Effects of non-commutativity of space-time on the stellar properties

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Quantum gravity phenomenology in the multi-messenger approach

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Motivation: why (sub-)stellar objects and MG/QG?

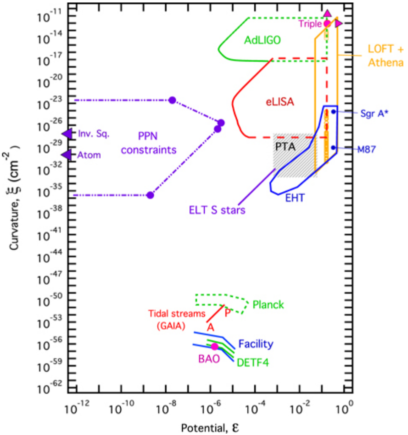
- MG/QG affects physics of non-relativistic objects[1]
- We understand the physics of those objects a bit better than physics of neutron stars and black holes
- In stars and giant planets, all four interactions are taking place in the regimes of temperatures and pressures a bit better understood
- The biggest impact of MG seem to be mainly related to the age of the particular objects ("objects older" than the Universe, formation of the Solar System, age determination techniques)[2]
- We are/we will be equipped with more and more accurate data[3], e.g. Cosmic Vision 2015-2025, Voyage 2050, James Webb & Nancy Grace Roman Space Telescopes,...

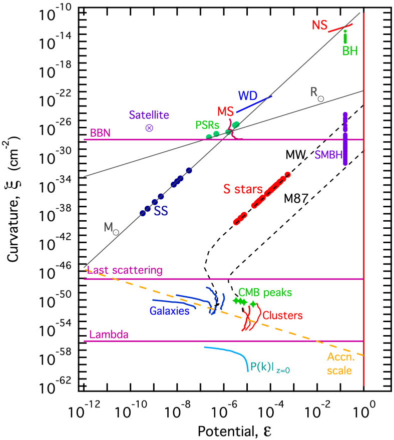
[1] G.J. Olmo, D. Rubiera-Gracia, AW, Stellar structure models in modified theories of gravity: lessons and challenges, Physics Reports 876 (2020); PRD 104 (2021) 2, 024045; A. Kozak, AW, Eur. Phys. J. C 81 (2021) 6, 492

[2] AW, PRD 103 (2021) 4, 044037; M. Benito, AW, PRD 103 (2021) 6, 064032; AW, PRD 104 (2021) 10, 104058; S. Kalita, L. Sarmah, AW, PRD 107 (2023) 4, 044072

[3] see current and near-future missions of ESA and NASA related to dwarf stars and (exo-)planets

The stellar and galaxy curvature regime not considered too much in MG/QG...





<u>Untested</u> regime in the **the galaxy and stellar physics regime**. It could potentially hide the onset of corrections to GR (T. Baker et al. 2015 ApJ 802, 63).

Curvature $\xi = \left(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\right)^{\frac{1}{2}} = \sqrt{48}\frac{GM}{r^3c^2}$

Potential $\varepsilon = \frac{GM}{rc^2}$

SPOILER: Astrophysical bounds on Generalized Uncertainty Principle[5]

Our bound when more realistic physics taken into account[4]

$$\beta_0 \leq 1.36 \times 10^{48}$$

experiment	ref.	upper bound on β
equivalence principle (pendula)	[240]	$\cancel{10^{20}}$ 10^{73}
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4 A. Pachol, AW, arXiv:2307.03520

5 See review by Bosso+ 2023, arXiv:2305.16193

Quantum gravity[6]

Observation 1:

Modifies Heisenberg uncertainty principle (GUP)

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}\Big(1 + \text{modification}\Big)$$

or/and dispersion relation

$$E^2 + p^2\Big(1 + \text{modification}\Big) = m^2$$

[6] LQG, Doubly Special Relativity, String Theory, Noncommunative geometry,...

Observation 2:

The weighted phase space volume is modified (D - dim of the phase space).

$$\frac{d^D \mathbf{x} d^D \mathbf{p}}{1 + \text{modification}}$$

Consequence: modified partition function ($z = e^{\mu / k_B T}$)

$$\ln \mathcal{Z} = \frac{V}{(2\pi\hbar)^3} \frac{g}{\pm 1} \int \ln \left(1 \pm z e^{-E/k_B T}\right) \frac{d^3 p}{1 + \text{modification}}$$

