Carrollian and Galilean limits of deformed symmetries in 3D gravity

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*__T. T.__, arXiv:2306.05409 [hep-th]
Kowalski-Glikman, Lukierski & **T. T.**, JHEP **09**, 096 (2020)

Outline:

1. Introduction

2. Kinematical algebras, r-matrices and quantum contractions
   - Classical Carrollian and Galilean symmetries
   - Deriving their coboundary deformations

3. Pictorial overview of (almost) all (coboundary) deformations
   - Comparing with the classifications of r-matrices
   - Carrollian and Galilean cases

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Context and motivation

Landscape of spacetime symmetries:

- Kinematical algebras, e.g. Poincaré, Carroll and Galilei; also their (central) extensions, e.g. Bargmann
- Asymptotic-symmetry algebras, e.g. BMS (and extensions); also their non-Lorentzian versions, e.g. Carroll-BMS and Galilei-BMS
- Quantum (Hopf-algebraic) deformations of both kinds of algebras, e.g. κ-Poincaré, as well as e.g. κ-BMS$_{\rm ext}$
- Non-Lorentzian versions of the latter, e.g. κ-Carroll and κ-Galilei

In 2+1 dimensions, with the cosmological constant Λ:

- Recently completed classification of (quantum) deformations
- The cases of $\Lambda \neq 0$ and $\Lambda = 0$ related by quantum contractions[a]
- Such deformations arise in the classical theory of (2+1)d gravity

[a] Kowalski-Glikman, Lukierski & **T. T.**, JHEP **09**, 096 (2020)

Context and motivation – non-Lorentzian kinematics

Carrollian symmetries:

- Associated with the Carroll (or ultrarelativistic) limit $c \to 0$
  - Ultralocality – trivial dynamics of free particles
  - Two Carroll limits of GR: “electric” and “magnetic”
  - Strong-gravity expansion, BKL conjecture, asymptotic silence[a]
- Symmetries of null hypersurfaces one dimension higher
  - Black-hole horizons, plane gravitational waves
  - BMS group $\cong$ a conformal extension of Carroll group

[a] Mielczarek & **T. T.**, PRD **96**, 024012 (2017)

Galilean symmetries:

- Associated with the Galilei (or “nonrelativistic”) limit $c \to \infty$
  - Weak-gravity expansion, gravitational waves research
- Algebraic/geometric structures “dual” to the Carrollian ones

Lorentz, Carroll and Galilei in (2+1)d

The brackets of Poincaré and (anti-)de Sitter algebras in (2+1)d can be written in a unified fashion (with $\Lambda = 0$, $\Lambda < 0$ or $\Lambda > 0$):

$$[J_0, K_a] = \epsilon_a{}^b K_b\,, \quad [K_1, K_2] = -J_0\,, \quad [J_0, P_a] = \epsilon_a{}^b P_b\,, \quad [J_0, P_0] = 0\,,$$
$$[K_a, P_b] = \delta_{ab} P_0\,, \quad [K_a, P_0] = P_a\,, \quad [P_1, P_2] = \Lambda\, J_0\,, \quad [P_0, P_a] = -\Lambda\, K_a\,. \quad (1)$$

Denoting $J := J_0$, $T_a := P_a$ and rescaling $Q_a := c\, K_a$, $T_0 := c\, P_0$, we take the limit $c \to 0$ to obtain Carroll / (anti-)de Sitter-Carroll algebra:

$$[J, Q_a] = \epsilon_a{}^b Q_b\,, \quad [Q_1, Q_2] = 0\,, \quad [J, T_a] = \epsilon_a{}^b T_b\,, \quad [J, T_0] = 0\,,$$
$$[Q_a, T_b] = \delta_{ab} T_0\,, \quad [Q_a, T_0] = 0\,, \quad [T_1, T_2] = \Lambda\, J\,, \quad [T_a, T_0] = \Lambda\, Q_a\,. \quad (2)$$

If we denote $J := J_0$, $T_0 := P_0$ and rescale $Q_a := c^{-1} K_a$, $T_a := c^{-1} P_a$, the limit $c \to \infty$ leads to Galilei / (anti-)de Sitter-Galilei algebra:

$$[J, Q_a] = \epsilon_a{}^b Q_b\,, \quad [Q_1, Q_2] = 0\,, \quad [J, T_a] = \epsilon_a{}^b T_b\,, \quad [J, T_0] = 0\,,$$
$$[Q_a, T_b] = 0\,, \quad [Q_a, T_0] = T_a\,, \quad [T_1, T_2] = 0\,, \quad [T_a, T_0] = \Lambda\, Q_a\,. \quad (3)$$

These are examples of contractions, which relate various kinematical (Lie) algebras with each other.

