Gravitational perturbations of noncommutative Schwarzschild black hole

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- Star-product
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Noncommutativity

- Noncommutative space
- Hopf algebra
- Star product
- Noncommutative differential geometry

Star-product

For two vector fields X and K we define

$$f \star g = f \, \exp\left(\frac{ia}{2}\left(\overleftarrow{\mathcal{L}}_K \overrightarrow{\mathcal{L}}_X - \overleftarrow{\mathcal{L}}_X \overrightarrow{\mathcal{L}}_K\right)\right) g$$
$$= f\, g + \frac{ia}{2}\Big(\mathcal{L}_K(f)\mathcal{L}_X(g) - \mathcal{L}_X(f)\mathcal{L}_K(g)\Big) + O(a^2),$$

where f and g are smooth functions on the spacetime manifold.

In spherical coordinates for $X = \partial_r$ and $K = \alpha\partial_t + \beta\partial_\varphi$ we have

$$[t, r]_\star = ia\alpha,$$
$$[\varphi, r]_\star = ia\beta.$$

Noncommutative differential geometry

Star-tensors are multilinear with respect to the star product, e.g.

$$T(f \star \partial_\mu, \partial_\nu) = f \star T(\partial_\mu, \partial_\nu).$$

The metric inverse satisfies

$$g_{\mu\alpha} \star g^{\alpha\nu} = g^{\nu\alpha} \star g_{\alpha\mu} = \delta^\nu_\mu.$$

Christoffel symbols, Riemann and Ricci tensor and Ricci scalar are given by

$$\begin{aligned}
\Gamma^\mu_{\nu\rho} &= \frac{1}{2} g^{\mu\alpha} \star (\partial_\nu g_{\rho\alpha} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}), \\
R_{\mu\nu\rho}{}^\sigma &= \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\beta_{\nu\rho} \star \Gamma^\sigma_{\mu\beta} - \Gamma^\beta_{\mu\rho} \star \Gamma^\sigma_{\nu\beta}, \\
R_{\nu\rho} &= R_{\mu\nu\rho}{}^\mu, \\
R &= g^{\mu\nu} \star R_{(\mu\nu)}.
\end{aligned}$$

Noncommutative vacuum Einstein equation is

$$R_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} \star R = 0 \implies R_{(\mu\nu)} = 0.$$

Linearized Schwarschild metric

To study perturbations of the Schwarzschild spacetime, we split the metric into background $\mathring{g}_{\mu\nu}$ and perturbation $h_{\mu\nu}$,

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}.$$

We then decompose the axial modes of $h_{\mu\nu}$ as

$$h_{t\theta} = \frac{1}{\sin\theta} \sum_{\ell,m} h_0^{\ell m}(r) \partial_\varphi Y_{\ell m}(\theta,\varphi) e^{-i\omega t},$$

$$h_{t\varphi} = -\sin\theta \sum_{\ell,m} h_0^{\ell m}(r) \partial_\theta Y_{\ell m}(\theta,\varphi) e^{-i\omega t},$$

$$h_{r\theta} = \frac{1}{\sin\theta} \sum_{\ell,m} h_1^{\ell m}(r) \partial_\varphi Y_{\ell m}(\theta,\varphi) e^{-i\omega t},$$

$$h_{r\varphi} = -\sin\theta \sum_{\ell,m} h_1^{\ell m}(r) \partial_\theta Y_{\ell m}(\theta,\varphi) e^{-i\omega t}.$$

The metric inverse is

$$g^{\mu\nu} = \mathring{g}^{\mu\nu} - \mathring{g}^{\mu\alpha} \star h_{\alpha\beta} \star \mathring{g}^{\beta\nu}.$$

We can now calculate the Christoffel symbols, Riemann and Ricci tensor up to the first order in $h_{\mu\nu}$ and noncommutativity parameter a.

Since $h_{\mu\nu} \propto e^{-i\omega t} e^{im\varphi}$, for $K = \alpha\partial_t + \beta\partial_\varphi$ we have

$$\mathcal{L}_K h_{\mu\nu} = i\lambda h_{\mu\nu},$$

where $\lambda = -\alpha\omega + \beta m$ is the eigenvalue of Killing field's action on the perturbation mode.

