The ultraviolet limit of the power spectrum and Lagrangian perturbation theory

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Based on: [JB, Z. Vlah, O. Antipin, *to appear*]

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Perturbation theory of LSS

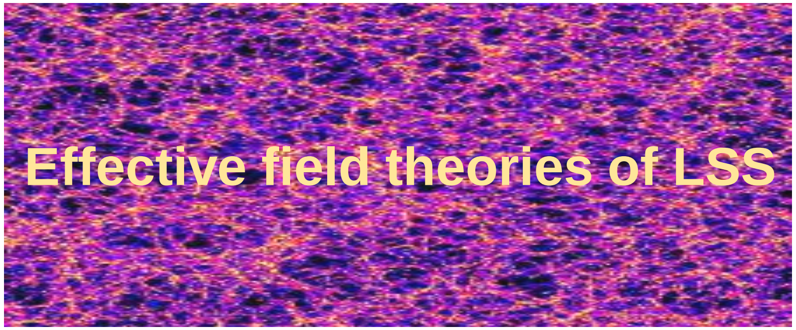
The study of the Large Scale Structure (LSS) has emerged as a core technique to understand the properties of the Universe.

LSS is mainly studied via numerical simulations.

However, it is important to develop analytical approaches to investigate the non-linear growth of cosmic structures.

Standard Perturbation Theory (SPT)

Lagrangian Perturbation Theory (LPT)



Effective field theories of LSS

Lagrangian perturbation theory

We consider cold dark matter interacting via Newtonian gravity.

For a fluid element at position **q** at some initial time, its position **x** at later times is written in terms of the displacement field ψ(**q**,t)

$$\mathbf{x}(\mathbf{q},t) = \mathbf{q} + \psi(\mathbf{q},t)$$

The matter power spectrum is

$$P(\mathbf{k}) + (2\pi)^3\delta_D(\mathbf{k}) = \int d^3q e^{i\mathbf{k}\cdot\mathbf{q}} \ \left\langle e^{i\mathbf{k}\cdot\Delta\psi(\mathbf{q})}\right\rangle$$

where $\Delta\psi(\mathbf{q}) \equiv \psi\left(\frac{\mathbf{q}}{2}\right) - \psi\left(-\frac{\mathbf{q}}{2}\right)$

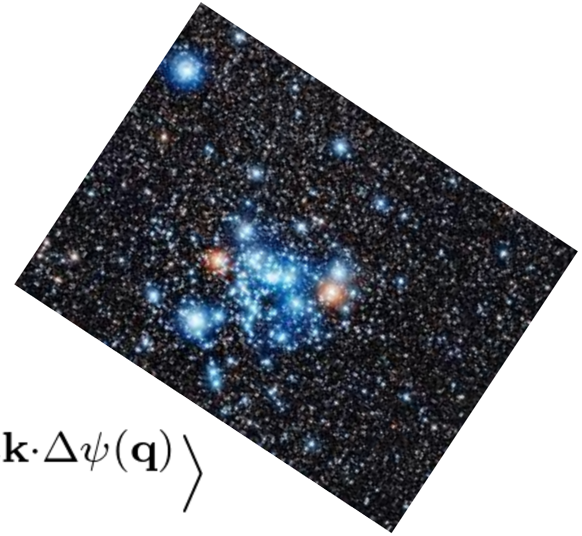
Lagrangian perturbation theory (LPT) expands ψ(q) as

$$\psi(\mathbf{q},t) = \psi^{(1)}(\mathbf{q},t) + \psi^{(2)}(\mathbf{q},t) + \psi^{(3)}(\mathbf{q},t) + \dots$$

and solves the EOM perturbatively.

Effective field theory [Z. Vlah, M. White, A. Aviles, JCAP 09 (20.15) 014]
[R. A. Porto, L. Senatore, M. Zaldarriaga, JCAP 05 (2014) 022]

The large k expansion



We consider the matter power spectrum

$$P(\mathbf{k}) + (2\pi)^3\delta_D(\mathbf{k}) = \int d^3q e^{i\mathbf{k}\cdot\mathbf{q}} \left\langle e^{i\mathbf{k}\cdot\Delta\psi(\mathbf{q})}\right\rangle$$

and expand the relative displacement field Δψ(q) as

$$\Delta\psi_i(\mathbf{q}) \sim \sum_{a=0} \frac{1}{2^{2a}(2a+1)!} M^{(2a+1)}_{i\mu_1\mu_2\ldots\mu_{2a+1}} q^{\mu_1} q^{\mu_2} \ldots q^{\mu_{2a+1}}$$

where $M^{(2a+1)}_{i\mu_1\mu_2\ldots\mu_{2a+1}} \equiv \left.\frac{\partial^{2a+1}\psi^i(\mathbf{q})}{\partial q_{\mu_1}\partial q_{\mu_2}\ldots\partial q_{\mu_{2a+1}}}\right|_{\mathbf{q}=0}$

