Consequences of quantum fluctuations for cosmology according to unimodular quantum gravity

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Program

- Use Unimodular Gravity as starting point for quantization.
- Focus on flat FLRW models with a scalar field representing the matter content.
- analyze the solutions and their spreading behaviour.
- try to give an estimate for the consequences of quantum fluctuations for cosmological predictions.

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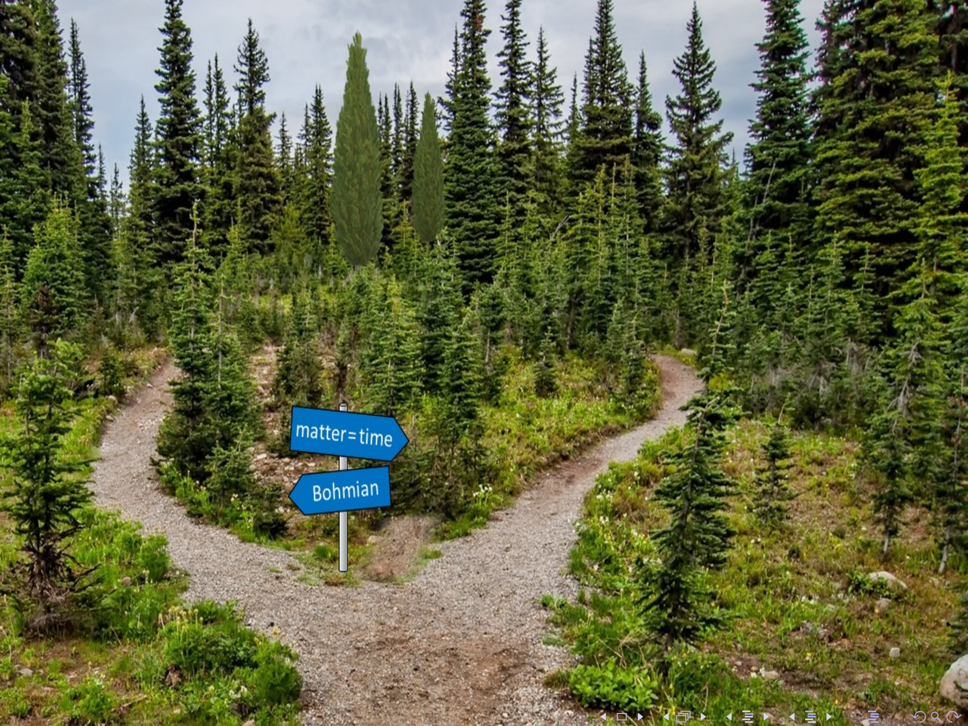
Structural problems: time vanishes, no positive definite scalar product , no unitary time evolution.


matter=time
Bohmian

1. The Bohmian Strategy

Time from guidance condition of the classical Hamilton-Jacobi Theory:

$$\frac{\partial L}{\partial \dot{q}} = p = \frac{\partial S}{\partial q}\,, \text{where} \quad \Psi = Re^{iS/\hbar}\,.$$

