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Entanglement entropy in conformal quantum mechanics

in collaboration with M. Arzano and D. Frattulillo

Based on [1]

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Introduction

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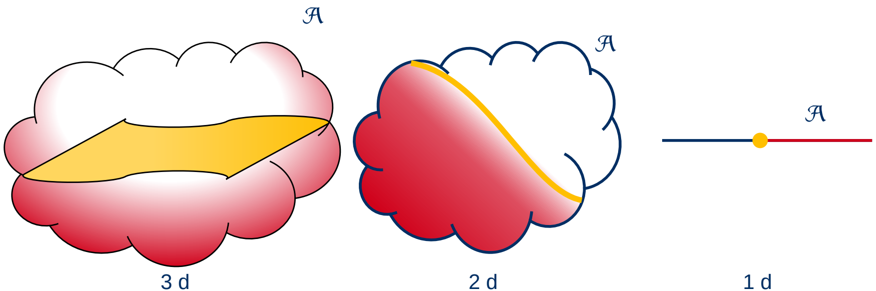

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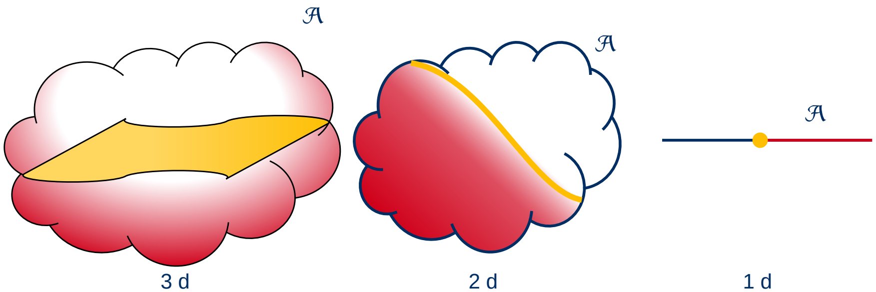

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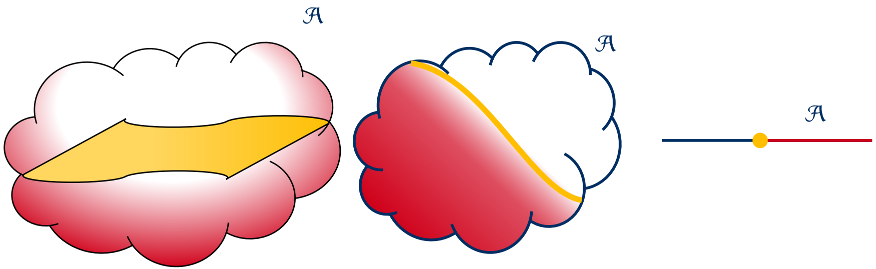

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the entanglement entropy is the **Von Neumann entropy** associated to ρ_L

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methods — correspondence between radial conformal symmetries in Minkowski space-time and time evolution in conformal quantum mechanics (CQM)

how — within CQM we identify states which have a thermofield double structure associated to these generators of time evolution

Conformal radial Killing vectors in Minkowski space-time

Minkowski metric in radial coordinates

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + \mathrm{dr}^2 + \mathrm{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega^2\right) \tag{1}$$

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close the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

$$[P_0, D_0] = P_0,\ [K_0, D_0] = -K_0,\quad [D_0, P_0] = 2D_0 \tag{4}$$

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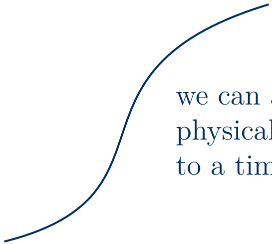
- combination of translations and special conformal transformations

$$S_0 = \frac{1}{2}\left(\alpha P_0 - \frac{K_0}{\alpha}\right) \tag{7}$$



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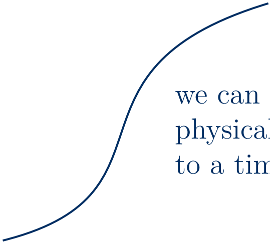
to a time-like worldline



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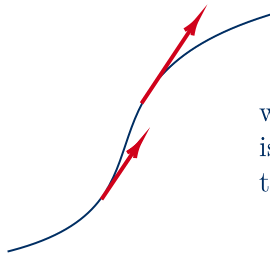