Why para-Euclidean and para-Poincaré?

Let us also show the brackets of (inhomogeneous) Euclidean algebra:

$$[J_3, K_a] = \epsilon_a{}^b K_b\,, \quad [K_1, K_2] = J_3\,, \quad [J_3, P_a] = \epsilon_a{}^b P_b\,, \quad [J_3, P_3] = 0\,,$$
$$[K_a, P_b] = -\delta_{ab} P_3\,, \quad [K_a, P_3] = P_a\,, \quad [P_1, P_2] = 0\,, \quad [P_3, P_a] = 0\,. \qquad (4)$$

It describes different kinematics but is related by the isomorphism

$$K_a \mapsto \Lambda^{-1/2} T_a\,, \quad P_a \mapsto \Lambda^{1/2} Q_a\,, \quad J_3 \mapsto J\,, \quad P_3 \mapsto T_0 \qquad (5)$$

with de Sitter-Carroll algebra, hence called the “para-Euclidean”. Meanwhile, Poincaré algebra is mathematically related by the isomorphism

$$K_a \mapsto |\Lambda|^{-1/2} T_a\,, \quad P_a \mapsto -|\Lambda|^{1/2} Q_a\,, \quad J_0 \mapsto J\,, \quad P_0 \mapsto T_0 \qquad (6)$$

with anti-de Sitter-Carroll algebra, hence called the “para-Poincaré”.

Meanwhile, the name “expanding/oscillating Newton-Hooke” is sometimes used for dS-Galilei/adS-Galilei algebra.

Coboundary (quantum) deformations

A Lie bialgebra is a Lie algebra $(\mathfrak{g}, [\,,])$ equipped with a cobracket:

$$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\,, \qquad (\mathfrak{g}, \delta) \text{ is a Lie coalgebra}\,,$$

$$\forall_{x,y\in\mathfrak{g}} : \delta([x, y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] + [\delta(x), y \otimes 1 + 1 \otimes y]\,. \quad (7)$$

The structure of a coboundary Lie bialgebra is determined by

$$r \in \mathfrak{g} \wedge \mathfrak{g}\,, \qquad \forall_{x\in\mathfrak{g}} : \delta(x) = [x \otimes 1 + 1 \otimes x, r]\,. \quad (8)$$

Such r is called a (antisymmetric) classical r-matrix and is actually an equivalence class with respect to automorphisms of $\mathfrak{g}$. Moreover, it is a solution of the classical Yang-Baxter equation

$$[[r, r]] = t\,\Omega\,, \qquad \Omega \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}\,,\ t \in \mathbb{C}\,, \quad (9)$$

where Ω is $\mathfrak{g}$-invariant and $[[\,,]]$ denotes Schouten bracket. r-matrix is quasitriangular if $t \neq 0$, or triangular if $t = 0$.

Quantum contractions and deformation parameters

To perform a (quantum) contraction of a quantum-deformed algebra, one not only needs to rescale the appropriate generators but also each deformation parameter q is rescaled to:

$$\hat{q} := q/\omega^2 \qquad \text{or} \qquad \tilde{q} := q/\omega \qquad \text{or} \qquad q = q\,; \tag{10}$$

with $\omega = |\Lambda|$ for $\Lambda \to 0$, and $\omega = c$ for $c \to 0$, and $\omega = c^{-1}$ for $c \to \infty$.

Obtaining the most general contraction limit may require a redefinition of parameters before their rescaling, e.g. γ, ς are replaced by $\tilde{\gamma}$, $\hat{\gamma}$, where $\varsigma = (\varsigma - \gamma) + \gamma =: c^2\hat{\gamma} + c\,\tilde{\gamma}$ **(see the next slide)**.

On the other hand, after the contraction, some parameters can be eliminated by an automorphism **(see the next slide)**.