We have

$$P(k) = \frac{1}{k^3}\langle\int d^3z\; e^{i\hat{k}^i z^j(\delta_{ij}+M^{(1)}_{ij})+\frac{i}{24k^2}\hat{k}^i z^j z^k z^l M^{(3)}_{ijkl}+\ldots}\rangle \sim \frac{1}{k^3}\sum_{i=0}\mathcal{C}_i\left(\frac{k_{\mathrm{UV}}}{k}\right)^{2i}$$

The large k expansion

$$\boxed{P(k) \sim \frac{1}{k^3} \sum_{i=0} \mathcal{C}_i \left(\frac{k_{\mathrm{UV}}}{k}\right)^{2i}}$$

The coefficients can be expressed in terms of probability distribution functions (PDFs) for the derivatives of ψ(**q**,t). E.g.

$$\mathcal{C}_0 = \langle \int d^3z\ e^{i\hat{k}^i z^j (\delta_{ij} + M^{(1)}_{ij})} \rangle = (2\pi)^3 \mathcal{P}_{M^{(1)}_{13}, M^{(1)}_{23}, M^{(1)}_{33}} [0, 0, -1]$$

[S. Chen, M. Pietroni, JCAP 06 (2020) 033]

The PDFs are dominated by the non-linear dynamics and can be computed in simulations.

In this talk we relate the large k expansion to LPT.

The Zel’dovich approximation

ZEL’DOVICH POWER SPECTRUM

$$(2\pi)^3\delta_D(k)+P_Z(k) = \int d^3q\, e^{i\vec{q}\cdot\vec{k}}\, e^{-\frac{1}{2}k_i k_j A_{ij}(\vec{q})} = \int d^3q\, e^{i\vec{q}\cdot\vec{k}}\, e^{-\frac{1}{2}k^2\left(X(q)+\mu^2 Y(q)\right)}$$

A_{ij} is the two-point function

$$\mu \equiv \frac{\mathbf{k}\cdot\mathbf{q}}{kq}$$

$$A_{ij}(\vec{q}) = \langle \Delta\Psi_i(\mathbf{q})\Delta\Psi_j(\mathbf{q})\rangle = X(q)\delta_{ij} + Y(q)\hat{q}_i\hat{q}_j$$

LINEAR POWER SPECTRUM

where e.g.

$$Y(q) = 2\xi_2(q) \qquad \xi_2(q) = \int_0^\infty \frac{dk}{2\pi^2}\, P_L(k) j_2(qk)$$

This is the leading order in LPT.

SPHERICAL BESSEL FUNCTION

Large k and LPT: leading order

We expand X(q) and Y(q) around q=0 as

$$X(q) = q^2 \sum_{n=0}^{\infty} x_n (k_{\rm UV} q)^{2n} \qquad Y(q) = q^2 \sum_{n=0}^{\infty} y_n (k_{\rm UV} q)^{2n}$$

$$x_n = 2\frac{(-1)^n}{(2n+5)\Gamma(2n+4)}\sigma_n$$

$$y_n = 2(n+1)x_n$$

$$\sigma_n \equiv \frac{k_{\rm UV}^2}{2\pi^2}\int_0^\infty P_L(k)\left(\frac{k}{k_{\rm UV}}\right)^{2n+2} dk$$

The Zel'dovich power spectrum becomes

$$P_Z(k) = \frac{2\pi}{k^3}\int_0^\infty \int_{-1}^{1} z^2 e^{i\mu z} \exp\left(-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{k_{\rm UV}}{k}\right)^{2n} z^{2n+2}\left(x_n + \mu^2 y_n\right)\right) d\mu\, dz$$

By expanding the above for large k we have

$$P(k) \sim \frac{1}{k^3}\sum_{i=0} \mathcal{C}_i \left(\frac{k_{\rm UV}}{k}\right)^{2i}$$

General coefficients

$$\boxed{P(k) \sim \frac{1}{k^3} \sum_{i=0} \mathcal{C}_i \left(\frac{k_{\mathrm{UV}}}{k} \right)^{2i}}$$

The leading coefficient is

$$\mathcal{C}_0 = 15\sqrt{5} \left(\frac{2\pi}{\sigma_0} \right)^{3/2} e^{-\frac{5}{2\sigma_0}}$$

The coefficients are non-perturbative in σ_0.

