Bootstrapping (metric affine) gravity

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Based on: [gr-qc: arXiv:2211.13056]

Rijeka, Croatia - 12/07/2023

Outline

1. Introduction
2. Bootstrapping Fierz-Pauli: General Relativity?
3. Extensions to the metric affine framework and higher derivative theories.
4. Subtleties in higher derivative theories of gravity.
5. Conclusions.

Bootstraping of Fierz-Pauli: Historical account

- **Problem:** is GR the only consistent nonlinear extensions of Fierz-Pauli?
- Lots of works: Gupta (1954) and Feynman (1962); Kraichnan (1955) and Huggins (1962); Deser (1970).
- Renewed interest after Padmanabhan (2004) critic to old works: Deser (2009), Butcher *et al* (2009).
- Do higher derivative theories theories can also be reconstructed from their linear versions? (Ortin 2017, Deser 2017)

The objectives of our paper:

- Clarify the construction for Fierz-Pauli and the uniqueness of the construction.
- Clarify the results for higher derivative theories of gravity.
- Extend the analysis to metric affine theories of gravity (try to constrain them by consistency arguments).

Linear gravity: Fierz-Pauli

- Let us begin to analyze gravity as a field theory.
- The starting point is to consider gravity as a massless spin-2 theory:

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \partial_\mu h^\mu{}_\nu \partial_\alpha h^{\alpha\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h \partial^\mu h.$$

Fierz-Pauli Proc.Roy.Soc.Lond.A 173 (1939) 211-232.

- Gauge invariance under linearly realized diffeomorphisms:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

- Ensures that only the two degrees of freedom propagate.

Another theory describing linear gravity: WTDiff

- There is another theory which describes the propagation of massless spin-2 particles, the so called WTDiff theory:

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\alpha h_{\mu\nu}\right)^2 + \partial_\mu h^\mu{}_\nu \partial_\alpha h^{\alpha\nu} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{3}{8}\partial_\mu h \partial^\mu h.$$

Álvarez *et al* Nucl.Phys.B 756 (2006) 148-170

- Its gauge symmetries are Weyl and TDiff transformations:

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi^T_\nu + \partial_\nu \xi^T_\mu + \frac{1}{2}\phi\eta_{\mu\nu},$$
$$\partial^\mu \xi^T_\mu = 0.$$

Another theory describing linear gravity: WTDiff

- Spoiler: The result of the self-coupling is Unimodular Gravity.

Another theory describing linear gravity: WTDiff

- Spoiler: The result of the self-coupling is Unimodular Gravity.
- But that is a story for another day...
- For a review and systematic comparison with GR, see: Class.Quant.Grav. **39** (2022) 24, 243001

Consistent non-linear extensions?

- Problem: including interactions into the picture without spoiling the propagation of only 2 degrees of freedom.
- The starting point are the Fierz-Pauli eoms:

$$\mathcal{D}^{\alpha\beta}{}_{\rho\sigma} h^{\rho\sigma} = 0.$$

- The equations are divergenceless (Bianchi identities, gauge invariance):

$$\partial_\alpha \left(\mathcal{D}^{\alpha\beta}{}_{\rho\sigma} h^{\rho\sigma} \right) = 0.$$

Consistent non-linear extensions?

- Any non-linear term that we add to the Fierz-Pauli action

$$\mathcal{L} = \partial h \partial h + \lambda h \partial h \partial h + \mathcal{O}(\lambda^2),$$

- Leads to equations of motion of the form

$$\mathcal{D}^{(\alpha\beta)}_{\rho\sigma} h^{\rho\sigma} = \lambda t^{(\alpha\beta)} + \mathcal{O}(\lambda^2).$$
$$t^{\alpha\beta} \sim \partial h \partial h + h \partial^2 h$$

- Needs to be consistent with the symmetric and divergenceless structure of the eoms:

$$\partial_\alpha \left(\mathcal{D}^{\alpha\beta}{}_{\rho\sigma} h^{\rho\sigma} \right) = 0 \to \partial_\alpha t^{\alpha\beta} = 0.$$

- At least on-shell to leading order $(\mathcal{D}^{\alpha\beta}{}_{\rho\sigma} h^{\rho\sigma} = 0)$

Candidate for $t^{\mu\nu}$?

- It has to be symmetric and divergenceless on-shell. A natural candidate would be the energy-momentum tensor.
- Problem: a Lorentz invariant, conserved and gauge invariant energy-momentum tensor does not exist for the gravitational field.
- Solution: work with a gauge-dependent energy-momentum tensor.
- Another problem: it is ambiguous. We can always add identically conserved terms to a conserved current.