Quantum contractions and automorphisms

If we transform a deformed algebra by a suitable automorphism, this may lead to a separate contraction limit, e.g. two representatives of the r-matrix class r_{IV} for anti-de Sitter algebra

$$r_{IV}(\gamma,\varsigma) = \gamma\left(J_0 \wedge K_2 - P'_0 \wedge P'_1 - K_1 \wedge P'_2\right) - \frac{\varsigma}{2}(J_0 - P'_1) \wedge (K_2 + P'_0)\,,$$
$$r^a_{IV}(\gamma,\varsigma) = -\gamma\left(J_0 \wedge P'_1 + K_2 \wedge P'_0 + K_1 \wedge P'_2\right) + \frac{\varsigma}{2}(J_0 - K_2) \wedge (P'_0 + P'_1) \quad (11)$$

($P'_\mu \equiv |\Lambda|^{-1/2} P_\mu$) are equivalent but their Carrollian contraction limits

$$r_{CIV}(\tilde\gamma,\tilde\varsigma) = \tilde\gamma\left(J \wedge Q_2 - T'_0 \wedge T'_1 - Q_1 \wedge T'_2\right) - \frac{\tilde\varsigma}{2}(J - T'_1) \wedge (Q_2 + T'_0)\,,$$
$$r_{CIVa}(\tilde\gamma,\hat\gamma) = -\tilde\gamma\left(J \wedge T'_0 + Q_1 \wedge T'_2 - Q_2 \wedge T'_1\right) - \hat\gamma\, Q_2 \wedge T'_0$$
$$\cong -\tilde\gamma\left(J \wedge T'_0 + Q_1 \wedge T'_2 - Q_2 \wedge T'_1\right) = r_{CIVa}(\tilde\gamma) \quad (12)$$

($T'_\mu \equiv |\Lambda|^{-1/2} T_\mu$), describing deformations of adSC algebra, are not. The corresponding automorphism of adS is not inherited by adSC. Automorphisms yield additional contraction limits also for dSG.

Trivialized/reduced deformations – examples

A classical r-matrix is determined up to an antisymmetric split-Casimir, i.e. such $\mathcal{C}_s \in \mathfrak{g} \wedge \mathfrak{g}$ that $\forall_{x \in \mathfrak{g}} : [x \otimes 1 + 1 \otimes x, \mathcal{C}_s] = 0$. We find that Galilei algebra has an antisymmetric split-Casimir

$$\mathcal{C}_{s1} := Q_1 \wedge T_1 + Q_2 \wedge T_2 \,, \tag{13}$$

while (anti-)de Sitter-Galilei algebra has both (13) and

$$\mathcal{C}_{s2} := Q_1 \wedge Q_2 - \Lambda^{-1} T_1 \wedge T_2 \,. \tag{14}$$

The quantum contraction limits are simplified by dropping such terms. Incidentally, r-matrix of the form (13) describes timelike κ-deformation.

Spacelike κ-deformation for Carroll and (a)dS-Carroll is reduced to:

$$r(\gamma) = \gamma \, (J_0 \wedge P_1 + K_2 \wedge P_0) \quad \xrightarrow[c \to 0]{} \quad r(\hat{\gamma}) = \hat{\gamma} \, Q_2 \wedge T_0 \,. \tag{15}$$

Classifications of deformations vs their contractions

Semisimple or inhomogeneous-(pseudo)orthogonal algebras have only coboundary deformations, which can be completely classified in terms of r-matrices. This has been achieved for 2+1-dimensional algebras:

- Poincaré (as well as Euclidean)[a],
- (anti-)de Sitter[b],
- (anti-)de Sitter-Carroll (isomorphisms with Poincaré/Euclidean)[c].

Quantum contractions of (anti-)de Sitter r-matrices in the limit:

- $\Lambda \to 0$, leading to Poincaré[d],
- $c \to 0$, leading to (a)dS-Carroll[c],

recover all r-matrix classes for a given target algebra, up to a few missing terms in some classes.

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[a] Stachura, JPA **31**, 4555 (1998)
[b] Borowiec, Lukierski & Tolstoy, JHEP **11**, 187 (2017)
[c] **T. T.**, arXiv:2306.05409 [hep-th]
[d] Kowalski-Glikman, Lukierski & **T. T.**, JHEP **09**, 096 (2020)

Deformations of Poincaré and (a)dS algebras

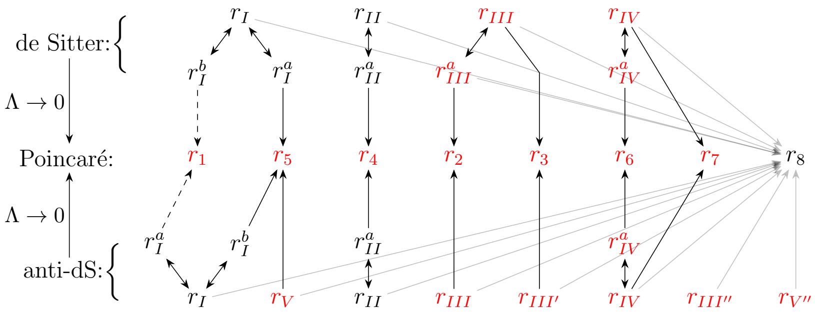

Figure: Quantum $\Lambda \to 0$ contractions relating all r-matrices for (anti-)de Sitter and Poincaré algebras; double-headed arrows denote automorphisms; arrows leading to r_8 are lightened for legibility; quasitriangular cases are red.