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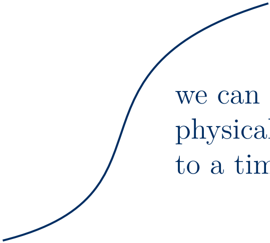

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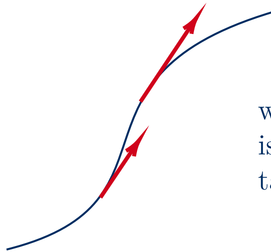

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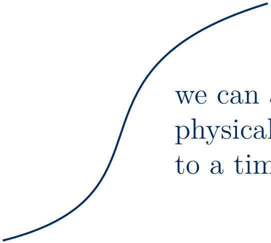
we will focus on **static** observers

$$\xi = \left(a\ t^2 + b\ t + c\right) \partial_t \tag{8}$$

generator of conformal transformations of the real (time) line



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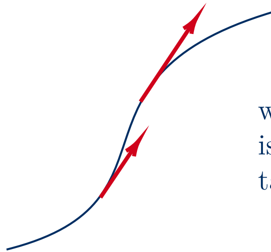

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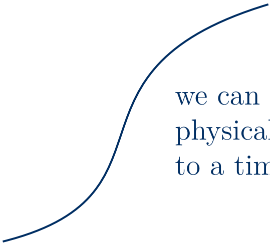
- D_0 time-like inside the future and past light-cone



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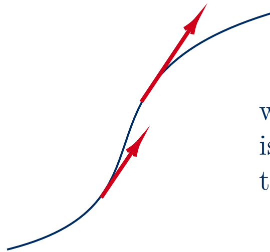

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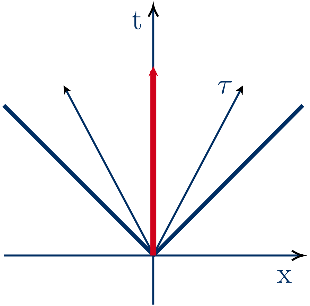

generates time evolution of a Milne observer in the Milne time τ

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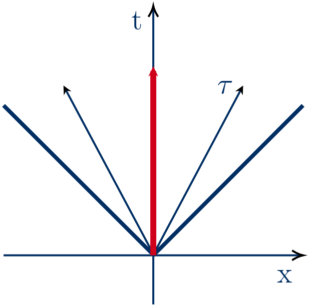

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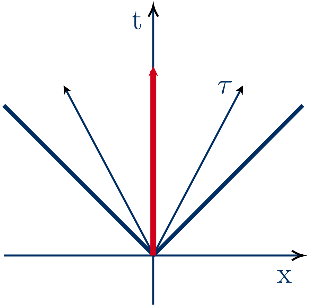

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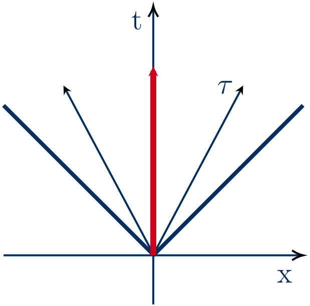

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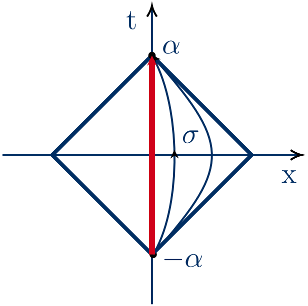

generates time evolution of a diamond observer in the diamond time σ

$\xi = D_0$ Milne space-time | $\xi = S_0$ causal diamond

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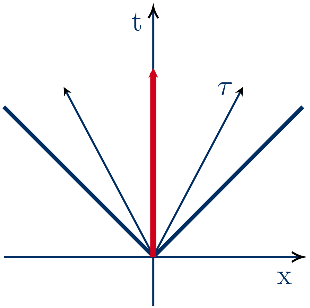

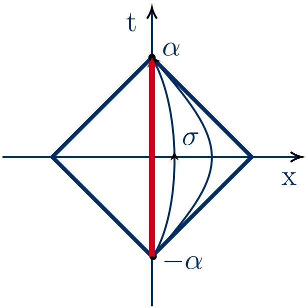

generates time evolution of a Milne observer in the Milne time τ

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$\exists$ an $SL(2,\mathbb{R})$ transformation mapping $S(t')$ into $D(t)$:

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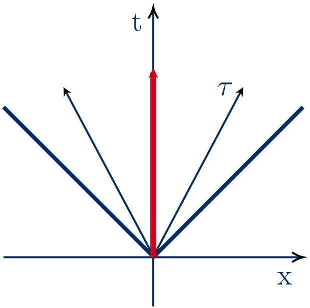