[S. Chen, M. Pietroni, JCAP 06 (2020) 033]

[S. Konrad, M. Bartelmann Mon.Not.Roy.Astron.Soc. 515 (2022) 2]

The coefficients can be written as

$$\mathcal{C}_n = \frac{1}{n!} \sum_{j=1}^{n} \left(-\frac{1}{2} \right)^j \int_{-1}^{1} Y_{n,j} \left(g_1, \ldots, g_{-j+n+1} \right) H_{j,n} \left(\mu, \sigma_0 \right) d\mu$$

$$g_n \equiv n! x_n \left(1 + 2\mu^2 (n+1) \right)$$

$$\begin{aligned} H_{j,n}\left(\mu, \sigma_1\right) = {} & 2^{j+n} 15^{j+n+1} \left(\left(2\mu^2 + 1 \right) \sigma_0 \right)^{-j-n-2} \left[\sqrt{30 \left(2\mu^2 + 1 \right) \sigma_0} \Gamma \left(j + n + \frac{3}{2} \right) \right. \\ & \left. \times {}_1F_1 \left(j + n + \frac{3}{2}; \frac{1}{2}; -\frac{15\mu^2}{4\sigma_0\mu^2 + 2\sigma_0} \right) - 30 i \mu \Gamma (j+n+2) \, {}_1F_1 \left(j + n + 2; \frac{3}{2}; -\frac{15\mu^2}{4\sigma_0\mu^2 + 2\sigma_0} \right) \right] \end{aligned}$$

Large k and LPT: 1-loop EFT

1-LOOP POWER SPECTRUM

$$(2\pi)^3\delta^D(k) + P_{\text{1-loop}}(k) = 2\pi \int_0^\infty \int_{-1}^{1} q^2 e^{ik\mu q} e^{-\frac{i}{6}k^3\mu^3 T(q)} e^{-\frac{1}{2}ik^3\mu V(q)} e^{-\frac{1}{2}k^2\left(X(q)+\mu^2 Y(q)\right)}\, dq\, d\mu$$

V(q) and T(q) come from the three-point function W_{ijk}

$$W_{ijk}(\vec{q}) = \langle \Delta\Psi_i(\mathbf{q})\Delta\Psi_j(\mathbf{q})\Delta\Psi_k(\mathbf{q})\rangle = V(q)\hat{q}_{\{i}\delta_{jk\}} + T(q)\hat{q}_i\hat{q}_j\hat{q}_k$$

We have e.g.

$$Y(q) = \frac{1}{\pi^2}\int_0^\infty \underline{j_2(qk)} \left(P_L(k) + \frac{9Q_1(k)}{98} + \frac{10R_1(k)}{21} + \underline{\alpha_1 k^2 P_L(k)} \right) dk$$

Bessel function Counterterms

and the same for X(q), V(q), T(q).

Large k and LPT: 1-loop EFT

We again expand X(q), Y(q), V(q), T(q) around q=0 as e.g.

$$Y(q) = q^2 \sum_{n=0}^{\infty} y_n \left(k_{\rm UV} q\right)^{2n} \qquad y_n = 2(n+1)\frac{(-1)^n}{(2n+5)\Gamma(2n+4)}\tau_{1,n}$$

$$\tau_{1,n} \equiv \frac{k_{\rm UV}^2}{2\pi^2}\int_0^\infty \left(P_L(k) + \frac{9}{98}Q_1(k) + \frac{10}{21}R_1(k) + \alpha_1 k^2 P_L(k)\right)\left(\frac{k}{k_{\rm UV}}\right)^{2n+2} dk$$

We are computing the coefficients of the large k expansion in LEFT.

$$P(k) \sim \frac{1}{k^3}\sum_{i=0} \mathcal{C}_i \left(\frac{k_{\rm UV}}{k}\right)^{2i}$$

$$\mathcal{C}_0 = 2\pi \int_0^\infty \int_{-1}^1 z^2 \exp\left(i\mu z - \frac{z^2\left(245\left(2\mu^2+1\right)\tau_{1,0} - 24 i\mu z\left(\left(5\mu^2+3\right)\tau_{2,0} - 7\tau_{3,0}\right)\right)}{7350}\right) d\mu\, dz$$