Deformations of Carroll and Galilei algebras

We derived the Carroll/Galilei r-matrices by quantum $c \to 0$ / $c \to \infty$ contractions of the Poincaré ones. Possibly, some deformations can not be obtained in this way. There may also exist non-coboundary deformations of these algebras[a].

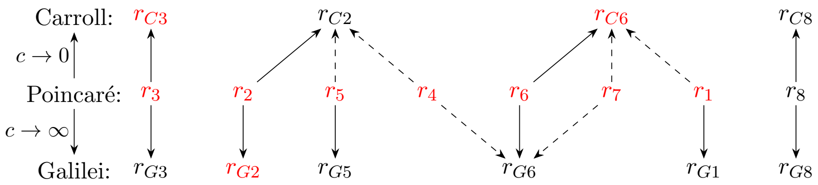

Figure: Quantum $c \to 0$ / $c \to \infty$ contractions relating all r-matrices for Poincaré with those obtained for Carroll/Galilei algebra; a dashed line means that a given contraction leads to a subclass; quasitriangular cases are red.

[a] Ballesteros et al., PLB **805**, 135461 (2020)

Deformations of (a)dS, (a)dSC and Carroll algebras

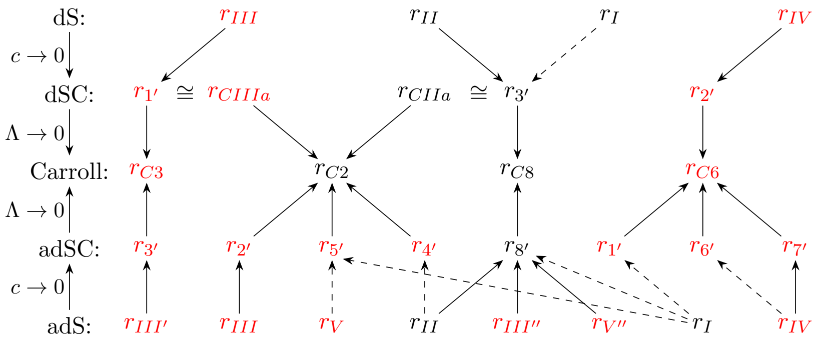

Figure: Quantum $c \to 0$ and $\Lambda \to 0$ contractions relating all r-matrices for (anti-)de Sitter and (a)dS-Carroll, and those obtained for Carroll algebra; a dashed line means that a given $c \to 0$ contraction leads to a subclass.

Deformations of (a)dS, (a)dSG and Galilei algebras

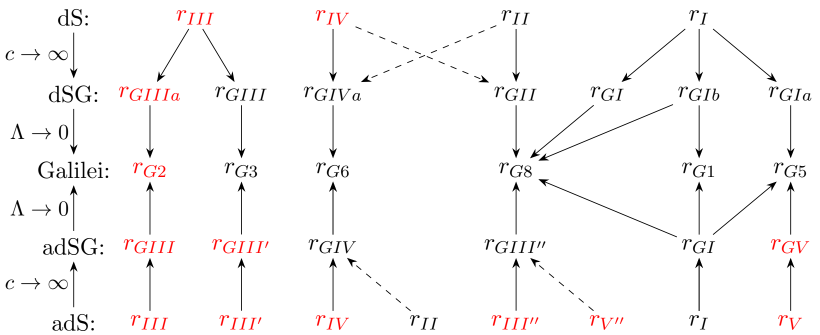

Figure: Quantum $c \to \infty$ and $\Lambda \to 0$ contractions relating all r-matrices for (anti-)de Sitter with those obtained for (a)dS-Galilei and Galilei algebras; a dashed line means that a given $c \to \infty$ contraction leads to a subclass.

Possibly, not all deformations of (a)dSG can be obtained by contractions and there may also exist non-coboundary ones.