Conclusion: Quantum Gravity modifies equations of state since

$$P = k_B T \frac{\partial}{\partial V} \ln \mathcal{Z},$$

$$n = k_B T \frac{\partial}{\partial \mu} \ln \mathcal{Z} \mid_{T,V},$$

$$U = k_B T^2 \frac{\partial}{\partial T} \ln \mathcal{Z} \mid_{z,V}$$

Observation 3: MG as an effective theory derived from QG

Snyder model

One can consider many realizations of deformed phase spaces which correspond to Snyder non-commutative space time $[\hat{x}_\mu, \hat{x}_\nu] = i\hbar\beta M_{\mu\nu}$. β is related with the minimal (Planck) length.
The deformation of the quantum-mechanical phase space (Heisenberg algebra), in the most general realization (parametrized by χ) of Snyder model, up to the linear order in the non-commutativity parameter β, can be written as:

$$[p_i, \hat{x}_k] \quad = \quad -i\hbar\delta_{ik}\left(1+\beta\left(\chi-\frac{1}{2}\right)p_j p_j\right) - 2i\hbar\chi\beta p_i p_k + O(\beta^2).$$

The original Snyder case is recovered for $\chi = 1/2$. The most popular GUP is given by $\chi = 1/2$ and $\chi = 0$.
The Heisenberg algebra generators $\hat{x}_i$ and p_j can be represented on momentum space wave functions $\phi(p)$ as:

$$\hat{x}_i\phi(p) = i\hbar\left(\left(1+\beta\left(\chi-\frac{1}{2}\right)p_k p_k\right)\frac{\partial}{\partial p_i} + 2\chi\beta p_i p_j\frac{\partial}{\partial p_j} + \gamma p_i\right)\phi(p),$$

$$p_i\phi(p) = p_i\phi(p)$$

γ is an arbitrary constant, which does not enter the commutation relations, but affects the definition of the scalar product in momentum space (physical choice for $\gamma = 0$).
To define symmetric operators the new inner product in momentum space must take the form:

$$\langle\psi,\phi\rangle = \int\frac{d^D p}{(1+\beta(3\chi-\frac{1}{2})p^2)^\alpha}\psi^*(p)\phi(p) =: \int\frac{d^D p}{(1+\omega p^2)^\alpha}\psi^*(p)\phi(p)$$

where $\alpha = \frac{\beta(2\xi+D\xi-\frac{1}{2})-\gamma}{\beta(3\chi-\frac{1}{2})}$ and $\omega = \beta(3\chi-\frac{1}{2})$. Note that for $\xi = 1/6$ there is no deformation in the measure.
One can introduce $\alpha\omega = \beta(5\xi-\frac{1}{2})-\gamma$ for D=3 which includes deformation parameter β and choice of realization χ.

Fermi-Dirac equation of state[7]

Let us consider a system of N fermions with the energy states E_i. The partition function in the grand-canonical ensemble

$$\ln Z = \sum_i \ln\left[1 + ze^{-E_i/k_BT}\right]$$

where T is the temperature, k_B Boltzmann constant, $z = e^{\mu/k_BT}$ while μ is the chemical potential.
Considering a large volume and 3D

$$\sum_i \to \frac{1}{(2\pi\hbar)^3}\int \frac{d^3x d^3p}{(1+\omega p^2)^\alpha},$$

the partition function (note that $f(E) = \left(1 + ze^{-E/k_BT}\right)^{-1}$ is the F-D distribution function)

$$\ln Z = \frac{V}{(2\pi\hbar)^3} g \int \ln\left[1 + ze^{-E/k_BT}\right]\frac{d^3p}{(1+\omega p^2)^\alpha}\ ,$$

where g is a spin of a particle, $V := \int d^3x$ is the volume of the cell (of the configuration space), while $E = (p^2c^2 + m^2c^4)^{1/2}$.
The pressure is given by

$$P = \frac{1}{\pi^2\hbar^3}\int \frac{1}{3}p^3\, {}_2F_1\left(\frac{3}{2},\alpha,\frac{5}{2},-p^2\omega\right) f(E)\frac{c^2p}{E}dp \overset{|\omega p^2|<<1}{=} \frac{1}{\pi^2\hbar^3}\int \frac{p^3}{3}\left(\sum_{k=0}^{\infty}\frac{(\alpha)_k\left(\frac{3}{2}\right)_k(-\omega p^2)^k}{\left(\frac{5}{2}\right)_k k!}\right) f(E)\frac{c^2p}{E}dp$$

[7] A. Pachol, AW, arXiv:2304.08215

Non-relativistic ($E \approx \frac{p^2}{2m_e}$) degenerate Fermi gas[8]

$$P = \frac{1}{3\pi^2\hbar^3}\int\left(\frac{(2m_eE)^{\frac{3}{2}}}{5} - \frac{3\alpha\omega}{35}(2m_eE)^{\frac{5}{2}}\right)f(E)dE.$$

In the limit $T \to 0$, the chemical potential $\mu \approx E_F$

$$f(E) = \begin{cases} 1 & \text{if } E \leq E_F \\ 0 & \text{otherwise.} \end{cases}$$

$$P_{T\to 0} = \frac{2}{5}vE_F^{\frac{5}{2}}\left(1 - \frac{3\alpha\omega}{7}(2m_e)E_F\right),$$

where we have defined $v = \frac{(2m_e)^{\frac{2}{3}}}{3\pi^2\hbar^3}$.