Symmetry constraints at one loop

At one-loop the combination $W_0 = T + 5V$ appears.

$$W_0(q) = \frac{1}{\pi^2}\int_0^\infty \left(\frac{-3}{7k}\right) j_1(qk)(Q_1(k)-3Q_2(k)+2R_1(k)-6R_2(k)+\alpha_2 k^2 P_L(k))\, dk$$

The small q expansion consistent with large k scaling is

$$W_0(q) = q^3 \sum_{n=0}^{\infty} w_n \, (k_{\rm UV} q)^{2n} \qquad w_n = \frac{3\sqrt{\pi}}{7}\frac{(-1)^{n+1}}{\Gamma(n+2)\Gamma\left(n+\frac{7}{2}\right)}\tau_{3,n}$$

$$\tau_{3,n} \equiv \frac{k_{\rm UV}^3}{2\pi^2}\int_0^\infty (Q_1(k)-3Q_2(k)+2R_1(k)-6R_2(k)+\alpha_2 k^2 P_L(k))\left(\frac{k}{2k_{\rm UV}}\right)^{2n+2} dk$$

However one would naively conclude that $W_0(q) \sim q$

$$J_1(qk) \underset{q\to 0}{\sim} \frac{qk}{3}$$

Solution

$$\int_0^\infty (Q_1(k) - 3Q_2(k) + 2R_1(k) - 6R_2(k) + \alpha_2 k^2 P_L(k))\; kdk = 0$$

$$\alpha_2 = 0$$

These hold for every $P_L(k)$ and come from symmetry.

Large k and LPT: the power spectrum

We start from the matter power spectrum

$$P(\mathbf{k}) + (2\pi)^3\delta_D(\mathbf{k}) = \int d^3q e^{i\mathbf{k}\cdot\mathbf{q}} \ \left\langle e^{i\mathbf{k}\cdot\Delta\psi(\mathbf{q})}\right\rangle$$

and use the cumulant theorem to expand it as

$$\left\langle e^{i\mathbf{k}\cdot\Delta\psi(\mathbf{q}))}\right\rangle = \exp\left[\sum_{N=1}^{\infty}\frac{(-i)^N}{N!}k^N\hat{k}_{\mu1}\dots\hat{k}_{\mu_N}\langle\Delta\Psi_{\mu_1}(\mathbf{q})\dots\Delta\Psi_{\mu_N}(\mathbf{q})\rangle_c\right] \equiv \exp\left[\sum_{N=1}^{\infty}\frac{(-i)^N}{N!}k^N\mathcal{B}_N(q,k_{\rm UV},\mu)\right]$$

where 

$$\mathcal{B}_N(q,k_{\rm UV},\mu)\sim q^N\sum_{n=0}^{\infty}b_{Nn}(\mu)(k_{\rm UV}q)^{2n}$$

The structure follows from dimensional analysis, translation invariance, and parity. We have

$$P(k) = \frac{2\pi}{k^3}\int_0^{\infty}\int_{-1}^{1} z^2 e^{i\mu z}\exp\left[\sum_{N=1}^{\infty}\frac{(-i)^N}{N!}\sum_{n=0}^{\infty}b_{Nn}(\mu)z^{N+2n}\left(\frac{k_{\rm UV}}{k}\right)^{2n}\right]d\mu\ dz$$

By expanding the exponential for small k_{UV}/k at fixed N, one generates the large k expansion with the corresponding coefficients C_i at the N-th order in LEFT

Integral constraints and counterterms

Consider odd N. Hence we can write

① $$\mathcal{B}_N(q,k_{\rm UV},\mu)\sim q^N\sum_{n=0}^{\infty}b_{Nn}(\mu)(k_{\rm UV}q)^{2n}$$

② $$\text{odd } N\text{:}\;\; \mathcal{B}_N=\sum_{i=0}^{\frac{N-1}{2}}\mu^{2i+1}\sum_{j=0}^{\frac{N-1}{2}}e_{i,2j+1}\Xi_{N,2j+1}(q,k_{\rm UV})$$

$$\mu\equiv\frac{\mathbf{k}\cdot\mathbf{q}}{kq}$$

where

$$\Xi_{N,l}(q,k_{\rm UV})=\int_0^{\infty}(f_{N,l}(k,k_{\rm UV})+\underbrace{[{\rm ct}]}_{}{}_{N,l}(k,k_{\rm UV}))\underbrace{j_l}_{}(kq)dk$$

Counterterms Bessel function

Consistency between ① and ② implies an infinite series of integral constraints for the functions and counterterms appearing in Lagrangian EFT.