Summary – special cases of deformations

The cases of particular interest are time- and spacelike κ-deformations, and the Lorentz double. They also survive under (almost) all quantum contractions ($\Lambda \to 0$, $c \to 0$, or $c \to \infty$) for both $\Lambda > 0$ and $\Lambda < 0$.

algebra	timelike κ-deformation	spacelike κ-deformation	Lorentz double
dSC	$r_{CIII}(\tilde{\gamma}_+) \cong r_{1'}(\gamma)$	$r_{CIIIa}(\hat{\gamma}_-) \cong r_{1'}(\theta_{12})$	$r_{CIV}(\tilde{\gamma}) \cong r_{2'}(\gamma)$
Carroll	$r_{C3}(\tilde{\gamma})$	$r_{C2}(\hat{\gamma})$	$r_{C6}(\tilde{\gamma})$
adSC	$r_{CIII'}(\tilde{\gamma}_+) \cong r_{3'}(\gamma)$	$r_{CIII}(\hat{\gamma}_+) \cong r_{2'}(\theta_{20})$	$r_{CIVa}(\tilde{\gamma}) \cong r_{7'}(\gamma)$
dS	$r_{III}(\gamma_+)$	$r^{a}_{III}(\gamma_-) \cong r_{III}(\gamma_-)$	$r_{IV}(2\gamma = \varsigma)$
Poincaré	$r_3(\gamma)$	$r_2(\gamma)$	$r_7(\gamma)$
adS	$r_{III'}(\gamma_+) \cong r_{III'}(\gamma_-)$	$r_{III}(\gamma_+) \cong r_{III}(\gamma_-)$	$r_{IV}(2\gamma = -\varsigma)$
dSG	0	$r_{GIIIa}(\tilde{\gamma}_-)$	$r_{GIVa}(2\hat{\gamma} = -\hat{\varsigma})$
Galilei	0	$r_{G2}(\tilde{\gamma})$	$r_{G6}(\hat{\gamma} = \hat{\varsigma})$
adSG	0	$r_{GIII}(\tilde{\gamma}_+) \cong r_{GIII}(\tilde{\gamma}_-)$	$r_{GIV}(2\hat{\gamma} = -\hat{\varsigma})$

Table: r-matrices (only $\neq 0$ deformation parameters) that characterize special cases of deformations, depending on a kinematical algebra; quasitriangular ones are red.

Abridged definition of the Hopf algebra

A Hopf algebra A is the vector space over a field K, equipped with a product (e.g. a Lie bracket) $\nabla : A \otimes A \to A$, satisfying the associativity

$$\nabla \circ (\nabla \otimes \mathrm{id}) = \nabla \circ (\mathrm{id} \otimes \nabla)\,; \tag{16}$$

a coproduct $\Delta : A \to A \otimes A$, satisfying the coassociativity

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta\,; \tag{17}$$

and an antipode $S : A \to A$, satisfying the relation

$$\nabla \circ (S \otimes \mathrm{id}) \circ \Delta = \nabla \circ (\mathrm{id} \otimes S) \circ \Delta = \mathbb{1}\,. \tag{18}$$

The tensor product of a pair of algebra representations (ρ_1, V_1), (ρ_2, V_2) (where $\rho_{1,2} : A \to \mathrm{GL}(V_{1,2})$) is given by $(\rho, V_1 \otimes V_2)$, such that

$$\rho(a)(v_1 \otimes v_2) = (\rho_1 \otimes \rho_2)(\Delta(a))(v_1 \otimes v_2)\,, \tag{19}$$

where $a \in A$, $v_{1,2} \in V_{1,2}$.

Example – the Hopf algebra corresponding to r_{III}

Denoting $H_0 \equiv H$, $H_1 \equiv \bar{H}$, $E_{0\pm} \equiv E_\pm$, $E_{1\pm} \equiv \bar{E}_\pm$ and $q_0 \equiv e^{\gamma/2}$, $q_1 \equiv e^{\bar{\gamma}/2}$, $\theta \equiv e^{\eta/4}$, we write down the deformed brackets

$$[H_k, E_{k\pm}] = E_{k\pm}\,, \qquad [E_{k+}, E_{k-}] = \frac{q_k^{2H_k} - q_k^{-2H_k}}{q_k - q_k^{-1}}\,, \tag{20}$$

where $k = 0, 1$. In the limit $q_k \to 1$ it reduces to $[E_{k+}, E_{k-}] = 2H_k$. Meanwhile, the coproducts have the form

$$\Delta(H_k) = H_k \otimes 1 + 1 \otimes H_k\,,$$
$$\Delta(E_{k\pm}) = E_{k\pm} \otimes q_k^{H_k} \theta^{\mp(-1)^k H_{k+1}} + \theta^{\pm(-1)^k H_{k+1}} q_k^{-H_k} \otimes E_{k\pm} \tag{21}$$

and antipodes

$$S(H_k) = -H_k\,, \qquad S(E_{k\pm}) = -q_k^{\pm 1} E_{k\pm}\,. \tag{22}$$

The dual of the subalgebra of translations are spacetime coordinates

$$[X_0, X_a] = 2\gamma\, X_a\,, \qquad [X_a, X_b] = 0\,, \quad a, b = 1, 2\,. \tag{23}$$