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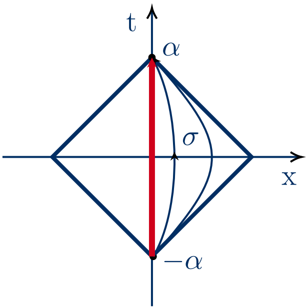

generates time evolution of a diamond observer in the diamond time σ

$$\exists \text{ an } SL(2,\mathbb{R}) \text{ transformation mapping } S(t') \text{ into } D(t): \qquad t' = \frac{\alpha(t-\alpha)}{t+\alpha} \tag{9}$$

Quantum mechanical counterpart

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how do we study time evolution of static observers in portions of Minkowski space-time using CQM?

Time evolution in Minkowski space-time using CQM

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$$\mathbf{L_0 = R}, \quad \mathrm{L}_\pm = \mathrm{S} \pm i\mathrm{D}, \quad \mathcal{C}^2 = \mathrm{R}^2 + \mathrm{R} - \mathrm{L}_-\mathrm{L}_+ \tag{11}$$

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R has discrete spectrum and normalised eigenvectors $|\mathrm{n}\rangle$

$$\mathrm{R}\,|\mathrm{n}\rangle = \mathrm{r_n}\,|\mathrm{n}\rangle \quad \text{with} \quad \mathrm{r_n} = \mathrm{r_0} + \mathrm{n}, \quad \mathrm{n} = 0, 1, \ldots, \quad \mathrm{r_0} > 0$$

$$\mathrm{L}_\pm\,|\mathrm{n}\rangle = \sqrt{\mathrm{r_n}(\mathrm{r_n} \pm 1) - \mathrm{r_0}(\mathrm{r_0} - 1)}\,|\mathrm{n} \pm 1\rangle$$

$$\langle \mathrm{m}|\mathrm{n}\rangle = \delta_{\mathrm{mn}}$$

$$\mathcal{C}^2\,|\mathrm{n}\rangle = \mathrm{r_0}(\mathrm{r_0} - 1)\,|\mathrm{n}\rangle$$

Time evolution in Minkowski space-time using CQM

$$\mathbf{L_0 = R}, \quad L_\pm = S \pm iD, \quad \mathcal{C}^2 = R^2 + R - L_- L_+ \tag{11}$$

R has discrete spectrum and normalised eigenvectors $|n\rangle$

$$R|n\rangle = r_n |n\rangle \quad \text{with} \quad r_n = r_0 + n, \quad n = 0, 1, \ldots, \quad r_0 > 0$$
$$L_\pm |n\rangle = \sqrt{r_n(r_n \pm 1) - r_0(r_0 - 1)}\, |n \pm 1\rangle$$
$$\langle m|n\rangle = \delta_{mn}$$
$$\mathcal{C}^2 |n\rangle = r_0(r_0 - 1)|n\rangle$$

change of basis $|\mathbf{n}\rangle \to |\mathbf{t}\rangle$ where

$$H|t\rangle = -i\frac{d}{dt}|t\rangle \tag{12}$$

$$\begin{cases} H\,|t\rangle = -i\frac{d}{dt}\,|t\rangle \\ D\,|t\rangle = -i\left(t\frac{d}{dt} + r_0\right)|t\rangle \\ K\,|t\rangle = -i\left(t^2\frac{d}{dt} + 2tr_0\right)|t\rangle \end{cases}$$

we set $r_0 = 1$ and obtain [4, 5]

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$$(n+1)\,\langle t|n\rangle = \langle t|R|n\rangle \implies \boxed{|t\rangle = \frac{1}{4}\left(\frac{a+it}{a-it}+1\right)^2 e^{-\frac{a+it}{a-it}\,L_+}\,|n=0\rangle} \tag{13}$$

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two sets of degrees of freedom as belonging to the domain of diamond and Milne time evolution and their complements

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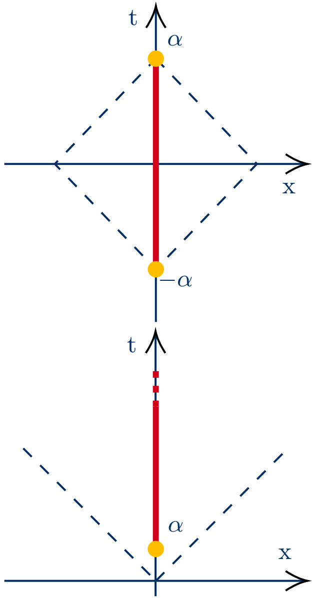
this entanglement entropy can be seen as the $0+1$-dimensional analogue of the entanglement entropy of a quantum field across space-time regions