Canonical energy-momentum tensor

- The standard computation from Noether theorem gives

$$t^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi^A} \partial_\nu \Phi^A - \mathcal{L} \eta^{\mu\nu},$$

- We can always add an identically conserved term of the form:

$$\Delta t^{\mu\nu} = \partial_\rho \chi^{[\rho\mu]\nu},$$

- To get another current that is identically conserved.

Example: Real scalar field

- The standard computation from Noether theorem gives

$$t^{\mu\nu} = -\partial^\mu \Phi \partial^\nu \Phi + \frac{1}{2}\eta^{\mu\nu}\partial\Phi^2,$$

- We can always add an identically conserved term of the form:

$$\Delta t^{\mu\nu} = \alpha \left(\partial^\mu \partial^\nu \Phi - \eta^{\mu\nu}\partial^2 \Phi\right),$$

- Arising from a superpotential

$$\chi^{\rho\mu\nu} = 2\alpha \partial^{[\rho}\Phi \eta^{\mu]\nu},$$

Hilbert's prescription

- Take flat spacetime action and replace the flat metric with an arbitrary curved metric and partial derivatives with covariant derivatives:

$$\eta_{\mu\nu} \to g_{\mu\nu}, \qquad \partial_\mu \to \nabla_\mu$$

- We get an action $S(\Phi; g_{\mu\nu})$ and define the energy momentum tensor as

$$t_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

Ambiguities in Hilbert's prescription

- We can always add non-minimal couplings in the generalization to a curved metric.
- Their variation does not need to be zero after particularizing for flat spacetime.
- They correspond to identically conserved terms.

$$S_{\mathrm{nm}}[g,\Phi] \xrightarrow{g=\eta} 0$$

$$\left.\frac{\delta S_{\mathrm{nm}}[g,\Phi]}{\delta g^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} \neq 0$$

Example: Free real scalar field

- Take real free scalar field Φ

$$S = -\frac{1}{2}\int d^n x\sqrt{-\eta}\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$$
$$\rightarrow S = -\frac{1}{2}\int d^n x\sqrt{-g}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$$

- Compute the energy momentum tensor to find again

$$t^{\mu\nu} = -\partial^\mu\Phi\partial^\nu\Phi + \frac{1}{2}\eta^{\mu\nu}\partial\Phi^2,$$

Example:Free real scalar field

- Consider adding a non-minimal coupling of the form:

$$S_{nm}[g, \Phi] = -\frac{\alpha}{2} \int d^n x \sqrt{-g} \Phi R(g),$$

- It again leads to the identically conserved current:

$$\Delta t^{\mu\nu} = \alpha \left(\partial^\mu \partial^\nu \Phi - \eta^{\mu\nu} \partial^2 \Phi \right),$$

Idea of the procedure

- **1.** Take as starting point $S^{(2)}(\eta, h)$.
- **2.** Compute $t^{\mu\nu}$ with all possible ambiguities by any procedure.

$$
\begin{aligned}
S^{(2)} \qquad &+ \ldots = S \\
\mathcal{D}^{\alpha\beta}{}_{\rho\sigma} h^{\rho\sigma} &+ \ldots = 0
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Idea of the procedure

- **3.** Demand that the tensor is derived from a term $S^{(3)}$ in the action (this fixes part of the ambiguities):

$$\frac{\delta S^{(3)}}{\delta h^{\mu\nu}} = t^{(2)\mu\nu}$$

$$\begin{aligned} S^{(2)} \qquad\quad &+ \lambda S^{(3)} + \ldots = S \\ \mathcal{D}^{\alpha\beta}{}_{\rho\sigma} h^{\rho\sigma} &+ t^{(2)\alpha\beta} + \ldots = 0 \end{aligned}$$

- **4.** This leads to a constraint between $S^{(2)}$ and $S^{(3)}$.

Idea of the procedure

- **5.** Now $S^{(3)}$ would give a contribution to the energy-momentum tensor:

$$\begin{aligned} S^{(2)} \quad &+ \lambda S^{(3)} \quad &&+ \ldots = S \\ \mathcal{D}^{\alpha\beta}{}_{\rho\sigma} h^{\rho\sigma} + t^{(2)\alpha\beta} &+ t^{(3)\alpha\beta} &&+ \ldots = 0 \end{aligned}$$

Idea of the procedure

- **6.** We want to derive it from an action $S^{(4)}$

$$\begin{aligned} S^{(2)} \qquad\quad &+ \lambda S^{(3)} + \lambda^2 S^{(4)} + \ldots = S \\ \mathcal{D}^{\alpha\beta}{}_{\rho\sigma} h^{\rho\sigma} &+ t^{(2)\alpha\beta} + t^{(3)\alpha\beta} + \ldots = 0 \end{aligned}$$

- **7.** Doing this recursively, we generate constraints between $S^{(n)}$ and $S^{(n+1)}$.