Poisson structure via the Fock-Rosly construction

$\mathfrak{g}$ equipped with r becomes the Lie algebra of a Poisson-Lie group of spacetime symmetries, dual to the particle phase space. At the same time, r determines the Hopf-algebraic deformation of $\mathfrak{g}$, providing the quantization of the theory. The consistency with 3D gravity requires

$$r = r_A + r_S\,, \quad r_S = \alpha\,(P_\mu \otimes J^\mu + J^\mu \otimes P^\mu) + \beta\,(\Lambda\, J^\mu \otimes J_\mu - P^\mu \otimes P_\mu)\;, \quad \alpha, \beta \in \mathbb{R}\,, \tag{24}$$

where r_S corresponds to the generalized form of the inner product in Chern-Simons action ($\beta = 0$ in the standard case), while r satisfies the homogeneous Yang-Baxter equation, hence r_A:

$$\begin{aligned}
[[r_A, r_A]] &= -[[r_S, r_S]] \\
&= -(\alpha^2 - \Lambda\beta^2)\left(\Lambda\, J_0 \wedge J_1 \wedge J_2 + \tfrac{1}{2}\,\epsilon^{\mu\nu\sigma}\, J_\mu \wedge P_\nu \wedge P_\sigma\right) \\
&\quad - 2\alpha\beta\left(\tfrac{1}{2}\,\Lambda\,\epsilon^{\mu\nu\sigma}\, J_\mu \wedge J_\nu \wedge P_\sigma + P_0 \wedge P_1 \wedge P_2\right)\,.
\end{aligned} \tag{25}$$

We call such a r_A to be FR-compatible and classify all of them in J. Kowalski-Glikman, J. Lukierski & **T. T.**, JHEP **09**, 096 (2020).

r-matrices of 3D (A)dS algebra relevant for gravity

Calculating the Schouten bracket $[[r_A, r_A]]$, we find that r-matrices are:

	FR-compatible $\forall\alpha, \beta$	FR-compatible for $\beta = 0$	FR-compatible for $\alpha, \beta \neq 0$
$\mathfrak{o}(3,1)$	r_{III}, r^a_{III}	r_{IV}, r^a_{IV}	
$\dot{\mathfrak{o}}(2,2)$	r_{III}	r_{IV}, r^a_{IV}	r_V
$\dot{\mathfrak{o}}'(2,2)$			r_{III}

Example – FR-compatible r-matrices of dS algebra:

$$\begin{aligned}
r_{III}(\gamma-\bar{\gamma},\gamma+\bar{\gamma},\eta;\Lambda) &= \tfrac{1}{2}(\gamma-\bar{\gamma})\left(J_1\wedge J_2-\Lambda^{-1}P_1\wedge P_2\right)\\
&\quad+\Lambda^{-1/2}\tfrac{1}{2}(\gamma+\bar{\gamma})\,(J_1\wedge P_2-J_2\wedge P_1)+\Lambda^{-1/2}\frac{\eta}{2}\,J_0\wedge P_0\,,\\
r^a_{III}(\gamma-\bar{\gamma},\gamma+\bar{\gamma},\eta;\Lambda) &= \Lambda^{-1/2}\tfrac{1}{2}(\gamma-\bar{\gamma})\,(J_0\wedge P_2-J_2\wedge P_0)\\
&\quad+\tfrac{1}{2}(\gamma+\bar{\gamma})\left(J_0\wedge J_2-\Lambda^{-1}P_0\wedge P_2\right)+\Lambda^{-1/2}\frac{\eta}{2}\,J_1\wedge P_1\,,\\
r_{IV}(\gamma,\varsigma;\Lambda) &= \gamma\left(J_1\wedge J_2-\Lambda^{-1/2}J_0\wedge P_0-\Lambda^{-1}P_1\wedge P_2\right)\\
&\quad+\frac{\varsigma}{2}\left(J_1-\Lambda^{-1/2}P_2\right)\wedge\left(J_2+\Lambda^{-1/2}P_1\right),\\
r^a_{IV}(\gamma,\varsigma;\Lambda) &= \Lambda^{-1/2}\gamma\,(J_0\wedge P_1-J_1\wedge P_0-J_2\wedge P_2)\\
&\quad+\Lambda^{-1/2}\frac{\varsigma}{2}\,(J_0-J_1)\wedge(P_0-P_1)\,.
\end{aligned}\tag{26}$$