Einstein equation

There are three nontrivial radial functions governing the $R_{(\mu\nu)}$. For black hole with Schwarschild radius R and for the quadrupole $\ell = 2$ they are

$$
\begin{aligned}
R_{(r\varphi)} = &\frac{1}{4r^2(r-R)^2}\Big[\, 4ir^4(r-R)\omega h_0 + 2r^2(r-R)\big(r^3\omega^2 - r(r-R)\big)h_1 - 2ir^5(r-R)h_0' \\
&+ \lambda a\Big(2ir^3\omega(r-2R)h_0 + \big(24r(r-R)^2 - 9(r-R)^2R - r^4R\omega^2\big)h_1 + ir^4R\omega h_0' + 2r(r-R)^3h_1'\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
R_{(t\varphi)} = &\frac{1}{4r^2}\Big[4r(R-3r)h_0 + 4ir^2\omega(r-R)h_1 + 2r^3(r-R)(i\omega h_1' + h_0'') \\
&+ \lambda a\Big((12r+R)h_0 + ir\omega(4r-3R)h_1 + r(4r-5R)h_0' + r^2R(i\omega h_1' + h_0'')\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
R_{(\theta\varphi)} = &-\frac{ir^3\omega}{r-R}h_0 - Rh_1 - r(r-R)h_1' \\
&+ \lambda a\Big(\frac{ir^2R\omega}{2(r-R)^2}h_0 - 3\frac{r-R}{r}h_1 - \frac{1}{2}Rh_1'\Big).
\end{aligned}
$$

By equating those functions to zero we obtain a system of three linearly dependent equations in h_0 and h_1. We express h_0 from the third equation and plug it into the first to get

$$
\begin{aligned}
&- r(r-R)(-4r^2 + 5R^2 + r^4\omega^2)h_1 + r^2(2r-5R)(r-R)^2 h_1' - r^3(r-R)^3 h_1'' \\
&+ \lambda a\Big(\frac{1}{2}R(-25r^2 + 52rR - 26R^2 + r^4\omega^2)h_1 - 3r(r-2R)(r-R)^2 h_1' - \frac{1}{2}rR(r-R)h_1''\Big) = 0.
\end{aligned}
$$

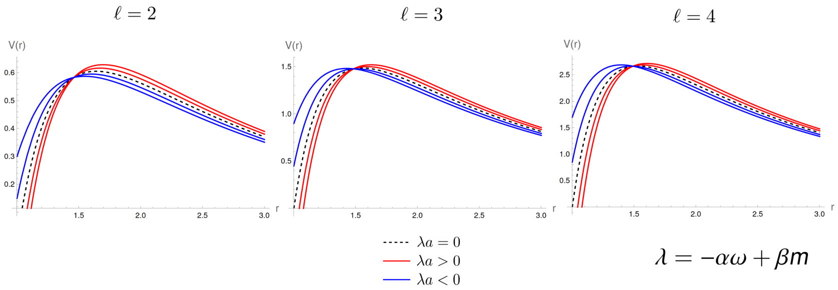
We introduce the modified tortoise coordinate and replace $h_1(r)$ with $W(r)$ as

$$
\begin{aligned}
\frac{dr}{dr_*} &= 1 - \frac{R}{r} + \lambda a\frac{R}{2r^2} \implies r_* = r + R\log\frac{r-R}{R} + \lambda a\frac{R}{r-R}, \\
h_1(r) &= \frac{r^2}{r-R}\Big(1 + \frac{\lambda a}{2}\Big(\frac{3}{r} - \frac{1}{r-R} + \frac{1}{R}\log\frac{r}{r-R}\Big)\Big)W(r).
\end{aligned}
$$

The equation becomes

$$\frac{d^2W}{dr_*^2} + \Big(\omega^2 - V(r)\Big)W = 0,$$

$$V(r) = \frac{(r-R)\Big(\ell(\ell+1)r - 3R\Big)}{r^4} + \lambda a \frac{r(3R-2r)\ell(\ell+1) + R(5r-8R)}{2r^5}.$$



$$\lambda = -\alpha\omega + \beta m$$

Figure: Plot of the potential with respect to the radial coordinate r for $\ell = 2, 3$ and 4. The blue lines correspond to $\lambda a = 0.1,\ 0.2$ and the red lines to $\lambda a = -0.1,\ -0.2$. Schwazschild radius is at $R = 1$ and the dashed line is potential without the noncommutativity corrections.

Asymptotic solutions are

$$r \to R: \quad \frac{d^2W}{dr_*^2} + \Big(\omega^2 + \lambda a \frac{3-\ell(\ell+1)}{2R^3}\Big) W = 0 \implies W \propto e^{\pm i\Omega r_*},$$
$$r \to \infty: \quad \frac{d^2W}{dr_*^2} + \omega^2 W = 0 \implies W \propto e^{\pm i\omega r_*}, \quad \Omega^2 = \omega^2 + \lambda a \frac{3-\ell(\ell+1)}{2R^3}.$$

For the angular noncommutativity ($\beta = 1 \implies \lambda = m$),

$$[\varphi, r]_\star = ia,$$

we observe Zeeman-like splitting of the potential.

For the time noncommutativity ($\alpha = 1 \implies \lambda = -\omega$)

$$[t, r]_\star = ia,$$

waves with different frequencies experience different potentials.

References

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