Conclusions



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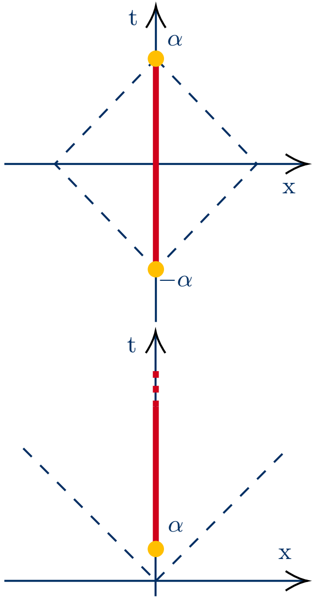

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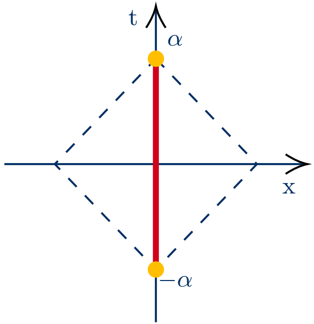

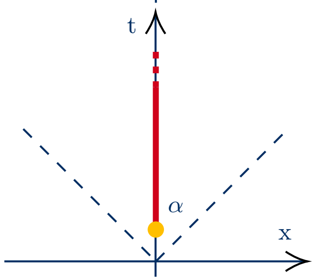

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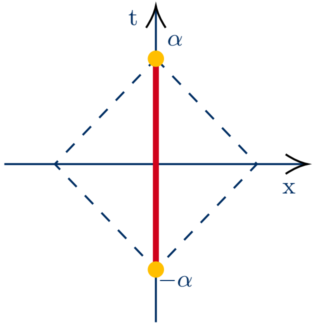

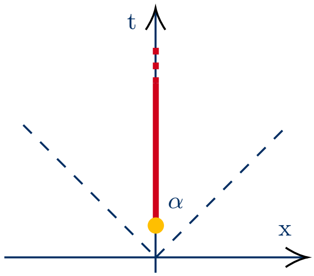

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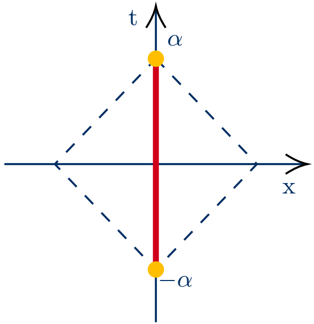

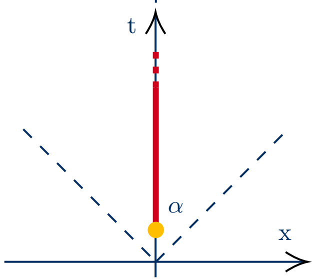

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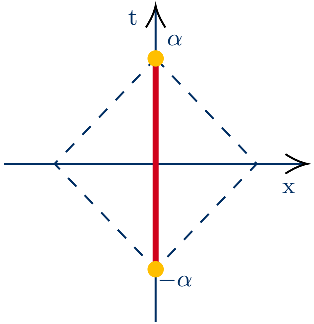

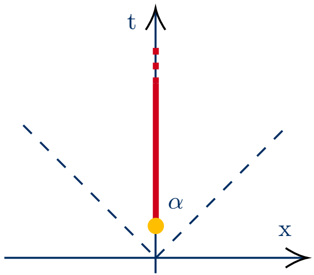

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- the result diverges logarithmically when the UV regulator is sent to zero as expected when the entangling boundary is **point-like**

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Thank you!