To be compared with P. K. Osei & B. J. Schroers, CQG **35**, 075006 (2018).

r-matrices of (A)dS algebra in the $\Lambda \to 0$ limit

Quantum IW contractions of r-matrices of (A)dS algebra lead to the following r-matrices of 3D Poincaré algebra:

r-matrix automorphism class[a]	$\mathfrak{o}(3,1)\downarrow$	$\dot{\mathfrak{o}}(2,2)\downarrow$	$\dot{\mathfrak{o}}'(2,2)\downarrow$
$r_1 = \chi\,(J_0 + J_1) \wedge J_2$	r_I^b	r_I^a	
$\hat{r}_2 = \hat{\gamma}\,(J_0 \wedge \mathcal{P}_2 - J_2 \wedge \mathcal{P}_0) + \frac{1}{2}\hat{\eta}\,J_1 \wedge \mathcal{P}_1$	$\hat{r}_{III}^a$	$\hat{r}_{III}$	
$\hat{r}_3 = \hat{\gamma}\,(J_1 \wedge \mathcal{P}_2 - J_2 \wedge \mathcal{P}_1) + \frac{1}{2}\hat{\eta}\,J_0 \wedge \mathcal{P}_0$	$\hat{r}_{III}$		$\hat{r}_{III}$
$\hat{r}_4 = \frac{1}{\sqrt{2}}\hat{\chi}\,(J_+ \wedge \mathcal{P}_1 - J_1 \wedge \mathcal{P}_+) - \hat{\varsigma}\,J_+ \wedge \mathcal{P}_+$	$\hat{r}_{II}^a$	$\hat{r}_{II}^a$	
$\hat{r}_5 = \frac{1}{2}\hat{\chi}J_1 \wedge (\mathcal{P}_0 + \mathcal{P}_2)$	$\hat{r}_I^a$	$\hat{r}_V$	
$\hat{r}_6 = \hat{\gamma}\,(J_0 \wedge \mathcal{P}_2 - J_2 \wedge \mathcal{P}_0 - J_1 \wedge \mathcal{P}_1) - \hat{\varsigma}\,J_+ \wedge \mathcal{P}_+$	$\hat{r}_{IV}^a$	$\hat{r}_{IV}^a$	
$\hat{r}_7 = \hat{\gamma}\,(J_0 \wedge \mathcal{P}_0 - J_1 \wedge \mathcal{P}_1 - J_2 \wedge \mathcal{P}_2)$	$\hat{r}_{IV}$	$\hat{r}_{IV}$	

(as well as the irrelevant cases $\sim P_\mu \wedge P_\nu$), where $J_+ \equiv \frac{1}{\sqrt{2}}(J_0 + J_2)$, $P_+ \equiv \frac{1}{\sqrt{2}}(P_0 + P_2)$. Only $\hat{r}_2$, $\hat{r}_6$ and $\hat{r}_7$ are relevant for 3D gravity, i.e.

$$[[r_1, r_1]] = [[\hat{r}_4, \hat{r}_4]] = [[\hat{r}_5, \hat{r}_5]] = 0\,,$$
$$[[\hat{r}_3, \hat{r}_3]] = \hat{\gamma}^2 \epsilon^{\mu\nu\sigma} J_\mu \wedge \mathcal{P}_\nu \wedge \mathcal{P}_\sigma\,,$$
$$[[\hat{r}_2, \hat{r}_2]] = [[\hat{r}_6, \hat{r}_6]] = [[\hat{r}_7, \hat{r}_7]] = -\hat{\gamma}^2 \epsilon^{\mu\nu\sigma} J_\mu \wedge \mathcal{P}_\nu \wedge \mathcal{P}_\sigma\,. \qquad (27)$$

[a] P. Stachura, J. Phys. A: Math. Gen. **31**, 4555 (1998)

Chern-Simons action of 3D gravity (with Λ)

Instead of the metric $g_{\alpha\beta}$, gravity can be described in terms of the vielbein $e_\alpha^{\ \mu}$ and spin connection $\omega_\alpha^{\ \mu\nu}$, defined as

$$e_\alpha^{\ \mu} e_\beta^{\ \nu} \eta_{\mu\nu} = g_{\alpha\beta} \,, \qquad \omega_\alpha^{\ \mu\nu} = e_\beta^{\ \mu} \partial_\alpha e^{\beta\nu} + e_\beta^{\ \mu} \Gamma^\beta_{\ \alpha\gamma} e^{\gamma\nu} \,. \tag{28}$$