- advantage: Time emerges naturally from the canonical structure of the theory
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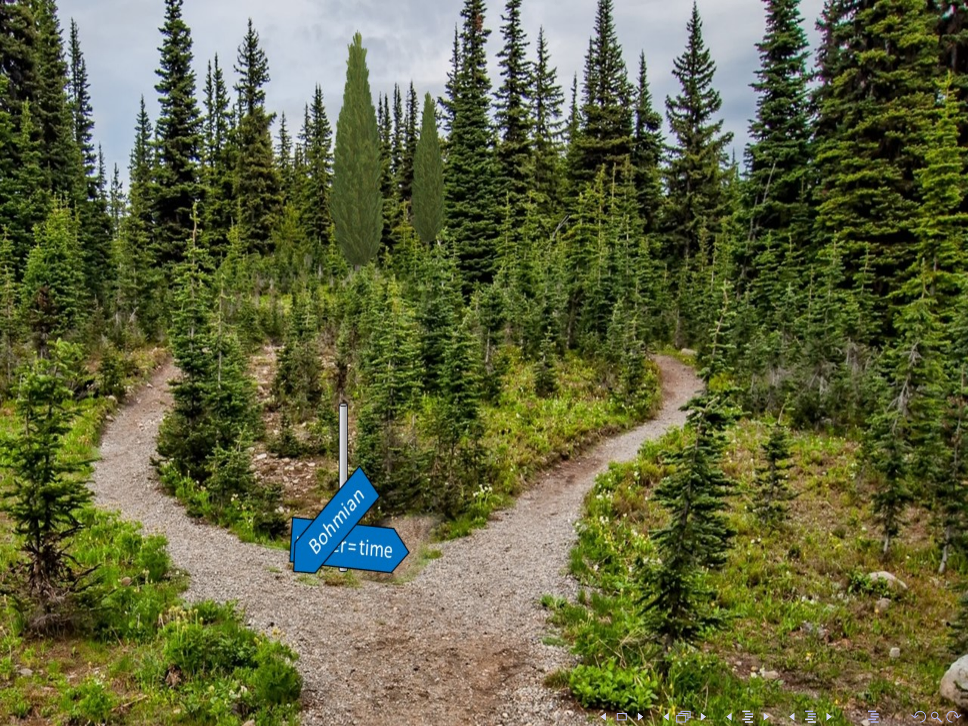
2. Matter as Clock

Using one matter variable as "time".

- advantage: Choice of scalar product and self-adjoint time evolution possible
- disadvantage: one canonical variable is taken out and declared as "time“.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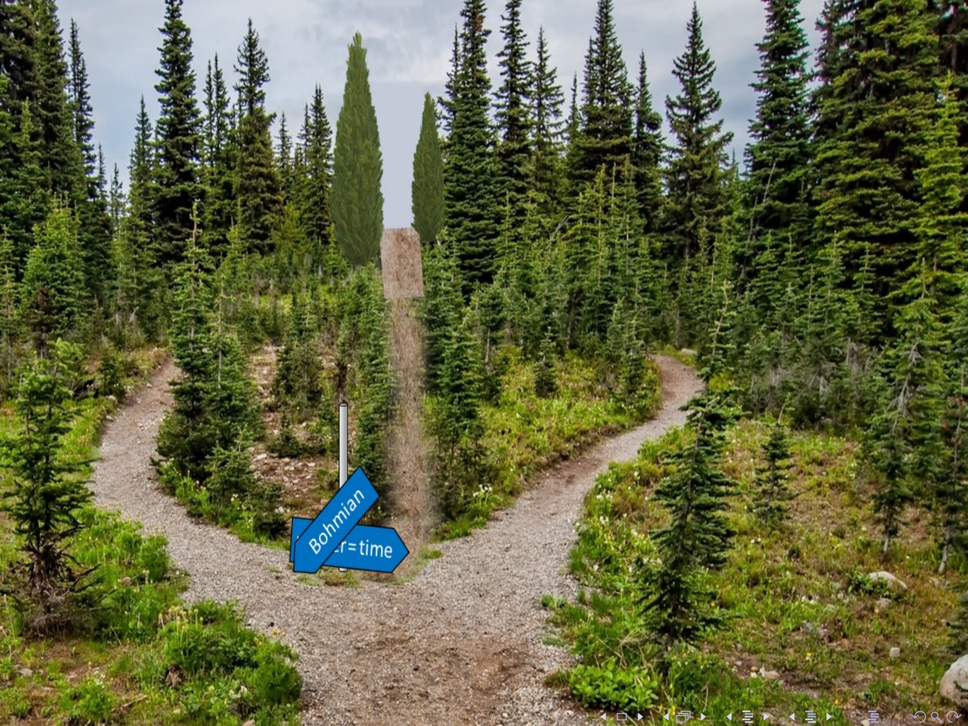
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Quantize Unimodular Gravity