Backup slides

Conformal invariance

Consider the most general scalar massless field Lagrangian in CFT in d dimensions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \mathrm{g}\phi^{\frac{2\mathrm{d}}{\mathrm{d}-2}} \tag{18}$$

invariance under the full conformal group

H = translations

D = dilations

K = special conformal transformations

$$[\mathrm{H},\mathrm{D}] = \mathrm{iH},\ [\mathrm{K},\mathrm{D}] = -\mathrm{iK},\ [\mathrm{D},\mathrm{H}] = 2\mathrm{iD}$$

$$\mathcal{C}^2_{\mathrm{CFT}} = \frac{1}{2}(\mathrm{HK}+\mathrm{KH}) - \mathrm{D}^2 \quad \text{Casimir} \tag{19}$$

these operators leave the action invariant and are thus constant of motion

$$\xi = \mathrm{a}\,\mathrm{H} + \mathrm{b}\,\mathrm{D} + \mathrm{c}\,\mathrm{K} \tag{20}$$

ξ can be used to study the evolution of the system

Classification of radial conformal Killing vectors

a, b and c determine the causal character of ξ according to $\Delta = b^2 - 4ac$ [3, 4]:

- $\Delta < 0$ elliptic transformations (rotations)

$$R_0 = \frac{1}{2}\left(\alpha P_0 + \frac{K_0}{\alpha}\right) \tag{21}$$

- $\Delta > 0$ hyperbolic transformations (Lorentz boosts):

$$D_0 \quad \text{and} \quad S_0 = \frac{1}{2}\left(\alpha P_0 - \frac{K_0}{\alpha}\right) \tag{22}$$

- $\Delta = 0$ parabolic transformations (null rotations): P_0 and K_0

More on the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

$\mathfrak{sl}(2,\mathbb{R})$ Lie algebra can be realized in terms of two sets of creation and annihilation operators $a_L^\dagger, a_R^\dagger, a_L, a_R$

$$L_0 = \frac{1}{2}\left(a_L^\dagger a_L + a_R^\dagger a_R + 1\right) \ , \quad L_+ = a_L^\dagger a_R^\dagger \quad \text{and} \quad L_- = a_L a_R \tag{23}$$

this shows that the ground state of the R-operator has a bipartite structure

$$|n=0\rangle = |0\rangle_L \otimes |0\rangle_R \ , \tag{24}$$

and that the $|t=0\rangle$ state can be written as

$$|t=0\rangle = e^{-a_L^\dagger a_R^\dagger} |0\rangle_L |0\rangle_R = \sum_n (-1)^n |n\rangle_L |n\rangle_R = -i \sum_n e^{i\pi L_0} |n\rangle_L |n\rangle_R \ . \tag{25}$$

More on the inertial vacuum

$\mathrm{H}\,|\mathrm{E}\rangle = \mathrm{E}\,|\mathrm{E}\rangle$ satisfy the conditions

$$\left\langle \mathrm{E} \middle| \mathrm{E}' \right\rangle = \delta(\mathrm{E}-\mathrm{E}') \quad \text{and} \quad \int_0^{+\infty} \mathrm{dE}\,|\mathrm{E}\rangle\langle\mathrm{E}| = \mathbb{1} \tag{26}$$

we can write $|\mathrm{t}\rangle$ as [4]

$$|\mathrm{t}\rangle = \mathrm{e}^{\mathrm{iHt}}\,|\mathrm{t}=0\rangle = \int_0^{\infty} \mathrm{dE}\,\frac{\alpha\sqrt{\mathrm{E}}}{2}\mathrm{e}^{\mathrm{iEt}}\,|\mathrm{E}\rangle \tag{27}$$

and obtain the overlap between $|\mathrm{E}\rangle$ and $|\mathrm{t}\rangle$

$$\langle \mathrm{t}|\mathrm{E}\rangle = \frac{\alpha\sqrt{\mathrm{E}}}{2}\mathrm{e}^{-\mathrm{iEt}} \tag{28}$$

the states $|\mathrm{E}\rangle$ are similar in spirit to the momentum eigenstates $|\mathbf{p}\rangle$ that one introduces in QFT in terms of which the action of a field operator $\phi(\mathrm{x})$ on the vacuum state is

$$\phi(\mathbf{x})\,|0\rangle = \int \frac{\mathrm{d}^3\mathrm{p}}{(2\pi)^3}\frac{1}{2\mathrm{E_p}}\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,|\mathbf{p}\rangle\ , \quad \text{where} \quad \left\langle \mathbf{p}\middle|\mathbf{p}'\right\rangle = 2\mathrm{E}_{\mathbf{p}}(2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{p}') \tag{29}$$