In (2+1)D, they neatly combine into a gauge field – with values in the local isometry algebra $\mathfrak{g}$ (3D Poincaré or (Anti-)de Sitter) – which is the Cartan connection

$$A = -\tfrac{1}{2} \epsilon^\mu_{\ \nu\sigma} \omega_\alpha^{\ \nu\sigma} J_\mu dx^\alpha + e_\alpha^{\ \mu} P_\mu dx^\alpha \,, \tag{29}$$

where J_μ, P_μ are generators of $\mathfrak{g}$. Then, the Einstein-Hilbert action can be written as the Chern-Simons theory action

$$S = \frac{1}{16\pi G} \int \left(\langle dA \wedge A \rangle + \frac{1}{3} \langle A \wedge [A, A] \rangle \right) \tag{30}$$

if the scalar product on $\mathfrak{g}$ is given by

$$\langle J_\mu, P_\nu \rangle = \eta_{\mu\nu} \,, \qquad \langle J_\mu, J_\nu \rangle = \langle P_\mu, P_\nu \rangle = 0 \,. \tag{31}$$

How to couple a conical defect

The massive point-particle spacetime interval looks as in vacuum:

$$ds^2 = (1 - \Lambda r^2)\, dt^2 - (1 - \Lambda r^2)^{-1} dr^2 - r^2 d\tilde{\phi}^2\,, \tag{32}$$

when the polar angle is rescaled to $\tilde{\phi} := (1-4Gm)\,\phi$ (a conical defect). Similarly, spin $\neq 0$ introduces a jump in the time coordinate.

If spacetime has the topology $\mathbb{R} \times \mathcal{S}$, the field A may be expressed as $A = A_t dt + A_\mathcal{S}$ and the action of gravity with a particle is

$$S = \int dt\, L = \frac{1}{16\pi G} \int dt \int_\mathcal{S} \left\langle \dot{A}_\mathcal{S} \wedge A_\mathcal{S} \right\rangle - \int dt \left\langle c_0 h^{-1} \dot{h} \right\rangle + \int dt \int_\mathcal{S} \left\langle A_t \left(\frac{1}{8\pi G} F_\mathcal{S} - h c_0 h^{-1} \delta^2(\vec{x} - \vec{x}_*)\, dx^1 \wedge dx^2 \right) \right\rangle . \tag{33}$$

Mass $\neq 0$ and spin of a particle are encoded by $\mathfrak{g} \ni c_0 = m\, J_0 + s\, P_0$, while a gauge group element h acting via $h c_0 h^{-1} = \mathbf{p} + \mathbf{j}$ determines its momentum $\mathbf{p} = p^\mu J_\mu$ and (generalized) angular momentum $\mathbf{j} = j^\mu P_\mu$.

Relating the gravitational and particle DOFs

A_t acts as a Lagrange multiplier imposing a constraint on the curvature of spatial connection $F_S = dA_S + [A_S, A_S]$:

$$F_{\mathcal{S}} = 8\pi G\, hc_0 h^{-1} \delta^2(\vec{x} - \vec{x}_*)\, dx^1 \wedge dx^2 \,. \tag{34}$$

From $F_{\mathcal{S}} = R_{\mathcal{S}} + T_{\mathcal{S}} + C_{\mathcal{S}}$ ($C_{\mathcal{S}}$ is the cosmological-constant term), it follows that the spatial Riemann curvature and torsion are given by

$$R_{\mathcal{S}} = -C_{\mathcal{S}} + 8\pi G\, \mathbf{p}\, \delta^2(\vec{x} - \vec{x}_*)\, dx^1 \wedge dx^2 \,,$$
$$T_{\mathcal{S}} = 8\pi G\, \mathbf{j}\, \delta^2(\vec{x} - \vec{x}_*)\, dx^1 \wedge dx^2 \,, \tag{35}$$

i.e. they vanish (on the constant background $R_{\mathcal{S}} = -C_{\mathcal{S}}$) everywhere except a singularity at the particle's worldline.
Alternatively, $\mathfrak{g}$ can be equipped with the scalar product

$$\langle J_\mu, P_\nu \rangle = 0 \,, \qquad \langle J_\mu, J_\nu \rangle = -\Lambda^{-1} \langle P_\mu, P_\nu \rangle = \eta_{\mu\nu} \,. \tag{36}$$

If our action is defined using it, **j** generates $R_{\mathcal{S}}$ and **p** generates $T_{\mathcal{S}}$.