[8] A. Pachol, AW, arXiv:2304.08215

Non-relativistic ($E \approx \frac{p^2}{2m_e}$) degenerate Fermi gas[10]

Let us use the definition of the measure of electron degeneracy ($u = (3\pi^2\hbar^3 N_A)^{\frac{2}{3}}/2m_e$)

$$\psi = \frac{k_B T}{E_F} = \frac{2m_e k_B T}{(3\pi^2\hbar^3)^{2/3}}\left[\frac{\mu_e}{\rho N_A}\right]^{2/3} \equiv u^{-1}k_B T\left[\frac{\mu_e}{\rho}\right]^{2/3}$$

to rewrite the pressure as a mixture of two polytropes (compare to MG case[9])

$$P_{T\to 0} = \frac{2}{5}vu^{\frac{5}{2}}\left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}}\left[1 - \frac{3u}{7}\alpha\omega(2m_e)\left(\frac{\rho}{\mu_e}\right)^{\frac{2}{3}}\right] =: K_1\rho^{\Gamma_1} - \alpha\omega K_2\rho^{\Gamma_2}$$

where $K_1 = \frac{2}{5}vu^{\frac{5}{2}}\mu_e^{-\frac{5}{3}}$, $\Gamma_1 = 5/3$ and $K_2 = \frac{12}{35}vu^{\frac{7}{2}}m_e\mu_e^{-\frac{7}{3}}$, $\Gamma_2 = 7/3$.

Interpretation from the bulk modulus (incompressibility)

$$B = \frac{dP}{d\ln\rho} = (1 - \frac{6}{7}\alpha\omega m_e E_F)\rho^{\frac{5}{3}},$$

For incompressible solids $B \to \infty$ ($\alpha\omega < 0$) while for infinitely compressible one $B = 0$ ($\alpha\omega > 0$).

9 Kim, H. C. (2014), PRD 89(6), 064001; AW, PRD 107 (2023) 4, 044025

10 A. Pachol, AW, arXiv:2304.08215

Non-relativistic stars[12]

Non-relativistic Poisson, hydrostatic equilibrium, and mass equations

$$\nabla^2\phi = 4\pi G\rho, \qquad \frac{d\phi}{dr} = -\rho^{-1}\frac{dP}{dr}, \qquad M = \int 4\pi' \tilde{r}^2\rho(\tilde{r})d\tilde{r}.$$

Applying our Fermi equation of state ($\varepsilon = \frac{6}{7}\left(\frac{3\pi^2\hbar^3}{N_A\mu_e}\right)^{\frac{2}{3}}\alpha\omega$)

$$P = K\rho^{\frac{5}{3}}\left[1 - \varepsilon\rho^{\frac{2}{3}}\right]$$

we can write down modified Lane-Emden equation

$$\frac{d}{d\xi}\left\{\xi^2\frac{d\theta}{d\xi}\left[1 - \epsilon\theta\right]\right\} = -\xi^2\theta^{\frac{3}{2}} \ ,$$

Such an equation also results from MG[11] with a non-modified polytrope ($\tilde{\epsilon} = \frac{7}{4}K\epsilon$)

$$\nabla^2\phi = 4\pi G\rho - \tilde{\epsilon}\nabla^2\rho^{\frac{4}{3}}$$

[11] For review, see G. Olmo, D. Rubiera-Garcia, AW, Phys. Rept. 876 (2020) 1-75

[12] A. Pachol, AW, arXiv:2307.03520

Hydrogen burning - minimal main sequence mass[15]

... is the lowest mass where the rate-limiting reaction for hydrogen burning can be sustained stably.
... "it is mass that a pre-main-sequence star must have in order to jump into the Main Sequence family"

It means that energy generated in the core is compensated by energy radiated from the surface, which corresponds to the mass where

$$\boxed{L_{\text{hydrogen burning}} = L_{\text{photosphere}}}$$

We want to use this fact for constraining modified gravity theories

The observational bound[13]: M-dwarf star G1 866C with the mass $(0.0930 \pm 0.0008) M_\odot$