An analogous result can be obtained for even N.

Integral constraints and counterterms

The integral constraints are

N IS EVEN

$$\int_0^\infty \left(\sum_{j=0}^{\gamma} e_{i,2j} a_{\gamma-j,2j} f_{N,2j}(k, k_{\mathrm{UV}}) \right) k^{2\gamma} dk = 0$$

$$\int_0^\infty \left(\sum_{j=0}^{\gamma} e_{i,2j} a_{\gamma-j,2j} [\mathrm{ct}]_{N,2j}(k, k_{\mathrm{UV}}) \right) k^{2\gamma} dk = 0$$

$$\gamma = 1, \ldots, \frac{N-2}{2}, \quad i = 0, \ldots, \frac{N}{2}$$

N IS ODD

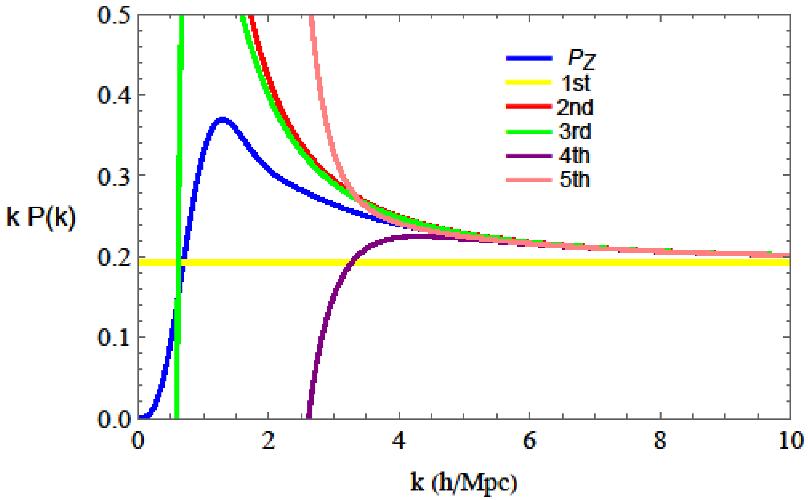
$$\int_0^\infty \left(\sum_{j=0}^{\gamma} e_{i,2j+1} a_{\gamma-j,2j+1} f_{N,2j+1}(k, k_{\mathrm{UV}}) \right) k^{2\gamma+1} dk = 0$$

$$\int_0^\infty \left(\sum_{j=0}^{\gamma} e_{i,2j+1} a_{\gamma-j,2j+1} [\mathrm{ct}]_{N,2j+1}(k, k_{\mathrm{UV}}) \right) k^{2\gamma+1} dk = 0$$

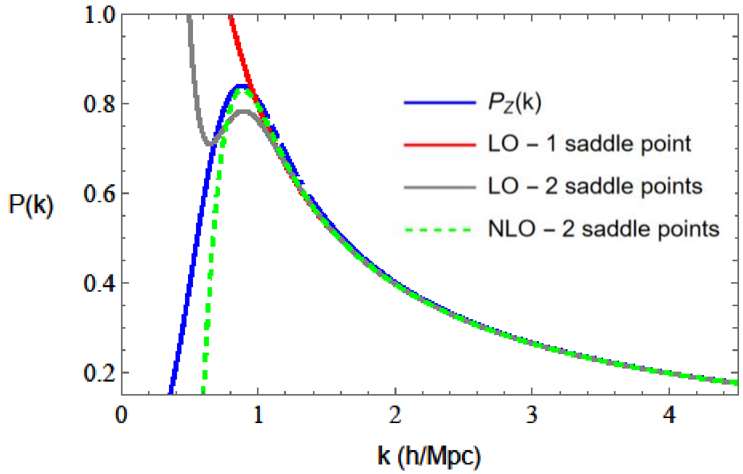
$$\gamma = 0, \ldots, \frac{N-3}{2}, \quad i = 0, \ldots, \frac{N-1}{2}$$

The a_{ij} are the coefficients of the Maclaurin series of the spherical Bessel functions.

Comparisons in d=1+1 dimensions



The Zel’vovich approximation (blue) and the large k expansion at various orders.



The large k expansion is a saddle point expansion (around q=0). Including subleading saddles improves the results.