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An alternative: Unimodular Gravity

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Quantize Unimodular Gravity

- fully equivalent to Einsteins Relativity on the classical level
- No need to reconstruct time- time does not vanish
- It was possible to define a scalar product and conditions for a self-adjoint time evolution for a flat Friedmann universe filled with a scalar field. $\surd$

Variational formulation of General Relativity - a reminder

$$\delta_{g_{\mu\nu}} \quad \left(\frac{1}{2\kappa} \int d^4x \sqrt{-g}\, R + S_{matter} \right)$$
$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

- Varying the action with respect to the metric.
- Getting Einsteins equations.

Unimodular gravity

$$\delta_{g_{\mu\nu}} \left(\frac{1}{2\kappa} \int d^4x \sqrt{-g}\, R + S_{matter} \right) \Bigg|_{\mathbf{g=-1}} = 0$$

$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} - \Lambda\, g_{\mu\nu}$$

- Varying the action with respect to the metric under the condition $det\, g_{\mu\nu} = g = -1$
- Getting Einsteins equations with an additional term
- Identifying Λ with the cosmological constant

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  with $\Psi \left[h_{ab}, t\right]$

- In the case of reduced models there is not any constraint at all.

The Model:

spacetime:

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2$$

$d\Omega_3^2 \ldots$ 3-dim. flat space

$$det\, g_{\mu\nu} \stackrel{!}{=} -1 \rightarrow N = 1/a^3$$

matter:

Lagrangian of the field ϕ

$$L_{matter} = N\, a^3 \left(\frac{\dot{\phi}^2}{2N^2c^2} - V(\phi) \right)$$

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- in general there exists no baratropic fluid equation for the scalar field
- exact equivalence between massless scalar field and stiff matter

$$c^2 \rho = p = \frac{\dot{\phi}^2}{2}$$

Unimodular Hamiltonian cosmology

Hamiltonian of a spatially flat Friedmann universe with scalar field :

$$H_{uni} = \frac{c^2}{2}\frac{p_\phi^2}{v_0 a^6} - \frac{c^2}{v_0 4\epsilon}\frac{p_a^2}{a^4} + v_0 V(\phi) . \qquad (\epsilon = 3c^4/(8\pi G) = 3/\kappa)$$

No Hamiltonian constraint, H_{uni} is a conserved quantity!

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Choice: $H_{uni} \equiv -\Lambda\epsilon v_0/3\,,$

so that Λ assumes the value of the cosmological constant in general relativity.

Unimodular Hamiltonian operator

- canonical quantization:

$$\hat{p}_a = -i\hbar\frac{\partial}{\partial a}\,, \quad \hat{p}_\phi = -i\hbar\frac{\partial}{\partial \phi}\,, \tag{2}$$

- factor ordering that yields a Laplace Beltrami operator

$$\Rightarrow \tag{3}$$

$$\widehat{H} = \frac{\hbar^2 c^2}{4 v_0 \epsilon}\frac{1}{a^5}\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{\hbar^2 c^2}{2 v_0}\frac{1}{a^6}\frac{\partial^2}{\partial \phi^2} + v_0 V(\phi)\,, \tag{4}$$

symmetric with respect to the measure $a^5 da\, d\phi$ (5)

Coordinate transformation:

$$A = a^3/3 \qquad B = \frac{3}{\sqrt{2\epsilon}}\phi \quad \Rightarrow$$

$$\widehat{H} = \frac{\hbar^2 c^2}{v_0 4\epsilon}\left\{\frac{1}{A}\frac{\partial}{\partial A}A\frac{\partial}{\partial A} - \frac{1}{A^2}\frac{\partial^2}{\partial B^2}\right\},$$

measure: $A\, dA\, dB$

Lightcone coordinates

$$u = Ae^{-B} \qquad v = Ae^{B} \,,$$

$$\widehat{H} = \frac{\hbar^2 c^2}{v_0 \, \epsilon} \frac{\partial^2}{\partial u \partial v} + v_0 V\left(\frac{u}{v}\right)$$