All orders analysis

- In general, doing this procedure is impossible in practice.
- Approach by Butcher *et al* (2009): do a reverse engineer exercise.
- Consider GR, expand on an arbitrary background to obtain all orders $g^{\mu\nu} \to \bar{g}^{\mu\nu} + h^{\mu\nu}$:

$$S_{\text{GR}}[g] = \sum_{n=2}^{\infty} \lambda^n S^{(n)}[\bar{g}, h]$$

- Notice that $S^{(n)}$ contains terms that vanish when we impose $\bar{g} = \eta$ (e.g. $R_{\mu\nu} h^{\mu\nu} h$).

All order analysis

- They showed that:

$$\frac{\delta S^{(n+1)}[\eta, h]}{\delta h^{\mu\nu}} \sim t_{\mu\nu}^{(n)}$$

$$t_{\mu\nu}^{(n)} \sim \left.\frac{\delta S^{(n)}}{\delta \bar{g}^{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

Butcher *et al* Phys.Rev.D **80** 084014, (2009)

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- This precisely shows that GR bootstraps since we have:

$$S^{(2)} + \lambda S^{(3)} + \lambda^2 S^{(4)} + \ldots = S$$

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Repeating the analysis for arbitrary metric theories:

- We were able to prove the same identities for an arbitrary metric theory of gravity:

$$S[g] = \sum_{n=2}^{\infty} \lambda^n S^{(n)}[\bar{g}, h]$$

$$\frac{\delta S^{(n+1)}[\eta, h]}{\delta h^{\mu\nu}} \sim t^{(n)}_{\mu\nu}$$

$$t^{(n)}_{\mu\nu} \sim \left. \frac{\delta S^{(n)}}{\delta \bar{g}^{\mu\nu}} \right|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

This means that any metric theory bootstraps

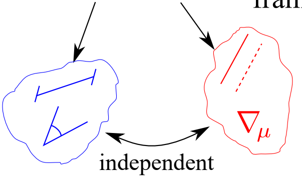
- This means that any metric theory bootstraps from its linearization.
- This also illustrates the non-uniqueness of the construction from bottom up.
- Fierz-Pauli is the linearizations of both: Lovelock and GR.
- Different choices of the energy-momentum tensor lead to different theories at the end.
- The ambiguities are present and they are crucial! This was somehow overlooked in the literature.

The metric-affine framework

$$(M, g_{\mu\nu}) \rightsquigarrow (M, g_{\mu\nu}, \Gamma_{\mu\nu}{}^{\rho})$$

Metric-Affine framework

$(M, g_{\mu\nu})$ induces $\mathring{\Gamma}_{\mu\nu}{}^{\rho}$



Γ, g

Curvature

$$R_{\mu\nu\rho}{}^{\lambda} := \partial_\mu \Gamma_{\nu\rho}{}^{\lambda} - \partial_\nu \Gamma_{\mu\rho}{}^{\lambda} + \Gamma_{\mu\sigma}{}^{\lambda}\Gamma_{\nu\rho}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\lambda}\Gamma_{\mu\rho}{}^{\sigma}$$

Torsion

$$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}$$

Nonmetricity

$$Q_{\mu\nu\rho} := -\nabla_\mu g_{\nu\rho}$$

Credit: Alejandro Jiménez-Cano

Summary of the other things we did

- **1.** Do the analysis in terms of the vielbein, instead of the metric (required to include fermions in the picture).
- **2.** Do the analysis for an arbitrary metric-affine theory including torsion and non-metricity (general connection).
- **3.** Include arbitrary matter content coupled to the metric and the general connection.

Results for metric affine theories

- Associated with Lorentz transformations we have the spin density current.
- We showed that any theory with a metric/vielbein couples to the energy-momentum tensor and dynamical torsion couples to the spin-density current order by order.
- Nonmetricity is tricky because the dilation-shear tensor to which it couples does not have a canonical counterpart.

Clarifying bootstrap of higher derivative theories

- Deser worked in Palatini formalism (metric and connection independent).
- Higher derivative gravities cannot be bootstrapped unless one imposes a constraint: connection = Levi-Civita.

Clarifying bootstrap of higher derivative theories

- Deser worked in Palatini formalism (metric and connection independent).
- Higher derivative gravities cannot be bootstrapped unless one imposes a constraint: connection = Levi-Civita.
- In higher derivative gravities, the Palatini formulation is not equivalent to metric formulation.
- Either one gives dynamics to the connection and bootstraps it or one needs to impose the constraint to propagate the same degrees of freedom.

Messages to take home

- The bootstrapping procedure is not unique in general due to the ambiguities. (Lovelock and GR)
- The connection also needs to be bootstrapped if it is dynamical.
- Torsion couples to the spin density current and metric perturbations to the energy-momentum tensor order by order.

Thanks for the attention!