The GR theoretical prediction[14]: $\sim 0.08 - 0.09 M_\odot$

[13] D. Segransan et al., Astron. Astrophys. 364 (2000) 665

[14] A. Burrows, J. Liebert, Rev. Mod. Phys. 65, 301 (1993)

[15] J. Sakstein, PRL 115 (2015) 201101; G. Olmo, D. Rubiera-Garcia, AW, PRD 100 (2019) 4, 044020

Light elements burning in stellar cores[17]

The magnitudes of thermonuclear reaction rates are predominantly governed by temperature and density, enabling us to estimate the energy generation rate through the application of power laws

$$\dot{\epsilon}_{pp} = \dot{\epsilon}_c \left(\frac{T}{T_c}\right)^s \left(\frac{\rho}{\rho_c}\right)^{u-1} ,$$

where the two exponents can be phenomenologically fitted as $s \approx 6.31$ and $u \approx 2.28$ at the transition mass of the core[16], while the function

$$\dot{\epsilon}_c = \dot{\epsilon}_0 \, T_c^s \, \rho_c^{u-1} \, \mathrm{ergs\, g^{-1} s^{-1}} ,$$

with $\dot{\epsilon}_0 \approx 3.4 \times 10^{-9}$ in suitable units.
Luminosity from the hydrogen burning:

$$L_{pp} = \int \dot{\epsilon}_{pp} \, dM = 1.54 \times 10^7 L_\odot \frac{\delta^{5.49} \, (1-\epsilon)^{3/2}}{\gamma^{16.46} \omega} M_{-1}^{11.97} \frac{\eta^{10.15}}{(\alpha_d + \eta)^{16.46}} ,$$

Modelling photosphere: ideal gas, T_{eff} given by matching interior and photospheric entropies, simply opacity model

$$L_{\text{ph}} = 4\pi\sigma R^2 T_{\text{eff}}^4 = 0.534 L_\odot \frac{M_{-1}^{1.31}}{\eta^{3.99} \gamma^{0.37} (\alpha_d + \eta)^{0.37} \kappa_{-2}^{1.18}}$$

[16] A. Burrows et al. Reviews of Modern Physics, 65(2), 301 (1993)

[17] A. Pachol, AW, arXiv:2307.03520

Testing and constraining the theory with MMSM [18]

$$M_{-1}^{\text{MMSM}} = 0.227\frac{\gamma^{1.51}\omega^{0.09}(\alpha_d+\eta)^{1.51}}{(1-\epsilon)^{0.14}\delta^{0.51}\eta^{1.33}\kappa_{-2}^{0.11}},$$

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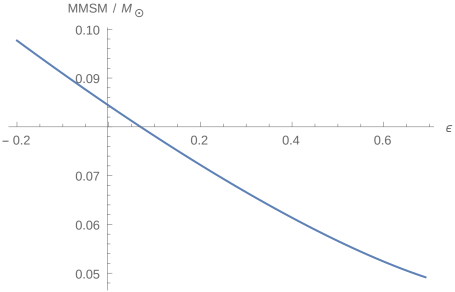

[18] A. Pachol, AW, arXiv:2307.03520

Our bound

$$\alpha\omega = \beta(5\chi - \frac{1}{2}) \geq -1.6 \times 10^{46}$$

- for $(5\chi - \frac{1}{2}) > 0$, implying $\chi > \frac{1}{10}$, we get:

  $$\beta \geq -\frac{1.6}{(5\chi - \frac{1}{2})} \times 10^{46},$$

  which is obvious as we assumed $\beta > 0$,
- for $(5\chi - \frac{1}{2}) < 0$, implying $\chi < \frac{1}{10}$, we get:

  $$\beta \leq \frac{1.6}{(\frac{1}{2} - 5\chi)} \times 10^{46}.$$

For example, choosing the value of the realization $\chi = 0$ (studied in many GUP models):

$$\beta_0 \leq 1.36 \times 10^{48},$$

where we have used a more common notation, that is, $\beta_0 = \beta M_P^2 c^2$.

Astrophysical bounds on Generalized Uncertainty Principle[19]

Our bound when more realistic physics taken into account (arXiv:2307.03520)

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[19] See review by Bosso+ 2023, arXiv:2305.16193

Summary and conclusions

- To obtain reasonable constraints, we need to consider more realistic description of matter
- We must be consistent in describing physical systems in different scales
- More research on matter properties in the QG/MG framework is necessary
- Tests of gravity with the use of stars and substellar objects (brown dwarfs, (exo)-planets, seismology)
- We should consider more realistic models on both sides: gravity and matter - rotating bodies, magnetic fields, ..., opacities (atmosphere)

Thanks!

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