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$u \in (0, \infty), v \in (0, \infty)$

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Classical Hamiltonian in light cone coordinates:

$$H = -\frac{c^2}{\epsilon v_0} p_u\, p_v + v_0 V\left(\frac{u}{v}\right)\,.$$

Schrödinger equation of unimodular quantum cosmology

$$\frac{\hbar^2 c^2}{v_0 \epsilon} \frac{\partial^2}{\partial u \partial v} \psi + v_0 V\left(\frac{u}{v}\right) \psi = i\hbar \frac{\partial}{\partial t} \psi$$

Conventional probability interpretation for unitary time evolution possible !

Condition for the unitary time evolution

requirement on the wavefunction:

$$\frac{d}{dt}\langle\psi|\widehat{H}^n|\psi\rangle = 0 \quad \text{for} \quad n = 2, 3, \ldots .$$

sufficient condition:

$$\psi(0, v, t) = C(t)f_1(v) \qquad \psi(u, 0, t) = C(t)f_2(u)\,,$$

where $f_1(x), f_2(x)$ are real functions with $f_1(0) = \pm f_2(0)$ and $C(t)$ is arbitrary.

Constructing solutions for an arbitrary scalar field

Search for eigenstates:

$$\frac{1}{v_0}\frac{\partial^2}{\partial u \partial v}\psi_\Lambda(u,v) + v_0 V\left(\frac{u}{v}\right) = -\frac{\Lambda \epsilon v_0}{3}\psi_\Lambda(u,v)$$

with the boundary conditions

$$\psi_\Lambda(0,x) = f_1(x) \qquad \psi_\Lambda(x,0) = f_2(x),$$

where $f_1(x), f_2(x)$ are real functions.
We construct wavepacket solutions by superposition

$$\psi(u,v,\tau) = \int_{-\infty}^{\infty} e^{i\,t\,\frac{\Lambda \epsilon v_0}{3}}\psi_\Lambda(u,v)F(\Lambda)\,d\Lambda\,.$$

We obtain for the time evolution at the edges

$$\psi(0,v,\tau) = C(\tau) f_1(v) \qquad \psi(u,0,\tau) = C(\tau) f_2(u)$$

$$\text{where} \qquad C(\tau) = \int_{-\infty}^{\infty} e^{i\,t\,\frac{\Lambda \epsilon v_0}{3}} F(\Lambda)\, d\Lambda\,.$$

- **The solutions meet the condition for the unitary time evolution!**
- **For late times: asymptotic boundary conditions**

$$\lim_{\tau\to\infty} \psi(u,0,t) = \lim_{\tau\to\infty} \psi(0,v,t) = 0\,.$$

Analysis of the dynamics of the expectation values

The Heisenberg equations

$$\frac{d}{dt}\langle\psi|\widehat{O}|\psi\rangle = -\frac{i}{\hbar}\langle\psi|\left[\widehat{O},\widehat{H}\right]|\psi\rangle \tag{6}$$

apply only in the asymptotic future

$$t \ll \frac{\Lambda\epsilon v_0}{\hbar} = \frac{3\Lambda v_0 c^4}{8\pi G\hbar}$$

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- dynamical analysis independent of concrete wavefunction is only possible for late times.
- in this quasiclassical time-regime, the expectation values obey the classical dynamics provided the uncertainties remain small.