$\xi = \mathrm{i}\ \mathrm{S}$

compactification proposal

$$\mathrm{L}_0 = \mathrm{i}\ \mathrm{S}, \quad \mathrm{L}_+ = \mathrm{i}\,(\mathrm{D}-\mathrm{R})\,, \quad \mathrm{L}_- = -\mathrm{i}\,(\mathrm{D}+\mathrm{R}) \tag{30}$$

$$\boxed{|\mathrm{t}\rangle = \frac{\alpha^2}{(\alpha+\mathrm{t})^2} e^{\frac{\mathrm{t}-\alpha}{\mathrm{t}+\alpha}\mathrm{L}_+}\,|\mathrm{n}=0\rangle} \tag{31}$$

eigenstate associated to an observer whose time evolution is defined by $\xi = \mathrm{i}\ \mathrm{S}$

$$\mathrm{i}\frac{\mathrm{S}}{\alpha} \mapsto \mathrm{i}\ \partial_\sigma \implies \langle\sigma_1|\sigma_2\rangle = -16\sin^{-2}\left(\frac{\sigma_1-\sigma_2}{2\alpha}\right) \xrightarrow{\alpha\mapsto \mathrm{i}\ \alpha} -16\sinh^{-2}\left(\frac{\sigma_1-\sigma_2}{2\alpha}\right)$$

$\alpha \mapsto \mathrm{i}\alpha$ maps the CQM description of $\xi = \mathrm{i}\ \mathrm{S}$ into the time evolution of a diamond observer!

The vacuum is perceived being in a thermal bath with $\boxed{\mathrm{T} = \frac{1}{2\pi\alpha}}$

$\xi = \mathrm{i}\,\mathrm{D}$

compactification proposal

$$\mathrm{L}_0 = \mathrm{i}\,\mathrm{D}, \quad \mathrm{L}_+ = -\mathrm{i}\,\alpha \mathrm{H}, \quad \mathrm{L}_- = -\mathrm{i}\,\frac{\mathrm{K}}{\alpha} \tag{32}$$

$$\boxed{|\mathrm{t}\rangle = \frac{\alpha^2}{2\mathrm{t}^2} \mathrm{e}^{-\frac{\alpha}{\mathrm{t}}\mathrm{L}_+} |\mathrm{n}=0\rangle} \tag{33}$$

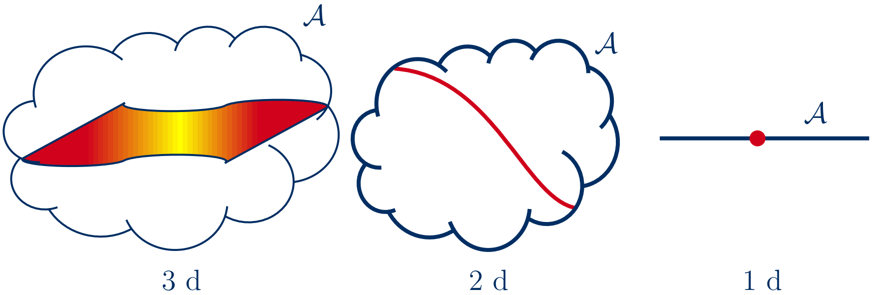
eigenstate associated to an observer whose time evolution is defined by $\xi = \mathrm{i}\,\mathrm{D}$

$$\mathrm{i}\frac{\mathrm{D}}{\alpha} \mapsto \mathrm{i}\,\partial_\tau \implies \langle \tau_1 | \tau_2 \rangle = -16 \sin^{-2}\left(\frac{\tau_1 - \tau_2}{2\alpha}\right) \xrightarrow{\alpha \mapsto \mathrm{i}\,\alpha} -16 \sinh^{-2}\left(\frac{\tau_1 - \tau_2}{2\alpha}\right)$$

$\alpha \mapsto \mathrm{i}\alpha$ maps the CQM description of $\xi = \mathrm{i}\,\mathrm{D}$ into the time evolution of a Milne observer!

The vacuum is perceived being in a thermal bath with $\boxed{\mathrm{T} = \frac{1}{2\pi\alpha}}$

Entanglement entropy



We want to compute the entanglement entropy [8] between two regions where the entangling surface (E.S.) is comprised of disconnected points

- in QFT the known result is [9]

$$S_{\mathcal{A}} \propto \frac{\text{Area}(\partial\mathcal{A})}{\epsilon^{d-2}} \xrightarrow[\text{point-like E.S.}]{\partial\mathcal{A}=\alpha} S_{\mathcal{A}} \propto \log\frac{\alpha}{\epsilon} \tag{34}$$

where Area($\partial\mathcal{A}$) is the area of the boundary of the region (e.s.) and ϵ is a UV cut-off