW}(\eta_0,k)}{\rho_{\rm c}(\eta_0)}=\frac{\rho_{\rm GW}(\eta_*,k)}{\rho_{\rm c}(\eta_*)}\left(\frac{a_*}{a_0}\right)^4\frac{\rho_{\rm c}(\eta_*)}{\rho_{\rm c}(\eta_0)}$$
$$=\Omega_{\rm GW}(\eta_*,k)\Omega^{(0)}_{\rm r}\frac{\rho_{{\rm r},*}a_*^4}{\rho_{{\rm r},0}a_0^4}, \tag{58}$$

where the index 0 denotes our present time and η_* is a reference time usually taken as the horizon crossing time when one considers that an enhanced energy perturbation with a characteristic scale k collapses to form a PBH and $\Omega_{\rm GW}\sim a^{-4}$.
Then, from $\rho_r=\frac{\pi^2}{15}g_{*\rho}T_{\rm r}^4$ and $T_{\rm r}\propto g_{*{\rm S}}^{-1/3}a^{-1}$:

$$\Omega_{\rm GW}(\eta_0,k)=\Omega^{(0)}_r\frac{g_{*\rho,*}}{g_{*\rho,0}}\left(\frac{g_{*{\rm S},0}}{g_{*{\rm S},*}}\right)^{4/3}\Omega_{\rm GW}(\eta_*,k), \tag{59}$$

where $g_{*\rho}$ and $g_{*{\rm S}}$ stand for the energy and entropy relativistic degrees of freedom.