Analysis of the uncertainty dynamics

- Consider an expansion of the terms depending on $V\left(\frac{u}{v}\right)$ about the (classical) expectation values
- take no higher order terms than $(\Delta u)^2, (\Delta v)^2, \Delta(u, v)$
- get a non-autonomous system of 10 linear equations for the uncertainties.

$$\frac{d}{dt}(\Delta u)^2 = -\frac{2\mu}{v_0}(\Delta(u, p_v)$$
$$\frac{d}{dt}(\Delta v)^2 = -\frac{2\mu}{v_0}(\Delta(v, p_u))$$
$$\ldots$$

$\mu = c^2/\epsilon = 2\pi G/(3c^2)$

It contains the time-dependent functions

$$V_{11}(t) = \frac{\partial^2 V}{\partial v^2}, \quad V_{22}(t) = \frac{\partial^2 V}{\partial u^2} \quad V_{12}(t) = \frac{\partial^2 V}{\partial v \partial u}$$

which are taken at the classical values $u(t), v(t)$.

Analysis of the dynamical system

$$\frac{d}{dt}\overrightarrow{\Delta} = \mathcal{M}(t) \cdot \overrightarrow{\Delta}$$

$$\overrightarrow{\Delta} = \Big\{ (\Delta u)^2, (\Delta v)^2, \Delta(u,v), (\Delta p_u)^2, (\Delta p_v)^2, \\ \Delta(p_u, p_v), \Delta(u, p_u), \Delta(v, p_v), \Delta v, p_u, \Delta\left(u, p_v\right) \Big\}$$

The analysis requires the behaviour of $\mathcal{M}(t)$ fot $t \to \infty$

$\Rightarrow$ **Knowledge of classical late time behaviour necessary!**

Results for uncertainty dynamics

- the stiff matter case:

$$\mathcal{M}_0 \equiv \mathcal{M}\Big|_{\mathbf{V=0}}$$

The system is autonomous. Uncertainties are growing with leading order t^2.

- the general case:

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1(t)$$

the system is unstable for $\int_{t_0}^{\infty} |\mathcal{M}_1(t)| < \infty$

- exponential potential:

$$V = V_0 e^{\lambda\sqrt{\kappa}\phi}$$

due to classical analysis (Copeland, Liddle, Wands (1998)): $\mathcal{M}_1(t)$ integrable $\rightarrow$ uncertainty dynamics unstable

Classical and not classical epochs

- early epoch: $t \ll \frac{3\Lambda v_0 c^4}{8\pi G\hbar}$
- intermediate epoch:
  quasiclassical time evolution according to classical equation of motion with growing uncertainties no Heisenberg equations, no Ehrenfest theorem
- late epoch
  Growing uncertainties destroy the quasiclassical time evolution
  ??
  to be analyzed

Open question

How should v_0 be determined?

Matter density fluctuations

- determine the Wigner transform of the matter density operator $\hat{\rho}$
- perform an expansion around the classical values up to order Δ^2.

From our analysis we will assume the uncertainties grow with leading order t^2

Dark matter hypothesis

$$\langle \rho \rangle = \rho_{cl} + \Delta\rho$$

Dark matter hypothesis

visible matter

$$\langle \rho \rangle = \rho_{cl} + \Delta\rho$$

total amount of matter

dark matter

Estimation

Dark matter ratio :

$$R(\tau) \equiv \frac{\Delta\rho(\tau)}{\rho_{cl}(\tau)}$$

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Rough estimation for a matter dominated universe and uncertainties $\sim t^2$

$$\frac{R(\tau_2)}{R(\tau_1)} = \left(\frac{\tau_2}{\tau_1}\right)^6 = \frac{(1+z_2)^9}{(1+z_1)^9}$$

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Conclusion: Increasing dark mater ratio for increasing uncertainties!

Space-time fluctuations and light rays

geodesic equation:

$$\frac{dr}{dt} = \frac{c}{a(t)} \tag{7}$$

$\rightarrow$ stochastic equation:

$$\frac{dr}{dt} = \frac{c}{a(t)} + \mathcal{A}(\mathrm{t}) \tag{8}$$

$\mathcal{A}$t related to quantum fluctuations $\rightarrow$ calculate possible intrinsic fluctuations of redshift measurements.

Outlook

- How can we compare v_0 with observations?
- calculate inhomogeneities with unimodular theory